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Monte Carlo methods for pricing financial options

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Abstract. Pricing financial options is amongst the most important and challenging problems in the modern financial industry. Except in the simplest cases, the prices of options do not have a simple closed form solution and efficient computational methods are needed to determine them. Monte Carlo methods have increasingly become a popular computational tool to price complex financial options, especially when the underlying space of assets has a large dimensionality, as the performance of other numerical methods typically suffer from the 'curse of dimensionality'. However, even Monte-Carlo techniques can be quite slow as the problem-size increases, motivating research in variance reduction techniques to increase the efficiency of the simulations. In this paper, we review some of the popular variance reduction techniques and their application to pricing options. We particularly focus on the recent Monte-Carlo techniques proposed to tackle the difficult problem of pricing American options. These include: regression-based methods, random tree methods and stochastic mesh methods. Further, we show how importance sampling, a popular variance reduction technique, may be combined with these methods to enhance their effectiveness. We also briefly review the evolving options market in India.

Keywords. Pricing financial options; Monte Carlo methods; importance sampling; options market.

1. Introduction

Derivative financial securities such as options are securities whose value is based upon the value of more basic underlying securities. For example, a European call option on an underlying security gives the holder a right (not an obligation; hence the name option) to purchase the underlying security at a specified date for a specified price. In practice, far more complex options whose value may depend upon many underlying securities have become increasingly prevalent in markets all over the world. Such options are traded in high volume in stock exchanges. Many sophisticated options with complex payoff structures are routinely traded by financial institutions, fund managers and corporate treasurers in the *over the counter* market.

Options can often be classified in two broad categories: call options and put options. A *call option* gives the holder the right to *buy*, whereas a *put option* gives the holder the right

to *sell* the underlying asset(s) by a certain date for a certain price. Generally speaking, payoffs from derivative securities are a function of the trajectory followed by the underlying asset prices up to a specified time. European options can be exercised only at a fixed time, while American options can be exercised at any time up to a fixed time. This early exercise opportunity embedded in American options makes the problem of pricing them far more challenging. Bermudan options are intermediate between European and American options in that they can be exercised at any of the finite number of specified set of times (more than 1). If the number of exercise opportunities is large enough and well spaced, the associated Bermudan option price closely approximates the price of an American option. Often, American options are numerically priced by pricing a Bermudan option that closely approximates it.

1.1 *No arbitrage principle*

Determining the right price for such options has been an important and a challenging problem in the modern world of finance. An important breakthrough in this regard was made by Black & Scholes (1973), where using the no-arbitrage principle they derive a partial differential equation (PDE) that helps price certain generic options. Since then, this principle has spurred enormous research in determining the prices of financial securities. Roughly speaking, the no-arbitrage pricing principle states that two portfolios of securities that have the same payoffs in every possible scenario, should have the same price. Otherwise, by buying low and selling high, sure profit, or 'arbitrage', can be created from zero investment. Thus, if in a market, \$5 can be exchanged for Rs. 250 (bought or sold, assuming zero transaction costs), then the correct exchange rate for \$10 is Rs. 500. By buying low and selling high, any other rate would lead to an arbitrage.

Quite amazingly, under very general conditions, the no-arbitrage based European option price can be expressed as the mathematical expectation of the payoff from the option discounted at the risk-free rate under a new probability measure referred to as the 'risk-neutral' probability measure (see, e.g., Duffie 1996, Karatzas & Shreve 1991). This measure has the property that it is equivalent to the original measure, i.e., they both give positive probability to the same set of outcomes. Moreover, under it, the rate of return for each security equals the risk-free rate of return. Intuitively, this fact may be understood as follows: The no-arbitrage principle unambiguously prices options regardless of risk preferences of the investors. Thus, even if the investor is risk-neutral, he prices the security at the same value as investors with different risk preferences. Fortunately, the risk-neutral investor has a straightforward procedure for pricing securities. He expects all securities to have the same rate of return, i.e., the risk-free rate of return, thus he expects the security prices to evolve under the risk-neutral probability measure. Furthermore, from his perspective, the value of any security is the expected payoff from the security under the risk-neutral measure discounted to its present value using the risk-free rate of return.

Also note that under the risk-neutral probability measure the discounted security prices on an average do not have a drift, i.e., they are martingales. Thus, in the existing literature, this measure is also referred to as the equivalent martingale measure. There exists enormous research (see, e.g., Harrison & Kreps 1979 and Harrison & Pliska 1981 for pioneering work in this area) that extends the above idea to express the price of European options as a mathematical expectation of a random variable under an appropriate probability measure under very general assumptions (see, e.g., Glasserman 2004). We take this representation as a starting point of our discussion on the application of Monte Carlo methods for evaluating such expectations.

1.2 *Monte Carlo methods*

Note that Monte Carlo methods for evaluating the mathematical expectation of a random variable often involve generating many independent samples of the random variable and then taking the empirical average of the sample as a point estimate of the expectation. The accuracy of this method is proportional to $\sigma/\sqrt{n}$ where σ^2 denotes the variance of each sample, and n denotes the number of samples generated. The key advantage of the Monte-Carlo methods is that given the value of σ, the computational effort (roughly proportional to the number of samples) needed to achieve desired accuracy is independent of the dimension of the problem, i.e., if one thinks of the expectation as an integral, then this is independent of the dimension of the space where the integrand is defined. In this respect, it differs from other numerical techniques for evaluating integrals whose performance typically deteriorates as this dimension increases. An alternative approach to pricing options is to numerically solve the partial differential equation (PDE) satisfied by their price function (see, e.g., Wilmott *et al* 1993). However, if the asset price dynamics is sufficiently complex, a PDE characterizing the option price may even fail to exist. When the underlying dimensionality of the problem is large, numerical techniques (such as finite difference methods) to solve the PDE's may no longer be practical.

Thus, for complex options based on multi-dimensional underlying assets, the Monte-Carlo method provides a promising pricing approach. However, in many cases the computational requirement to get good accuracy can be prohibitive. To improve the efficiency of Monte Carlo methods to price options, several variance reduction techniques have been proposed in the recent literature. We briefly review some of the more promising techniques including the control variate, antithetic variate, stratification and importance sampling techniques.

1.3 *Pricing American or Bermudan options*

Recall that an American option can be exercised at any time up to a specified time. Thus, one may associate with such an option an exercise policy, i.e., a prescription that specifies in every scenario whether to hold on to or to exercise the option. Using the no-arbitrage principle, the value of the option under each such policy can be expressed as an expectation of a random variable. The rational price for the American option equals that of the policy having the maximum value (otherwise, an arbitrage can be created, see Duffie 1996). Finding this policy and hence the value of the option is a difficult problem. It can be seen that the option price as a function of time and state satisfies a set of dynamic programming backward recursion equations. A number of Monte Carlo methods have been recently designed to exploit this representation by approximately solving the backward recursion equations. We review three such methods:

(1) Regression-based methods proposed in Carriere (1996), Tsitsiklis & Van Roy (2001), and Longstaff & Schwartz (2001).
(2) Random tree methods proposed by Broadie & Glasserman (1997), and
(3) Stochastic mesh methods proposed by Broadie & Glasserman (2004).

The above techniques do not get unbiased estimates of the option price. To counter this drawback, these techniques often attempt to develop two estimates close to each other, one biased low and the other biased high. The low biased estimate is obtained easily as any guess for the optimal policy (the one with the maximum value) is sub-optimal and hence its value estimates a lower bound to the value of the optimal policy.

Recently the concepts of additive and multiplicative duality associated with American and Bermudan options have been developed that attempt to develop accurate upper bounds for option prices (see Andersen & Broadie 2001, Haugh & Kogan 2001, Jamshidian 2003, Bolia *et al* 2004). In this survey we briefly review these as well.

Bolia *et al* (2004) show how the variance reduction technique of importance sampling may be fruitfully combined with the regression-based methods. The novelty of our paper is that we extend their results to devise importance sampling techniques for random tree methods and stochastic mesh methods.

In the models to price American options, the state space may be partitioned into two sets: The *exercise region* or the set of states where it is optimal to exercise the policy, and the set of states where it is optimal to hold on to the option. In many cases the boundary separating the two sets has a nice structure. Significant literature exists that focuses on approximately learning the boundary dividing these two sets (see, e.g., Bossaerts 1989, Grant *et al* 1996, Li & Zhang 1996, Garcia 2003 and Wu & Fu 2003). However, in this survey we do not further review this approach.

It is noteworthy that the methods described in this paper illustrate an emerging body of literature to efficiently price American options. However, pricing of many complex multi-dimensional options may still take a prohibitive amount of computational time even with the approaches described. Thus, there remains a need for further research into development of efficient methodologies for pricing such options.

The structure of this paper is as follows: We first give examples of multi-dimensional options prevalent in the industry in §2. To appreciate how the no arbitrage principle guarantees the existence of a risk-neutral measure that can be used to price options we see its application in a simple one period Binomial tree example in §3. In §4 we describe the quantitative framework for conducting Monte Carlo simulation to price options. Popular variance reduction techniques and their performance on simple examples are discussed in §5. We review the mathematical framework for pricing Bermudan options (approximations to American options) in §6. In §7, 8 and 9 respectively, we discuss regression-based methods, random tree methods and stochastic mesh methods for pricing Bermudan options. In §10, we briefly review the additive and multiplicative duality in the context of Bermudan options. Finally, we discuss the evolution and the status of the derivatives market in India in §11.

2. Examples of multi-dimensional options

As indicated earlier, Monte Carlo methods typically find the greatest use in pricing multi-dimensional options (calls and puts). In this section we describe some of the commonly used multi-dimensional European puts and calls. By allowing early exercise for the European options, analogous Bermudan and American options may be constructed. Here, let $(S_1(t), S_2(t), \ldots, S_d(t))$ denote the price of d assets at time t. Let T denote the time to maturity of the option.

- *Basket option*: A basket option is an option whose return profile is based upon the average performance of a pre-set basket of underlying assets. If $(c_i, i = 1, 2, \ldots, d)$ denote the weight of each asset in the basket and K the strike price, then, the payoff of the basket call option is given by

$$\max\{[c_1 S_1(T) + c_2 S_2(T) + \ldots + c_d S_d(T)] - K, 0\}.$$

Typical examples are options based on a portfolio of related assets–bank stocks, Asian stocks, economic sector-based indices, and currencies. It is an ideal financial instrument for investors who wish broad exposure to a region, sector etc. This may also be of interest to a manufacturer who needs to purchase a collection of goods. A basket call option on these goods protects him against price hike in these goods.

- *Outperformance option*: These are options based on the relative performance of the underlying assets. Examples include option on the maximum or minimum of several assets. The payoff of a min put option, for instance, is given by

$$\max\{K - \min\{S_1(T), S_2(T), \ldots, S_d(T)\}, 0\}.$$

- *Barrier options*: Barrier options are a family of options that either come alive or die when an underlying asset reaches predetermined trigger points (barriers). A two-asset barrier option can have a payoff of the form

$$\mathbf{I}\left\{\min_{i=1,\ldots,n} S_2(t_i) < b\right\} \max\{S_1(T) - K, 0\},$$

where $0 \leq t_1 < \ldots \leq t_n = T$ and $\mathbf{I}\{\cdot\}$ denotes an indicator function. This is a down-and-in call on $S_1(\cdot)$ that knocks in (i.e. comes alive) when $S_2(\cdot)$ drops below a barrier b at any of the specified times. In this example, $S_1(\cdot)$ can be thought of as an underlying stock, and $S_2(\cdot)$ the level of an equity index. There can be many variations of this structure, knock-in feature being replaced by a knock-out (dying of the option) feature, for instance. Barrier options are attractive to buyers as they can be considerably cheaper than the associated option where the condition of the triggering barrier event is removed (so that the payoffs from the two are equal under certain scenarios and the latter may have a positive payoff while the former has a payoff of zero, in the other scenarios).

- *Average strike rate option*: This is an option where the strike price equals the average of the underlying index over the life of the option. The strike price can therefore be calculated only at the maturity of the option. Typically, the payoff of a put option looks like,

$$\max\{\bar{S}_2(T) - S_1(T), 0\},$$

where $S_1(\cdot)$ could be an individual asset and $\bar{S}_2(\cdot)$ denotes the average of an index over the life of the option.

- *Quantos*: A quanto (or a *cross-currency derivative*) is a cash-settled derivative that has an underlier denominated in one ("foreign") currency but settles in another ("domestic") currency at a fixed exchange rate. It is thus sensitive to both the asset price and an exchange rate. As an example, consider an option to buy a stock denominated in a foreign currency with the strike price fixed in the foreign currency but the payoff to be made in the domestic currency. If $S_1(\cdot)$ denotes the stock price and $S_2(\cdot)$ the exchange rate (quantity of domestic currency required per unit of foreign currency), then the payoff in the domestic currency is given by

$$S_2(T) \max\{S_1(T) - K, 0\}.$$

Quantos are attractive because they shield the purchaser from exchange rate fluctuations.

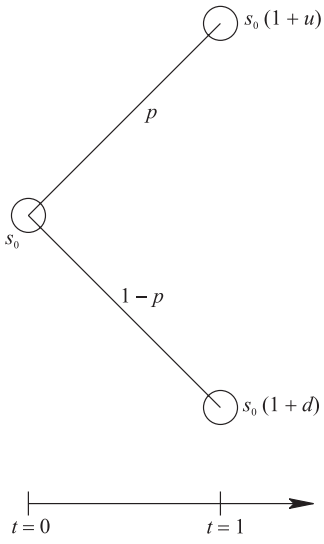

Figure 1. Binomial model.

3. One-period binomial tree model

The binomial tree approach to price options was first proposed by Cox, *et al* (1979). We consider a simple one-period binomial tree model and explain how the no-arbitrage principle guarantees the existence of the risk-neutral measure in this setting and show that the price of an option in this setting is simply the expectation of the discounted pay-off under this risk-neutral measure. In this setting we say that an arbitrage occurs if zero investment leads to no-loss with probability 1 and a positive profit with positive probability.

Consider an asset whose value at time zero equals s_0. Suppose that at time 1 two scenarios *up* and *down* can occur: the stock price takes value $(1+u)s_0$ in scenario up and $(1+d)s_0$ otherwise, where $u > d$. Suppose that scenario up occurs with probability $p > 0$ and scenario $down$ with probability $1-p$. These probabilities are referred to as *market probabilities*. Further suppose that in this economy there exists a risk-free asset whose value is 1 at time zero and it becomes $1+r$ in both the scenarios in time 1. We assume that an unlimited amount of both these assets can be borrowed or sold without any transaction costs.

First note that the no-arbitrage principle implies that $d < r < u$. Otherwise if $r \leq d$, borrowing amount s_0 at the risk-free rate and purchasing the risky security, the investor earns at least $s_0(1+d)$ in time period one where his liability is $s_0(1+r)$. Thus with zero investment he is guaranteed sure profit (at least with positive probability if $r = d$), violating the no-arbitrage condition. Similarly, if $r \geq u$, then by short-selling the stock the investor gains amount s_0 which he invests in the risk-free asset. In time 1 he gains $s_0(1+r)$ while his liability is at most $s_0(1+u)$ (the price at which he can buy back the risky security to close the short position), thus leading to arbitrage.

Now consider an option that pays amount O_u in the up scenario and amount O_d in the down scenario. To price it using the no-arbitrage principle we construct a portfolio of the risky and the risk-free asset that replicates the payoff of the option in the two scenarios. Then the value of this portfolio gives us the correct price of the option.

Suppose we purchase amount α of the risky security and invest amount β in the risk-free security, where α and β are allowed to take negative values and are chosen so that the resulting

payoff matches the payoff from the option at time one in the two scenarios. That is

$$\alpha(1+u)s_0 + \beta(1+r) = O_u,$$

and

$$\alpha(1+d)s_0 + \beta(1+r) = O_d.$$

Thus,

$$\alpha = \frac{O_u - O_d}{s_0(u-d)},$$

and

$$\beta = \frac{O_d(1+u) - O_u(1+d)}{(u-d)(1+r)}.$$

The value of the portfolio and hence of the option equals

$$\alpha s_0 + \beta = \frac{1}{1+r}\left(\frac{(r-d)}{u-d}O_u + \frac{(u-r)}{u-d}O_d\right).$$

Note that this price is independent of the value of the market or physical probability vector $(p, 1-p)$. Thus, p could be 0·01 or 0·99, it will not in any way affect the price of the option. Set $\hat{p} = (r-d)/(u-d)$, then due to the no-arbitrage principle $(\hat{p}, 1-\hat{p})$ denotes a probability vector. The value of the option equals

$$\frac{1}{1+r}\left(\hat{p}O_u + (1-\hat{p})O_d\right) = \hat{E}\left(\frac{1}{1+r}O\right) \tag{1}$$

where $\hat{E}$ denotes the expectation under the probability $(\hat{p}, 1-\hat{p})$ and $1/(1+r)O$ denotes the discounted value of the random pay-off at time 1, discounted at the risk-free rate. Interestingly,

$$s_0 = \frac{1}{1+r}\left(\hat{p}(1+u)s_0 + (1-\hat{p})(1+d)s_0\right),$$

so that the probability $(\hat{p}, 1-\hat{p})$ is characterized by the fact that under it the risky security earns the rate of return r. Clearly, these are the probabilities the risk-neutral investor would use and (1) denotes the price that the risk-neutral investor would assign to the option (blissfully unaware of the no-arbitrage principle!). Thus, the no-arbitrage principle leads to a pricing strategy in this simple setting that the risk-neutral investor would in any case follow. This result generalizes in far more mathematically complex models of asset price movement (discussed in the next section). In fact, it can be proved that the existence of a risk-neutral measure (also called *equivalent martingale measure*) is implied by the absence of arbitrage and vice-versa (see, e.g., Harrison & Kreps 1979). This fundamental theorem of asset pricing allows us to compute the price of an option as the mathematical expectation of the payoff from the option discounted at the risk-free rate under the risk-neutral probability measure, thus making the pricing problem amenable to Monte-Carlo analysis.

Another fundamental theorem of asset pricing considers necessary and sufficient conditions in the absence of arbitrage under which there exists a unique equivalent martingale measure

(under which discounted security price processes are martingales). This corresponds to models under which the market is *complete*, i.e., where any random payoff structure (i.e., payoff amount is a function of the observed scenario) can be replicated by a portfolio obtained by suitable time and scenario dependent trading of basic securities. Note that in the above one-period binomial tree example, any payoff (O_u, O_d) could be replicated by a portfolio of appropriate amount of risky and risk-free security, i.e., this market is complete.

4. Mathematical framework and simulation methodology

We now describe a general model for asset price movement during the time interval $[0, T]$. Suppose that there are d assets. Let $S_i(t)$ denote the price of asset i at time t and $S(t) = (S_i(t) : i \leq d)$. Suppose that $(B_i(t) : i \leq m, 0 \leq t \leq T)$ denote m independent standard Brownian motions that model the uncertainty in the evolution of the asset prices as given by the stochastic differential equations (SDE) (see, e.g., Oksendal 2003 for an introduction to SDEs)

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = a_i(S(t), t)\mathrm{d}t + \sum_{j \leq m} b_{ij}(S(t), t)\mathrm{d}B_j(t). \tag{2}$$

Here $a_i(S(t), t)$ denotes the instantaneous drift of the asset and $b_{ij}(S(t), t)$ measures the instantaneous sensitivity to the jth source of uncertainty for $i = 1, \ldots, d$ and $j = 1, \ldots, m$ (see, e.g., Karatzas & Shreve 1991 for conditions on $(a_i(\cdot, \cdot) : i \leq d)$ and on $(b_{ij}(\cdot, \cdot) : i \leq d, j \leq m)$ to ensure the existence and uniqueness of the solution to the SDE). The probability measure governing the movement of asset prices is referred to as the market measure. Readers unfamiliar with SDE's may view (2) as equivalent to its discretized approximation as in (5) discussed later. Further, suppose that amongst the d assets there exists one risk-free asset with a drift $r(\cdot, \cdot)$ and sensitivity to sources of uncertainty set to zero for all j and t. Then it can be seen that (see, e.g., Duffie 1996, Karatzas & Shreve 1991) under the no-arbitrage and some additional regularity conditions, the market is complete and there exists a unique risk-neutral measure (or equivalent martingale measure) under which the asset prices evolve according to the SDE's

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = r(S(t), t)\mathrm{d}t + \sum_{j \leq m} b_{ij}(S(t), t)\mathrm{d}B_j(t), \tag{3}$$

so that each security has an instantaneous drift equal to $r(\cdot, \cdot)$. Here, $(B_j(\cdot) : j \leq m)$ are again a collection of independent standard Brownian motions. Let $\tilde{E}$ denote the expectation operator under the risk-neutral measure. Then, it can be shown that due to the no-arbitrage principle, the price of any derivative security that pays an amount A at time T equals

$$\tilde{E}\left(\exp\left(-\int_0^T r(S(t), t)\mathrm{d}t\right)A\right).$$

This can be estimated via Monte-Carlo simulation. To keep the discussion simple, unless otherwise mentioned, we assume that the risk-free rate is constant and equals r.

Suppose that the payoff of the derivative security is a function of the asset prices at times $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$. For example, consider an Asian call option on an asset i

whose discounted payoff equals

$$\exp(-rT)\left(\frac{1}{n}\sum_{k=1}^{n} S_i(t_k) - K\right)^+,$$

where K denotes the strike price of the option. In such cases, if each $b_{ij}(S(t), t)$ is not a constant, it may be difficult to generate samples of $(S(t_k) : k \leq n)$ via simulation such that their joint distribution matches the corresponding joint distribution of the solution of the SDE's (3). Fortunately, when $b_{ij}(S(t), t) = b_{ij}$, a constant, then this is feasible. In that setting, we may set (by applying Ito's lemma to $\log S_i(t)$)

$$S_i(t_{k+1}) = S_i(t_k)\exp\left(r - \frac{1}{2}\sum_{j\leq m} b_{ij}^2\right)(t_{k+1} - t_k) + (t_{k+1} - t_k)^{1/2}\sum_{j\leq m} b_{ij}N_{k+1,j}), \tag{4}$$

where $N_{k,j}$ for each k and j are independent Gaussian random variables with mean zero and variance one (hereafter, we let $N(\mu, \sigma^2)$ denote a Gaussian random variable with mean μ and variance σ^2). In this case, the $(S(t_k) : k \leq n)$ so generated have the joint distribution that matches the corresponding joint distribution of the solution of the SDE's (3). In a general case, where r and b_{ij} are functions of $(S(t), t)$, generating exact distributions at the selected times may not be practical and one resorts to generating the approximate time-discretized version of the SDE using Euler's scheme. This amounts to setting

$$\begin{aligned} S_i(t_{k+1}) = {} & S_i(t_k) + S_i(t_k)r(S_i(t_k), t_k)(t_{k+1} - t_k) \\ & + S_i(t_k)(t_{k+1} - t_k)^{1/2}\sum_{j=1}^{m} b_{ij}(S_i(t_k), t_k)N_{k+1,j}. \end{aligned} \tag{5}$$

This method has a discretization error as the generated joint distributions differ from those corresponding to the SDE. To reduce the error, larger number of well-spaced times may be chosen so that accuracy is achieved at the cost of greater computational effort. We refer the reader to Glasserman (2004) for a discussion on improved discretization methods.

The Monte-Carlo simulation in this setting may be implemented as follows: Generate samples $N_{k,j}$ for $k = 1, \ldots, n$ and $j = 1, \ldots, m$ (see any standard simulation text for algorithms for generating $N(0, 1)$ random variable (r.v.), e.g., Law & Kelton 1991) and use them to generate samples of $(S_i(t_k) : i = 1, \ldots, d; k = 1, \ldots, n)$ using (4) or (5), whichever is appropriate. These are then used to arrive at the sample value of the discounted payoff from the option, call it $\hat{A}_1$. This process is repeated independently, say l times so that independent samples $(\hat{A}_i : i \leq l)$ are obtained. The point estimate of the (un-discounted) option price is given by

$$\bar{A} = \frac{1}{l}\sum_{i\leq l} \hat{A}_i.$$

Let σ_X^2 denote the variance of r.v. X and σ_X its standard deviation. Then from the central limit theorem, an approximate $100(1-\alpha)\%$ confidence interval for A equals

$$\bar{A} \pm z_{1-\alpha/2}(\sigma_{\hat{A}}/\sqrt{l}),$$

where $z_{1-\alpha}$ is the solution to the equation,

$$P(N(0,1) > z) = \alpha.$$

In practice $\sigma^2_{\hat{A}}$ is not known and is estimated from the generated samples by

$$\bar{\sigma}^2_{\hat{A}} = \frac{1}{l-1}\sum_{i\le l}(\hat{A}_i - \bar{A})^2,$$

and $\bar{\sigma}_{\hat{A}}$ replaces $\sigma_{\hat{A}}$ in the approximate confidence interval. This is asymptotically valid as $\bar{\sigma}_{\hat{A}}/\sigma_{\hat{A}} \to 1$ a.s. as $l \to \infty$.

Thus, the accuracy of estimation is proportional to $\sigma_{\hat{A}}/\sqrt{l}$. Suppose that computational effort c is expended at this estimation and suppose that each sample takes an average of computational effort τ in generating a sample $\hat{A}_i$. Then, roughly speaking, the number of samples generated equals about c/τ so that the accuracy of estimation is proportional to $(\sigma_{\hat{A}}\sqrt{\tau})/\sqrt{c}$. (see, Glynn & Whitt 1992, where this argument is made rigorous in an asymptotic framework).

Therefore, two different estimators with similar bias characteristics (e.g., both are unbiased) may be compared based on the figure of merit '$\sigma^2_{\hat{A}}\tau$'. In the next section, we discuss some promising variance-reduction techniques. These involve developing another estimator for the performance measure of interest that may in many cases have significantly lower variance than the original estimator. Typically, the average time to generate a sample does not differ much between the two techniques so that comparison amongst variances gives a fair measure of the comparison of the performance of the two estimators.

5. Variance reduction techniques

In this section, we discuss some popular variance reduction techniques, namely, the control variate, antithetic variate, stratification and importance sampling technique. We illustrate their effectiveness by applying them to pricing simple options.

5.1 *The control variate method*

Suppose we want to estimate the expectation EX of r.v. X and for this purpose we generate independent identically distributed samples $(X_1, \ldots, X_n)$ of X. Then

$$\bar{X}_n = \frac{1}{n}\sum_{i\le n} X_i,$$

is an unbiased and consistent estimator of EX. Suppose as a by-product of our simulation we also generate independent identically distributed samples $(Y_1, \ldots, Y_n)$, where each Y_i has mean zero and is correlated with X_i. Then $\bar{X}_n + \bar{Y}_n$ and $\bar{X}_n - \bar{Y}_n$ are two other unbiased and consistent estimators of EX. In fact, for any constant β,

$$\bar{X}_n + \beta\bar{Y}_n \tag{6}$$

is an unbiased and consistent estimator of EX. Then we look to choose β that provides the smoothest estimate, i.e., the estimate with the least variance. If X_i and Y_i are positively

correlated and hence $\bar{X}_n$ and $\bar{Y}_n$ are positively correlated, one expects β to be negative so that if the two are large or small together, their difference displays less variability. Similarly, If the two are negatively correlated, one expects β to be positive as when one is large the other is small and the sum may be expected to be less variable. To get the optimal value of β, note that the variance of (6) equals

$$\frac{1}{n}\sigma_X^2 + \beta^2\frac{1}{n}\sigma_Y^2 + 2\beta\frac{1}{n}\rho_{XY}\sigma_X\sigma_Y,$$

where ρ_{XY} denotes the correlation between X and Y. By differentiating it can be seen that the β^* minimizing the variance equals

$$-\rho_{XY}(\sigma_X/\sigma_Y).$$

For this β^*, the variance can be seen to equal

$$(1-\rho_{XY}^2)\sigma_X^2/n,$$

so that higher the absolute correlation $|\rho_{XY}|$, more the variance reduction. In practice, $(Y_i : i \geq 1)$ may correspond to a sequence of input variables whose mean is known and hence can be subtracted away so that the resultant r.v. has mean zero. The simplest case is to use the underlying uniform (0,1) variates used to generate random variates from other distributions, subtracting $1/2$ from the observed average so that its mean is zero. However, the correlation between this and the output quantity may not be strong, leading to little or no improvements. The examples below illustrate cases where it is easy to identify $(Y_i : i \geq 1)$ so that each Y_i has mean zero and is highly correlated with the X_i's.

In standard terminology, such sequence of r.v. are referred to as control variates. The β^* above typically is not known *a priori*. An initial set of samples of $((X_i, Y_i) : i \leq m)$ may be generated and used to estimate β^* by

$$\frac{\frac{1}{m}\sum_{i\leq m} X_iY_i - (\frac{1}{m}\sum_{i\leq m} X_i)(\frac{1}{m}\sum_{i\leq m} Y_i)}{\frac{1}{m}\sum_{i\leq m} Y_i^2 - (\frac{1}{m}\sum_{i\leq m} Y_i)^2},$$

and this may then be used in subsequent independently generated samples to estimate EX. This has the drawback that initial computational effort is needed to estimate β^* and that this is a 'noisy' estimator of β^*. Alternatively, a set of independent samples of (X_i, Y_i) may be generated to estimate EX using an estimate of β^* determined from these samples itself. While this has the advantage over the previous approach that no extra initial computational effort is needed, its drawback is that the resulting estimator may not be an unbiased estimator of EX. We refer the reader to Glasserman (2004) and Bratley *et al* (1987) for more discussion on this. They also discuss the generalizations involving multiple controls.

Example 1. Suppose that a stock price $(S_t : 0 \leq t \leq T)$[1] follows the single-dimensional SDE

$$\mathrm{d}S_t/S_t = r\mathrm{d}t + b\mathrm{d}B(t), \tag{7}$$

[1]The notation $S_i(t)$ used in §4 to describe the asset price differs from the one used here and in later sections. The earlier notation allowed us to highlight the time t as well as the dimensional index $i \in \{1, 2, \ldots, d\}$. Henceforth, unless otherwise mentioned, we focus on a single-dimension and make t a subscript of S, thus using S_t to denote the asset price at time t.

under the risk-neutral measure, where $B(\cdot)$ is standard one-dimensional Brownian motion. Consider the problem of estimating the price of an Asian call option whose undiscounted payoff at time T equals

$$\left(\frac{1}{n}\sum_{k=1}^{n} S_{t_k} - K\right)^+,$$

where K is the strike price and $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$. Samples of the stock price at times $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$ may be generated by starting with S_0 and setting

$$S_{t_{k+1}} = S_{t_k} \exp\left[(r - 1/2b^2)(t_{k+1} - t_k) + b(t_{k+1} - t_k)^{1/2} N_{k+1}\right]$$

where N_k are $N(0, 1)$ r.v. Kemna & Vorst (1990) note that a similar call option based on the geometric mean of the stock price at the n times, is tractable. Specifically, consider an option with an undiscounted pay-off

$$\left(\left(\prod_{k=1}^{n} S_{t_k}\right)^{1/n} - K\right)^+. \tag{8}$$

Since,

$$\left(\prod_{k=1}^{n} S_{t_k}\right)^{1/n} = S_0 \exp\left[\frac{1}{n}(r - 1/2b^2)T + \frac{b}{n}\sum_{k\le n} N_k\right],$$

it is log-normally distributed, so that the expectation of the undiscounted option price,

$$\tilde{E}\left[\left(\left(\prod_{k=1}^{n} S_{t_k}\right)^{1/n} - K\right)^+\right],$$

is easy to evaluate using the Black–Scholes formula (shown in example 2). Then (8) with its mean subtracted can serve as a control variate for the Asian option. Kemna & Vorst (1990) report reduction in the standard deviation of the order of 10–100 by using (8) as a control variate on some typical examples.

Example 2. For a stock price modelled as in example 1, consider the problem of pricing a European call option on a stock that may pay dividends. Suppose that the dividend can be paid at the first instant amongst $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$, when the stock price exceeds a specified threshold K_1 and suppose that the dividend amount in that case equals D. Let A denote the event that the dividend is paid and let $\tau = \inf_{t\in\{t_0,\ldots,t_n\}} \{t : S_t \ge K_1\}$ denote the time at which the dividend is paid (let it equal ∞ if A does not occur). Let K_2 denote the strike price of the option. The undiscounted payoff on a European call option equals

$$(S_T - e^{r(T-\tau)} D I(A) - K_2)^+,$$

where S_T is the asset price at T as obtained from the underlying geometric brownian motion, $I(A)$ is the indicator function of event A (it equals 1 if A occurs and zero otherwise). The payoff from this option may be expected to be highly correlated to the pay-off on the corresponding

option if the stock is assumed to be non-dividend paying, i.e., if the undiscounted payoff equals $(S_T - K_2)^+$. This option may be priced by observing that S_T is log-normally distributed, resulting in the well-known Black–Scholes formula for the price c, i.e.,

$$c = S_0 \Phi\left(\frac{\log(S_0/K_2) + (r + b^2/2)T}{b\sqrt{T}}\right)$$
$$- K_2\, e^{-rT}\, \Phi\left(\frac{\log(S_0/K_2) + (r + b^2/2)T}{b\sqrt{T}} - b\sqrt{T}\right),$$

where $\Phi(\cdot)$ denotes the standard Normal cumulative distribution function (see, e.g., Duffie 1996). In table 1, we report the variance reduction achieved using this control variate for a typical example in one dimension. The variance reduction achieved is quite large as in this case $(S_T - e^{r(T-\tau)} DI(A) - K_2)^+$ is highly correlated with $(S_T - K_2)^+$. Empirically, we found that the correlations between the two equal 0·99.

5.2 *Antithetic variate method*

Now we discuss the antithetic variate method. Like the control variate method, it is very easy to implement in fairly general situations. It also combines well with other variance reduction techniques. The essential idea behind this method is straightforward. Consider two continuous random variables X_1 and X_2. The variance of their sum is

$$\sigma^2_{X_1+X_2} = \sigma^2_{X_1} + \sigma^2_{X_2} + 2\rho_{XY}\sigma_{X_1}\sigma_{X_2}.$$

Thus, if the two are negatively correlated then the variance of the sum is less than the variance if they are independent. Intuitively, this is so since typically when X_1 is large, X_2 is small and vice versa, so that their sum is less variable than if they were independent.

Suppose that X_1 and X_2 have distribution functions $F_1(\cdot)$ and $F_2(\cdot)$ respectively. It is well known that if U is uniformly distributed, then $F_i^{-1}(U)$ is distributed as X_i. It is well known that $F_1^{-1}(U)$ and $F_2^{-1}(1-U)$ achieve the largest negative correlation possible amongst distributions with marginals corresponding to X_1 and X_2 (see, e.g., Nelsen 1999). This provides a useful guideline for selecting antithetic variates. It is also important to note that if N is a sample of a Gaussian random variable with mean 0, variance σ^2, then $-N$ also has the same distribution and has a correlation of -1 with N.

Thus, in Monte Carlo simulation of financial options rather than generating i.i.d. samples $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_l$, better variance characteristics are obtained if $l/2$ independent pairs of r.v. $(\hat{A}_1, \hat{A}_2), \ldots, (\hat{A}_{l-1}, \hat{A}_l)$ are generated so that within a pair negative correlation exists while each generated r.v. has the same marginals as before. We illustrate the application of this method through the following example:

Example 3. Again consider example 2. To generate a pair of negatively correlated pair of samples for the first sample we set

$$S_T = S_0 \exp\left[(r - 1/2b^2)T + b\sum_{k\le n-1}(t_{k+1} - t_k)^{1/2} N_{k+1}\right],$$

and for the second sample we set

$$\tilde{S}_T = S_0 \exp\left[(r - 1/2b^2)T - b\sum_{k\le n-1}(t_{k+1} - t_k)^{1/2} N_{k+1}\right],$$

Table 1. Performance of variance reduction techniques.

n	Naive price (95 % CI)	VR price (95 % CI)	VRF	TF	CRF
Control variate vs naive					
2	32·928 (32·883, 32·973)	33·101 (33·097, 33·105)	118	1·5	78·7
10	32·114 (32·068, 32·160)	32·209 (32·204, 32·214)	100	1·4	71·4
50	31·676 (31·630, 31·723)	31·798 (31·793, 31·802)	102	1·4	72·8
Antithetic vs naive					
2	33·105 (33·060, 33·151)	33·099 (33·089, 33·110)	18	1·0	18·0
10	32·183 (32·136, 32·229)	32·204 (32·193, 32·216)	17	1·2	14·2
50	31·824 (31·777, 31·871)	31·794 (31·783, 31·806)	17	1·3	13·1
Combined vs naive					
2	33·092 (33·046, 33·137)	33·099 (33·096, 33·101)	340	2·0	170·0
10	32·237 (32·191, 32·284)	32·203 (32·200, 32·206)	229	2·3	99·6
50	31·754 (31·707, 31·801)	31·798 (31·794, 31·801)	204	2·2	92·7

Option parameters: $r = 6\%$, $T = 1$, $K_1 = 140$, $b = 0{\cdot}2$, $K_2 = 100$, $S_0 = 130$, $D = 0{\cdot}05 * S_0 = 6{\cdot}5$. The results have been obtained for $M = 10^6$ sample paths. n indicates number of exercise opportunities up to expiration, "Naive price" indicates the estimate of option price without using any variance reduction technique, "VR price" indicates the estimate obtained by applying the variance reduction technique (VRT), figures in brackets are the 95% confidence intervals, variance reduction factor (VRF) is the ratio of naive variance to variance of the estimate based on the VRT, (time factor) TF is the ratio of time taken for VRT estimate to naive time per simulated path. The overall computational effort reduction factor (CRF) is given by VRF/TF.

then it can be seen that the two outputs $(S_T - e^{r(T-\tau)} DI(A) - K_2)^+$ and $(\tilde{S}_T - e^{r(T-\tau)} DI(A) - K_2)^+$, are negatively correlated. This fact is true more generally: Suppose that $f : \Re^n \to \Re$ is increasing in each of its arguments, then given random variables $N_1, \ldots, N_n$, it follows that

$$E[f(N_1, \ldots, N_n) f(-N_1, \ldots, -N_n)] \leq E[f(N_1, \ldots, N_n)] E[f(-N_1, \ldots, -N_n)],$$

(see, e.g., Esary *et al* 1967). We compare the performance of naive simulation with the antithetic variate method and also with the combined control variate and antithetic variate method in table 1. We conclude that the use of both, the control variate and the antithetic variate technique results in significant variance reduction, and when both the techniques are combined, we see further improvement in performance.

5.3 *Stratified sampling*

Suppose that we wish to find an expectation of r.v. $Z = f(A, X)$ where A is a random vector and X is a random variable. Under naive simulation one generates independent samples $(A_1, X_1), (A_2, X_2), \ldots, (A_n, X_n)$ of (A, X) and forms an estimator,

$$\frac{1}{n} \sum_{i \leq n} f(A_i, X_i)$$

of EZ. Consider the sets $B_1, B_2, \ldots, B_m$ that partition the real line (so that they are mutually disjoint and $\cup B_i = \Re$). Then,

$$EZ = \sum_{i \le m} E(Z|X \in B_i) p_i,$$

where $p_i = P(X \in B_i)$. Suppose that p_i are known or can be cheaply computed. Moreover, samples of Z can be generated using the probability obtained by conditioning on $\{X \in B_i\}$, call it P_{B_i}, for each i. Then, the problem of estimating EZ may be reduced to that of estimating each $E(Z|X \in B_i)$. This procedure of estimation is referred to as stratified sampling.

Suppose that each sample takes a unit of computational effort under P and under P_{B_i}. Consider the comparison of two approaches: In the first case n samples of Z are generated using the unconditional probability P. In the second approach np_i (ignoring the technicalities associated with np_i not being an integer) samples are generated under P_{B_i} for each i. The resulting variance of the estimator in the first case equals:

$$Var(Z)/n.$$

To see the variance of the estimator in the second case, let $(Z_{ij} : j \le np_i)$ denote the generated samples in this case for $i = 1, \ldots, m$. Then the estimator equals

$$\sum_{i \le m} \left[\frac{1}{np_i} \sum_{j \le np_i} Z_{ij} \right] p_i.$$

All the samples are independent, so it is easily seen that its variance equals

$$\frac{1}{n} \sum_{i \le m} \sigma_Z^2(B_i) p_i = \frac{1}{n} E[\sigma_Z^2(B_i)],$$

where $\sigma_Z^2(B_i)$ denotes the variance of Z under P_{B_i}.

Again, it is well known and easily checked that

$$\sigma_Z^2 = E[\sigma_Z^2(B_i)] + Var[E(Z|X \in B_i)],$$

so that the stratified estimator is an improvement over the original estimator.

Example 4. We illustrate the use stratified sampling to generate samples from a *heavy tailed distribution* (see, e.g., Glasserman 2004 for a definition). Specifically we consider generating n samples of r.v. X with polynomially decreasing tail distribution function

$$\bar{F}(x) = 1/x^{\alpha},$$

for all $x \ge 1$, where α is a positive constant. For such a distribution, with a small probability, r.v. may take extremely large values and unstratified sampling involving a small number of samples may under-represent such samples. Stratified sampling, on the other hand, can be used to force exactly one sample between the $(i-1)$th and ith percentile, $i = 1, \ldots, n$ and thus give a more representative empirical distribution. The algorithm for generating n stratified samples involves generating n independent samples of U, uniformly distributed r.v. taking values in $[0, 1]$. Call these, $U_1, U_2, \ldots, U_n$. Using the inverse transform method (based on the observation that $\bar{F}(X)$ is distributed as U so that $1/U^{1/\alpha}$ is distributed as X), set

$$X_i = \left[\frac{n}{i + U_i - 1} \right]^{1/\alpha}, \; i = 1, \ldots, n,$$

to get the required stratified samples.

Rather then selecting np_i samples from the distribution P_{B_i}, one may select nq_i such samples for each i and then select the probability vector $(q_i : i \leq m)$ so that the variance is minimized. The resulting optimization problem is easily solved by

$$q_i = \frac{p_i \sigma_Z^2(B_i)}{\sum_{j \leq m} p_j \sigma_Z^2(B_j)}.$$

In practice, $\sigma_Z^2(B_i)$ is typically not known and may be estimated from the data generated in a pilot run. We use the stratification technique in our discussion of the stochastic mesh method. We refer the reader to Glasserman (2004) for some further applications of the stratification technique to price options.

5.4 *Importance sampling*

Importance sampling involves estimating an expectation of a r.v. by generating sample paths using a probability measure, different from the original one, that emphasizes the important paths from the point-of-view of estimation. The generated estimator is then unbiased using the 'likelihood ratio' or the Radon-Nikodym derivative of the original measure with respect to the new measure. If done correctly, it can lead to orders of magnitude of variance reduction, especially in estimation of rare event probabilities (see, e.g., Heidelberger 1995, Juneja & Shahabuddin 2005). However, it needs to be implemented carefully as a bad choice of importance sampling distribution can lead to a large increase in the variance of the estimator.

We illustrate the basic idea in pricing an option. Again suppose that we simulate asset prices corresponding to (4) or (5) at discrete times

$$0 = t_0 < t_1 < t_2 < \ldots < t_n = T.$$

Let $f_{t_i}(x, \cdot)$ denote the pdf governing the distribution of $S_{t_{i+1}}$ given $S_{t_i} = x$. Suppose that for each i and x, $f^*_{t_i}(x, \cdot)$ denotes another pdf such that $f^*_{t_i}(x, y) > 0$ whenever $f_{t_i}(x, y) > 0$. Then the original probability measure P is absolutely continuous w.r.t. the new probability measure, call it P^* (suppose that the process remains Markov under P as well as P^*).

The key idea of importance sampling is to use P^* to generate the samples $(S_{t_i} : i \leq n)$, compute the corresponding option value payoff $\hat{A}$ and set the sample output from the simulation equal to $\hat{A}L_n$ where

$$L_n = \frac{f_0(S_{t_0}, S_{t_1})}{f_0^*(S_{t_0}, S_{t_1})} \frac{f_1(S_{t_1}, S_{t_2})}{f_1^*(S_{t_1}, S_{t_2})} \cdots \frac{f_{n-1}(S_{t_{n-1}}, S_{t_n})}{f_{n-1}^*(S_{t_{n-1}}, S_{t_n})}.$$

Average of many such independent samples provides an estimator of $E_P(\hat{A})$. It can be easily seen that $E_P(\hat{A}) = E_{P^*}(\hat{A}L_n)$ so that the resultant estimator is unbiased. The main problem is to select P^* so that the variance of each sample $\hat{A}L_n$ is minimized. Later in Section 6.2 we show an explicit form of the zero variance measure P^*, i.e., a collection of densities $(f_i^*(\cdot, \cdot) : i \leq n)$ such that $\hat{A}L_n = E_P(\hat{A})$ along every sample path that may be generated via simulation. This has the desirable property that the Markov process $(S_{t_i} : i \leq n)$ remains Markov under it. However, it requires explicit knowledge of option prices at all states of the Markov process and hence is un-implementable. Later in the settings of Bermudan options, we discuss how the densities associated with this measure may be approximated in an implementable way. In the remaining part of the section we discuss a desirable trait in an importance sampling measure and using it develop a good importance sampling measure in two cases.

Suppose that under P^*, there exists a $k < 1$ such that

$$L_n \hat{A} \leq k\hat{A}, \tag{9}$$

a.s. (so that L_n may be greater than k when $\hat{A} = 0$). In that case,

$$E_{P^*}[L_n^2 \hat{A}^2] = E_P[L_n \hat{A}^2] \leq k E_P[\hat{A}^2].$$

The ratio of variance of $L_n \hat{A}$ under P^* and $\hat{A}$ under P equals

$$\frac{E_{P^*}[L_n^2 \hat{A}^2] - E_P(\hat{A})^2}{E_P(\hat{A}^2) - E_P(\hat{A})^2} \leq \frac{k[E_P[\hat{A}^2] - E_P(\hat{A})^2] - (1-k)E_P(\hat{A})^2}{E_P(\hat{A}^2) - E_P(\hat{A})^2} \leq k.$$

Thus, one gets guaranteed reduction in the variance by a factor of k whenever (9) holds.

Example 5. Consider again example 2. Recall that the dividend D can be paid at the first instant amongst $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$, when the stock price exceeds a specified threshold K_1 and A denotes the event that the dividend is paid. K_2 denotes the strike price of the option. The undiscounted payoff on a European call option equals

$$(S_T - e^{r(T-\tau)} D I(A) - K_2)^+,$$

where τ denotes the time at which the dividend is paid. Again, S_T may be set to

$$S_0 \exp\left[(r - 1/2b^2)T + b \sum_{k \leq n-1} (t_{k+1} - t_k)^{1/2} N_{k+1}\right],$$

where each N_k is an independent N(0,1) random variable. To keep the notation simple and without any essential loss of generality suppose that $t_k = k$ for each k. Then

$$S_T = S_0 \exp\left[(r - 1/2b^2)T + b \sum_{0 \leq k \leq T-1} N_{k+1}\right]. \tag{10}$$

More generally suppose that

$$S_T = S_0 \exp\left(\sum_{k=1}^{T} Y_k\right),$$

for i.i.d. random variables Y_k. This holds true in many models, including in (10) where $Y_k = (r - b^2/2) + bN_k$. We now arrive at an importance sampling distribution for Y_k's so that guaranteed variance reduction is obtained. Let $f(\cdot)$ denote the pdf of Y_k. Consider a pdf $f_\theta(\cdot)$ obtained by exponentially twisting $f(\cdot)$ by θ, i.e.,

$$f_\theta(x) = \exp(\theta x - \Lambda(\theta)) f(x),$$

where $\Lambda(\theta)$ denotes the log-moment generating function evaluated at θ. For the r.v. $N(\mu, \sigma^2)$, it equals $\mu\theta + \sigma^2\theta^2/2$. Suppose that $(Y_k : k \leq T)$ are generated using the pdf $f_\theta(\cdot)$. (There is a strong theoretical basis for using exponentially twisted distributions when sums of random

variables are involved, see, e.g., Juneja 2001, Juneja & Shahabuddin 2005). Then the output sample under importance sampling equals

$$e^{-rT}(S_T - e^{r(T-\tau)}DI(A) - K_2)^+ \exp\left(-\theta \sum_{k=1}^{T} Y_k + T\Lambda(\theta)\right).$$

Suppose that the option is deep out of the money so that

$$S_0 < K_2.$$

Then along the set $S_T > K_2$

$$\sum_{k=1}^{T} Y_k \geq \log(K_2/S_0).$$

Let the rhs be denoted by δ. Then

$$\begin{aligned}(S_T - e^{r(T-\tau)}DI(A) - K_2)^+ &\exp\left[-\theta \sum_{k=1}^{T} Y_k + T\Lambda(\theta)\right] \\ &\leq (S_T - e^{r(T-\tau)}DI(A) - K_2)^+ \exp(-\theta\delta + T\Lambda(\theta)).\end{aligned}$$

Let θ^* denote the unique solution to

$$\Lambda'(\theta) = \delta/T,$$

(this equals $(1/\sigma^2)\,((\delta/T) - \mu)$ when N_k are $N(\mu, \sigma^2)$). It is easily checked that the likelihood ratio

$$L_T = \exp(-\theta^*\delta + T\Lambda(\theta^*)) < 1,$$

Thus, using $f_{\theta^*}(\cdot)$ for importance sampling, due to (9), we see guaranteed performance improvement of amount $\exp(-\theta^*\delta + T\Lambda(\theta^*))$ over naive simulation.

Example 6. Now consider a one-dimensional up-and-in European put option. This a minor variant of the option considered by Boyle *et al* (1997). We discuss how a more complex implementation of importance sampling proves effective in this setting.

Suppose that $K_1 > K_2$. The option pay-off equals

$$(K_2 - S_T)^+ I(\max_i S_{t_i} > K_1).$$

Thus, the option becomes active if the stock price exceeds K_1 at any of the specified times. Let $\tau = \inf\{i : S_{t_i} > K_1\}$. Set it equal to n if $S_{t_i} \leq K_1$ for all i. Again suppose that $t_k = k$ for each k and let $T = n$. Furthermore, as in example 5, suppose that

$$S_T = S_0 \exp\left(\sum_{k=1}^{T} Y_k\right),$$

for i.i.d. random variables Y_k. Again we develop an importance sampling strategy that guarantees variance reduction. Consider the following strategy: Generate Y_k's using the distribution

obtained by exponentially twisting the original distribution by an amount α for $k \leq \tau$. For $k > \tau$ twist by an amount $-\beta$. Exact values of α and β will be determined later, but it is intuitively clear that $\alpha > 0$ to increase the likelihood of the threshold K_1 being exceeded (e.g., it is easy to see that the mean of Y_k under $f_\alpha(\cdot)$, its original pdf twisted by amount α, equals $\Lambda'(\alpha)$ and is an increasing function of α), and $\beta > 0$ to increase the likelihood of asset price reaching below K_2, given that it has exceeded K_1. The output from this importance sampling strategy along the set $\{\tau < T\}$ equals

$$(K_2 - S_T)^+ \exp\left(-\alpha \sum_{k \leq \tau} Y_k + \tau \Lambda(\alpha) + \beta \sum_{k > \tau} Y_k + (k - \tau)\Lambda(-\beta)\right). \tag{11}$$

Note that $\sum_{k=1}^{\tau} Y_k \geq \log(K_1/S_0) \equiv \mu$ and $\sum_{k=\tau+1}^{T} Y_k \leq -\log(K_1/K_2) \equiv -\nu$. Now by setting $\Lambda(\alpha) = \Lambda(-\beta)$, we get the following uniform bound on (11) along the set $\{\tau < T\}$

$$(K_2 - S_T)^+ \exp(-\alpha\mu - \beta\nu + T\Lambda(-\beta)).$$

Thus, to get the smallest bound, we minimize

$$-\alpha\mu - \beta\nu + T\Lambda(-\beta),$$

subject to the constraint $\Lambda(\alpha) = \Lambda(-\beta)$. Suppose that for every β, $\alpha(\beta)$ solves this constraint. Then, by the Implicit Function Theorem (see, e.g., Sundaram 1999)

$$\mathrm{d}\alpha(\beta)/\mathrm{d}\beta = -\Lambda'(-\beta)/[\Lambda'(\alpha(\beta))].$$

So at the minimum point $(\alpha(\beta^*), \beta^*)$,

$$\mu[\Lambda'(-\beta^*)]/[\Lambda'(\alpha(\beta^*))] - \nu = T\Lambda'(-\beta^*).$$

So that

$$\mu/[\Lambda'(\alpha(\beta^*))] - \nu/[\Lambda'(-\beta^*)] = T.$$

When each Y_i is normally distributed with mean $r - b^2/2$ and variance b^2, we have $\Lambda(\theta) = (r - (b^2/2))\theta + (b^2\theta^2/2)$. Solving for $\alpha^* = \alpha(\beta^*)$ and β^*, we obtain,

$$\alpha^* = \frac{1}{b^2}\left[\left(\frac{\mu + \nu}{T}\right) - \left(r - \frac{b^2}{2}\right)\right], \text{ and } \beta^* = \frac{1}{b^2}\left[\left(\frac{\mu + \nu}{T}\right) + \left(r - \frac{b^2}{2}\right)\right].$$

Then we have the bound,

$$(K_2 - S_T)^+ \exp\left[\frac{1}{2b^2}\left[\left(r - \frac{b^2}{2}\right)\left\{2(\mu - \nu) - T\left(r - \frac{b^2}{2}\right)\right\} - \frac{(\mu + \nu)^2}{T}\right]\right],$$

and guaranteed variance reduction by a factor

$$\exp\left[\frac{1}{2b^2}\left[\left(r - \frac{b^2}{2}\right)\left\{2(\mu - \nu) - T\left(r - \frac{b^2}{2}\right)\right\} - \frac{(\mu + \nu)^2}{T}\right]\right],$$

using (9), where it should be noted that when $r > b^2/2$ and the option is deep out-of-money (i.e., $S_0 > K_2$), $(\mu - \nu) < 0$ and, therefore, the argument of the exponential function is negative giving a value less than one to the likelihood ratio. Hence, again variance reduction is guaranteed.

6. Pricing Bermudan options

6.1 *Mathematical framework*

We now construct a mathematical framework for pricing Bermudan options. Recall that American options are approximately priced by discretizing the times of exercise and estimating the price of the resulting Bermudan option.

Again, without essential loss of generality suppose that the option can be exercised only at $T+1$ times $0, 1, 2, \ldots, T$. Denote the underlying security prices by $(S_t \in \Re^d : t = 0, 1, \ldots, T)$. In addition, the description of the state at time t may include variables such as the value of stochastic interest rates and volatilities, and supplementary path dependent information so that the resulting process is Markov. Thus, each S_t may take values in a more general space denoted by $\mathcal{S}$. The value of the option at time t if exercised at that time, is denoted by $g_t : \mathcal{S} \to \Re^+$ (i.e., its exercise value or intrinsic value). Let $\mathcal{T}_k$ denote the set of stopping times taking value in $\{k, k+1, \ldots, T\}$ (recall that τ is a stopping time w.r.t. $\{S_k\}$ if $\{\tau = k\}$ is determined by observing $\{S_1, \ldots, S_k\}$). Note that each stopping time represents an exercise strategy by the owner of the option at time period k. Let

$$J_k(s) = \sup_{\tau \in \mathcal{T}_k} E[g_\tau(S_\tau)|S_k = s], \quad s \in \mathcal{S}, \tag{12}$$

where, the expectation is taken under the risk-neutral measure. Then $J_k(s)$ is the value of the option at time k given that the option is not exercised before time k. The initial state $S_0 = s_0$ is fixed and known. So, our pricing problem is to evaluate $J_0(s_0)$. This formulation is sufficiently general to include discounted payoffs through appropriate definition of the $\{S_k\}$ and $\{g_k\}$ (see Glasserman 2004, p.425), and hence these are not explicitly stated. It can be shown that (see, e.g., Duffie 1996) there exists an optimal exercise policy specified by a collection of increasing stopping times $(\tau_k^* : k \le T)$ where

$$\tau_k^* = \inf\{m \ge k : g_m(S_m) \ge J_m(S_m)\}.$$

Then,

$$J_k(s) = E[g_{\tau_k^*}(S_{\tau_k^*})|S_k = s]$$

a.s. Note that knowing the optimal policy corresponds to knowing at each state at each time whether it is optimal to exercise the option or to hold on to it.

Suppose that the pdf of S_{k+1} conditioned on $S_k = s$ evaluated at y is given by $f_k(s, y)$ under the risk-neutral measure (for example, the expression for $f_k(s, y)$ can be easily derived using (4) or (5) if either is appropriate). Let $Q_k(s)$ denote the conditional expectation

$$E[J_{k+1}(S_{k+1})|S_k = s] = \int_{\mathcal{S}} J_{k+1}(y) f_k(s, y) \mathrm{d}y. \tag{13}$$

We refer to $Q = (Q_k(s) : s \in \mathcal{S}, k \le T-1)$ as continuation value functions as $Q_k(s)$ denotes the value of the option at time k at state s if it is not exercised at time k.

It is not feasible to evaluate $J_k(s)$ by evaluating expectation $E[g_\tau(S_\tau)|S_k = s]$ for each stopping time τ in (12). Fortunately, it can be shown that the value functions $J = (J_k(s) : s \in \mathcal{S}, k \le T)$ satisfy the following intuitively plausible backward recursions (see, e.g., Shirayev 1978):

$$\begin{aligned} J_T(s) &= g_T(s), \\ J_k(s) &= \max\{(g_k(s), Q_k(s)\}. \end{aligned} \tag{14}$$

Define $P_k H(\cdot)$ as

$$(P_k H)(s) \stackrel{\Delta}{=} E[H(S_{k+1})|S_k = s] = \int_{\mathcal{S}} H(u) f_k(s,u)\mathrm{d}u.$$

Then, an alternative set of recursions satisfied by the continuation value function $Q = (Q_k(s) : s \in \mathcal{S}, k \leq T-1)$ is given by:

$$\begin{aligned} Q_{T-1}(s) &= (P_{T-1} g_T)(s), \\ Q_k(s) &= (P_k \max(g_{k+1}, Q_{k+1}))(s), \end{aligned} \tag{15}$$

for $k = 0, 1, 2, \ldots, T-2$.

Evaluating the value functions using these recursions would require discretizing state space and then solving these recursions approximately. However, even this becomes computationally unviable due to state-space blow-up when the dimension of the underlying process is large. In §7, 8 and 9 we describe simulation techniques that approximately solve these backward recursions to estimate the value function and the associated optimal exercise policy $(\tau_k^* : k \leq T)$. Often, these follow a two-step procedure. First, the optimal policy is learnt approximately. Thereafter, this policy is evaluated using the standard Monte Carlo procedure. Note that the value function corresponding to the policy learnt, lower bounds the price of the option.

We now discuss how importance sampling may be used to aid in efficiently estimating the Bermudan option price. In particular, we see that a zero variance importance sampling estimator always exists although it requires a-priori knowledge of the value functions. Though not implementable, the form of zero variance estimator is useful as, once we develop approximations for the value functions, these in turn provide an implementable approximate zero variance estimator.

6.2 *Importance sampling and zero-variance measure*

Let $\tau^* = \tau_0^*$ denote an optimal stopping time for our problem, i.e., suppose $J_0(s_0) = E[g_{\tau^*}(S_{\tau^*})]$. Suppose this stopping time is known, and to avoid trivialities suppose $\tau^* > 0$. Then, recall that a naive estimate of $J_0(s_0) = Q_0(s_0)$ is obtained by taking an average of independent identically distributed samples of $g_{\tau^*}(S_{\tau^*})$. Suppose, we generate these samples using the importance sampling pdf's $(\tilde{f}_t(s,\cdot) : s \in \mathcal{S}, t \leq T-1)$ such that $\tilde{f}_t(s,u) > 0$ whenever $f_t(s,u) > 0$ and vice versa for each s, u and t. Let $\tilde{P}$ denote the resultant measure (let P denote the original measure). Then the unbiased importance sampling (IS) estimator of $J_0(s_0)$ is obtained by taking an average of independent, identically distributed samples of

$$\frac{f_0(s_0,S_1)}{\tilde{f}_0(s_0,S_1)} \frac{f_1(S_1,S_2)}{\tilde{f}_1(S_1,S_2)} \cdots \frac{f_{\tau^*-1}(S_{\tau^*-1},S_{\tau^*})}{\tilde{f}_{\tau^*-1}(S_{\tau^*-1},S_{\tau^*})} g_{\tau^*}(S_{\tau^*}) \tag{16}$$

(see, e.g., Glynn & Iglehart 1989).

We assume that $Q_t(s) > 0$ for all $s \in \mathcal{S}$ and for $t \leq T-1$. Now suppose that the importance sampling distribution P^* corresponds to:

$$f_t^*(s,u) = \frac{f_t(s,u) J_{t+1}(u)}{Q_t(s)} \tag{17}$$

for each s, u and t. Equation (13) confirms the validity of $f_t^*(s,\cdot)$ as a pdf. Since $Q_t(S_t) = J_t(S_t)$ when $\tau^* > t$ and $J_{\tau^*}(S_{\tau^*}) = g_{\tau^*}(S_{\tau^*})$, it is easy to see that P^* is a zero-variance

measure as (16), with f_t^* replacing $\tilde{f}_t$, reduces to $Q_0(s_0) = J_0(s_0)$ a.s. (P^* is called a zero-variance measure, because under it, the unbiased estimator given by (16) is a constant - $Q_0(s_0)$ - i.e., has zero variance). See, e.g., Ahamed *et al* (2004) for further discussion of zero variance estimators in Markov chain settings.

In §7.3 we use regression-based approximations of $J_t(\cdot)$ to develop approximations for the densities associated with the zero-variance measure.

7. Regression-based methods

Tsitsiklis & Van Roy (2001) and Longstaff & Schwartz (2001) propose very similar methods based on least squares regression to price American options. The essential idea is to approximate the continuation value functions by a a linear combination of carefully chosen known basis functions. Then the problem of learning the continuation value functions reduces to a much lower dimension problem of learning the parameters in the linear combination (see, e.g., Bertsekas & Tsitsiklis (1996) for further discussion on such function approximation techniques). These parameters are obtained by regression from the samples generated via simulation. We now explain these ideas more concretely.

Let $\phi_k : \mathcal{S} \to \Re$ for $1 \leq k \leq K$ denote a set of basis functions. Consider a parameterized value function $\hat{Q} : \mathcal{S} \times \Re^K \to \Re$ that assigns values $\hat{Q}(s, r)$ to state $s \in \mathcal{S}$ where $r \in \Re^K$ and

$$\hat{Q}(s, r) = \sum_{k=1}^{K} \phi_k(s) r(k), \tag{18}$$

Using simulated paths, we find parameters $r_0^*, r_2^*, \ldots, r_{T-1}^*$ so that

$$\hat{Q}(s, r_t^*) \approx Q_t(s)$$

for each s and t. Tsitsiklis & Van Roy (2001) develop a methodology for learning the value functions, while Longstaff & Shwartz (2001) consider a variation that focuses on learning the optimal policy. Empirically it is seen that the estimated value function corresponding to the approximate optimal policy (evaluated using Monte Carlo simulation) is very close to the optimal value function, while the value function learnt using regression methods can be quite inaccurate.

7.1 *Learning the value function*

This algorithm involves generating M sample paths $(s_{m,t} : t \leq T, m \leq M)$ of the process $(S_t : t \leq T)$ using the original measure P.

The parameter r_{T-1}^* is estimated as follows:

$$\begin{aligned} r_{T-1}^* &= \arg\min_r \sum_{m=1}^{M} \left[g_T(s_{m,T}) - \sum_{k=1}^{K} \phi_k(s_{m,T-1}) r(k) \right]^2 \\ &= \left(\sum_{m=1}^{M} \phi(s_{m,T-1}) \phi(s_{m,T-1})^\top \right)^{-1} \sum_{m=1}^{M} \phi(s_{m,T-1}) g_T(s_{m,T}), \end{aligned} \tag{19}$$

where $\phi(\cdot)$ denotes the column vector $(\phi_k(\cdot) : k \leq K)$, and $A^\top$ denotes the transpose of matrix A. Here, $g_T(s_{m,T})$ is used as an unbiased proxy for the conditional expectation $Q_{T-1}(s_{m,T-1})$. The resultant approximation $\sum_{k=1}^{K} \phi_k(s) r^*_{T-1}(k)$ provides a reasonable approximation to $Q_{T-1}(s)$. We refer the reader to Tsitsiklis & Van Roy (2001) for asymptotic properties of this estimator as $M \to \infty$. In the same spirit, using the approximation $\hat{Q}(\cdot, r^*_{T-1})$ for $Q_{T-1}(\cdot)$ along each generated path m, $r^*_{T-2}, \ldots, r^*_1$ can be found using

$$r^*_t = \arg\min_r \sum_{m=1}^{M} \left[\max\{g_{t+1}(s_{m,t+1}), {r^*_{t+1}}^\top \phi(s_{m,t+1})\} - \sum_{k=1}^{K} \phi_k(s_{m,t}) r(k) \right]^2$$

$$= \left(\sum_{m=1}^{M} \phi(s_{m,t}) \phi(s_{m,t})^\top \right)^{-1} \sum_{m=1}^{M} \phi(s_{m,t}) \max\{g_{t+1}(s_{m,t+1}), {r^*_{t+1}}^\top \phi(s_{m,t+1})\}, \tag{20}$$

since $\max\{g_{t+1}(s_{m,t+1}), {r^*_{t+1}}^\top \phi(s_{m,t+1})\}$ provides a random proxy for the continuation value $Q_t(s_{m,t})$. Finally, the estimate of the true option price $J_0(s_0)$ is

$$\hat{J}_0(s_0) = \frac{1}{M} \sum_{m=1}^{M} \max\{g_1(s_{m,1}), {r^*_1}^\top \phi(s_{m,1})\}. \tag{21}$$

As mentioned earlier, the estimate of the option price obtained above can be significantly inaccurate even for single-dimension options. Also, as the direction of the bias in the estimate is not *a priori* clear, it cannot be used as a lower or an upper bound.

7.2 *Learning the optimal policy*

Recall that the option price equals the expected value of the payoff under a policy that maximizes this expectation. Therefore, the expectation corresponding to an approximation to the optimal policy is provably lower biased. We now describe the regression-based algorithm due to Longstaff & Schwartz (2001) to approximate an optimal policy.

Again generate M sample paths $(s_{m,t} : t \leq T, m \leq M)$ of the process $(S_t : t \leq T)$ using the original measure P.

The parameters $r^*_1, \ldots, r^*_{T-1}$ are found recursively:

$$r^*_{T-1} = \arg\min_r \sum_{m=1}^{M} \left[g_T(s_{m,T}) - \sum_{k=1}^{K} \phi_k(s_{m,T-1}) r(k) \right]^2$$

Then, using the approximation $\hat{Q}(\cdot, r^*_{T-1})$ for $Q_{T-1}(\cdot)$ along each generated path m we can approximately evaluate when to exercise the option, given that we have not exercised it till time $T-2$. Call this time $\tau_{m,T-2}$.

Recursively, consider time t. Suppose that we know $\tau_{m,t}$, the time to exercise the option along path m, given that we have not exercised it till time t. Then, parameters r^*_t are found as a solution to the least squares problem:

$$r^*_t = \arg\min_r \sum_{m=1}^{M} \left[g_{\tau_{m,t}}(s_{m,\tau_{m,t}}) - \sum_{k=1}^{K} \phi_k(s_{m,t}) r(k) \right]^2.$$

Note that if $\tau_{m,t}$ is a realization of the optimal stopping time, then $g_{\tau_{m,t}}(s_{m,\tau_{m,t}})$ above is an unbiased sample of the continuation value $Q_t(x_{m,t})$ (and hence a reasonable proxy).

For each t, Longstaff & Schwartz (2001) also observe improved performance if they restrict only to paths m that are in-the-money, i.e., for which, $g_t(s_{m,t}) > 0$. Once the stopping times on each path has been estimated, an estimate $\hat{Q}_0(s_0)$ of $Q_0(s_0)$ maybe set to the average of the payoffs along each path where the payoff at a given time equals the payoff at the first instant a decision to stop is encountered. Alternatively, one may generate more independent paths and use the estimate of Q functions provided by the parameters $(r_1^*, \ldots, r_{T-1}^*)$ above, to decide the first instant to stop at each generated path. Again, the average of payoffs over independent path provides an estimate of a lower bound to the option price. Empirically, it has been seen that the computational effort required to learn the optimal policy, i.e., to learn $(r_1^*, \ldots, r_{T-1}^*)$, is much less compared to the effort required to evaluate the expected payoff under this policy. Intuitively, this is true as slight errors in coming up with the optimal policy has little impact on the value of the policy. Thus when the instantaneous exercise value is very close to the continuation value, either decision has little impact on the expected payoff.

It is noteworthy that for a single-dimensional American put, Longstaff & Schwartz (2001) use the following four basis functions:

- 1, i.e., a constant
- $\exp(-s/2)$
- $\exp(-s/2)(1-s)$
- $\exp(-s/2)(1-2s-s^2/2)$

However, they also found that under their methodology, the form of basis functions has little effect on the estimate of the option price.

7.3 *Importance sampling for regression-based methods*

We now discuss how importance sampling may be applied to the regression-based methods. Our discussion follows that in Bolia *et al* (2004). As noted above, an approximately optimal policy can be learnt with relatively little effort (we refer to this as the first phase). Main effort is needed in the second phase for evaluating the expected payoff from this policy. We discuss how importance sampling may be useful in the second phase. Note that for determining approximate zero variance estimator, we need to find approximation $\tilde{J}$ for the value functions J. Furthermore, to generate samples and compute the likelihood ratio, it is desirable that the integral $(P_t \tilde{J}_{t+1})(s)$ be known explicitly and also that we may be able to sample cheaply from the associated densities $\tilde{f}$, where

$$\tilde{f}_t(s,u) = \frac{f_t(s,u)\tilde{J}_{t+1}(u)}{(P_t\tilde{J}_{t+1})(s)}, \quad s,u \in \mathcal{S}.$$

To achieve this, we consider a parameterized value function $\hat{J} : \mathcal{S} \times \Re^K \to \Re$ that assigns values $\hat{J}(s,x)$ to state s where again, $x \in \Re^K$, and

$$\hat{J}(s,x) = \sum_{k=1}^{K} \phi_k(s)x(k). \tag{22}$$

We choose each $\phi_k(\cdot)$ so that $(P_t\phi_k)(s)$ can be explicitly evaluated and it is easy to generate samples from the probability density functions

$$\frac{f_t(s,u)\phi_k(u)}{(P_t\phi_k)(s)}.$$

We estimate parameters $x_1^*, x_2^*, \ldots, x_T^*$, under non-negativity constraints, so that $\hat{J}(s, x_t^*) \approx J_t(s)$ for each s and t.

We modify the approach in §7.2 to determine the parameters $x_1^*, \ldots, x_T^*$. Set

$$x_T^* = \arg\min_{x\geq 0} \sum_{m=1}^{M} \left[g_T(s_{m,T}) - \sum_{k=1}^{K} \phi_k(s_{m,T}) x(k) \right]^2.$$

Note that we now use the non-negative least squares method (as in Lawson & Hanson 1974).

The parameters x_t^* for $t \leq T-1$ are found after parameters r_t^* have been determined as in §7.2. Knowing r_t^* allows us to determine whether to exercise the option at state $s_{m,t}$ or not by comparing $g_t(s_{m,t})$ and $\hat{Q}(s_{m,t}, r_t^*)$ for each m. Then, $\tau_{m,t-1}$ (i.e., the time to exercise the option along path m, given that it has not been exercised up to time $t-1$) is known for each m. Set

$$x_t^* = \arg\min_{x\geq 0} \sum_{m=1}^{M} \left[g_{\tau_{m,t-1}}(s_{m,\tau_{m,t-1}}) - \sum_{k=1}^{K} \phi_k(s_{m,t}) x(k) \right]^2.$$

Again, if $\tau_{m,t-1}$ is a realization of the optimal stopping time, then $g_{\tau_{m,t-1}}(s_{m,\tau_{m,t}})$ above is an unbiased sample of $J_t(s_{m,t})$.

Once $(\hat{J}(s, x_t^*) : s \in \mathcal{S}, t \leq T)$ are known, we start the second phase of the algorithm involving importance sampling to evaluate the price of the option. The importance sampling probability densities are given by

$$\hat{f}_t(s,u) = \frac{f_t(s,u)\hat{J}(u, x_{t+1}^*)}{\int_{u\in\mathcal{S}} f_t(s,u)\hat{J}(u, x_{t+1}^*)\mathrm{d}u}.$$

This may be re-expressed as

$$\hat{f}_t(s,u) = \sum_{k\leq K} p_k^*(s) \frac{f_t(s,u)\phi_k(u)}{(P_t\phi_k)(s)}, \tag{23}$$

where

$$p_k^*(s) = \frac{x_{t+1}^*(k)(P_t\phi_k)(s)}{\sum_{k\leq K} x_{t+1}^*(k)(P_t\phi_k)(s)}.$$

Note that $p_k^*(s) \geq 0$ and $\sum_{k\leq K} p_k^*(s) = 1$. Hence, if we can easily generate a sample from the pdf $[f_t(s,\cdot)\phi_k(\cdot)]/[(P_t\phi_k)(s)]$, then this makes generation from $\hat{f}_t(s,\cdot)$ also straightforward.

Bolia *et al* (2004) empirically demonstrate significant variance reduction on a simple one-dimensional example. In their example, the density $f_t(s,\cdot)$ is log-normally distributed and the basis functions have the form,

$$\phi_k(y) = \exp[\alpha_{1k}\log^2 y + \alpha_{2k}\log y],$$

for appropriately chosen $(\alpha_{1k}, \alpha_{2k})$ for each k. This results in $\hat{f}_t(s,\cdot)$ equalling a non-negative mixture of log-normal distributions so that using it, it is easy to generate samples and to compute the value of the likelihood ratio along each generated path.

8. Random tree algorithm

We now briefly describe the random tree algorithm (RTA) proposed by Broadie & Glasserman (1997). It involves generating a sample tree instead of a sample path via simulation. This amounts to starting at state s_0 at time 0 and generating b independent samples (or 'branches') of the underlying asset prices at time 1. From each generated sample at time 1, b independent samples are generated at time 2 and so on until time T. Thus the computational effort is $O(b^T)$ (i.e., grows exponentially with the number of time periods). Assuming that the empirical distribution corresponding to the generated tree is the true one, the dynamic programming backward recursion is used to compute the option price. It is shown using Jensen's inequality that the resultant estimator has an upper bias that reduces to zero as $b \to \infty$. A methodology is developed to compute another lower biased estimator where again the bias reduces to zero as $b \to \infty$. The algorithm is described in detail in §8.1. The main drawback of this algorithm is that the computational effort increases exponentially in the number of time steps T.

8.1 *Description of the algorithm*

The simulated trees are parameterized by b, the number of branches per node. The state variables are simulated at T exercise times. Starting from state $S_0 = s_0$, b independent samples $(s_1^{i_1} : i_1 = 1, \ldots, b)$ of S_1, the asset prices at time 1, are generated. For every i_1, b independent samples $(s_2^{i_1 i_2} : i_2 = 1, \ldots, b)$ at time 2 are generated using the distribution of S_2 conditioned on $S_1 = s_1^{i_1}$. Proceeding in this manner for each $i_1, i_2, \ldots, i_{T-1}$, b samples $(s_T^{i_1 i_2 \ldots i_T} : i_T = 1, \ldots, b)$ at time T are generated using the distribution of S_T conditioned on $S_{T-1} = s_{T-1}^{i_1 i_2 \ldots i_{T-1}}$. Figure 2 below provides an illustration of a tree with $b = 3$ and $T = 2$.

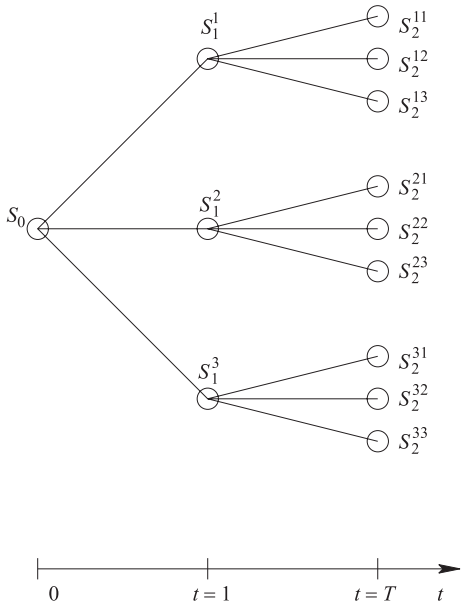

Figure 2. Random tree.

Note that given $s_t^{i_1,\ldots,i_t}$, $s_{t+1}^{i_1\ldots i_t j}$, $j = 1, \ldots, b$, are conditionally independent of each other and of all $s_u^{i'_1\ldots i'_u}$ with $u < t$ or $i_t \neq i'_t$. Each sequence,

$$s_0, s_1^{i_1}, s_2^{i_1 i_2}, \ldots, s_T^{i_1\ldots i_T},$$

is a realization of the Markov Chain $\{S_t : t = 0, 1, \ldots, T\}$, and two such sequences evolve independently of each other, once they differ in some i_t.

8.2 *High and low bias estimators*

Once a random tree is generated, the backward recursions (14) applied to the empirical distribution corresponding to the generated tree are used to compute the upper and lower biased estimates of the option price at every node in the tree. Let $U_t^{i_1 i_2\ldots i_t}$ and $L_t^{i_1 i_2\ldots i_t}$ denote the upper and lower biased estimators, respectively at the node $s_t^{i_1 i_2\ldots i_t}$.

8.2a *High estimator:* Recall that $g_t(s)$ denotes the pay-off from the option if it is exercised at time t at state s. The upper biased estimators are obtained as follows:

$$\begin{aligned} U_T^{i_1 i_2\ldots i_T} &= g_T(s_T^{i_1 i_2\ldots i_T}) \\ U_t^{i_1 i_2\ldots i_t} &= \max\left\{ g_t(s_t^{i_1 i_2\ldots i_t}), \frac{1}{b}\sum_{j=1}^{b} U_{t+1}^{i_1\ldots i_t j} \right\}, \end{aligned} \tag{24}$$

for $t = 0, 1, \ldots, T-1$. Thus, U_0 (here and throughout a quantity subscripted by $t = 0$ is understood to have no superscripts; thus $U_t^{i_1 i_2\ldots i_t}$ becomes U_0 at $t = 0$) is an estimator of the desired option price $J_0(s_0)$. We also denote it by $U_0(b)$ to emphasize its dependence on b.

Under the regularity conditions $E[\,|g_t(S_t)|^p\,] < \infty$ for some $p > 1$, it can be proved that

$$E[U_0(b)] \to J_0(s_0),$$

as $b \to \infty$, i.e., the estimator $U_0(b)$ is asymptotically unbiased. Note that by Jensen's inequality, given that $s_{T-1}^{i_1, i_2, \ldots, i_{T-1}} = s$,

$$E \max(g_{T-1}(s), \frac{1}{b}\sum_{j=1}^{b} U_T^{i_1 i_2\ldots i_T}) \geq \max\{g_{T-1}(s), Q_{T-1}(s)\} = J_{T-1}(s),$$

where, recall that $Q_{T-1}(s) = E\left[\frac{1}{b}\sum_{j=1}^{b} U_T^{i_1 i_2\ldots i_T} | s_{T-1}^{i_1, i_2, \ldots, i_{T-1}} = s\right]$ denotes the continuation value at time $T-1$ at state s. For finite b, by recursively applying the Jensen's inequality in the manner described above, $U_0(b)$ can be shown to be biased high, i.e.,

$$E[U_0(b)] \geq J_0(s_0).$$

We refer the reader to Broadie & Glasserman (1997) for a rigorous proof.

8.2b *Low estimator:* We define the lower estimator L recursively as follows. At time T, the estimator at the node $s_T^{i_1 i_2\ldots i_T}$ is given by

$$L_T^{i_1 i_2\ldots i_T} = g_T(s_T^{i_1 i_2\ldots i_T}). \tag{25}$$

Note that this estimate is an unbiased estimate of the option price. Now, consider a node $s_{T-1}^{i_1 i_2 \ldots i_{T-1}}$ at time $T-1$. Divide the b successor nodes from this node into two sets, the jth node $s_T^{i_1 i_2 \ldots i_{T-1} j}$ and the remaining $b-1$ nodes. Note that

$$\frac{1}{b-1} \sum_{\substack{i=1 \\ i \neq j}}^{b} L_T^{i_1 \ldots i_{T-1} i}$$

provides an unbiased estimate of the continuation value. An exercise policy that directs immediate exercise of the option at $s_{T-1}^{i_1 i_2 \ldots i_{T-1}}$ if

$$g_t(s_t^{i_1 i_2 \ldots i_{T-1}}) \geq \frac{1}{b-1} \sum_{\substack{i=1 \\ i \neq j}}^{b} L_T^{i_1 \ldots i_{T-1} i},$$

and continuation otherwise, is clearly sub-optimal (recall that optimal policy is the one that gives us the maximum value of the option price given only current information, and therefore, by definition, any policy based only on the current information is sub-optimal). Note that $\tilde{L}_{T-1}^{i_1 \ldots i_{T-1}}(j)$ provides an unbiased sample of continuation value under this policy, where $\tilde{L}_{T-1}^{i_1 \ldots i_{T-1}}(j)$ at all nodes at $T-1$ is given by

$$\tilde{L}_{T-1}^{i_1 \ldots i_{T-1}}(j) = \begin{cases} g_{T-1}(s_{T-1}^{i_1 i_2 \ldots i_{T-1}}), & \text{if } g_{T-1}(s_{T-1}^{i_1 i_2 \ldots i_{T-1}}) \geq \frac{1}{b-1} \sum_{\substack{i=1 \\ i \neq j}}^{b} L_T^{i_1 \ldots i_{T-1} i}, \\ L_T^{i_1 \ldots i_{T-1} j}, & \text{if } g_{T-1}(s_{T-1}^{i_1 i_2 \ldots i_{T-1}}) < \frac{1}{b-1} \sum_{\substack{i=1 \\ i \neq j}}^{b} L_T^{i_1 \ldots i_{T-1} i}, \end{cases} \tag{26}$$

and is therefore lower biased. Recursively, consider node $s_t^{i_1 i_2 \ldots i_t}$ at time $t \leq T-2$. Now, dividing the b successor nodes from this node into two sets, the jth node $s_{t+1}^{i_1 i_2 \ldots i_t j}$ and the remaining $b-1$ nodes, $[1/(b-1)] \sum_{\substack{i=1 \\ i \neq j}}^{b} L_{t+1}^{i_1 \ldots i_t i}$ provides an estimate of the continuation value. Moreover, an exercise policy based on this continuation value (as discussed above) is again sub-optimal. Therefore, an estimate of option value $\tilde{L}_t^{i_1 \ldots i_t}(j)$ realised by this policy, i.e.,

$$\tilde{L}_t^{i_1 \ldots i_t}(j) = \begin{cases} g_t(s_t^{i_1 i_2 \ldots i_t}), & \text{if } g_t(s_t^{i_1 i_2 \ldots i_t}) \geq \frac{1}{b-1} \sum_{\substack{i=1 \\ i \neq j}}^{b} L_{t+1}^{i_1 \ldots i_t i}, \\ L_{t+1}^{i_1 \ldots i_t j}, & \text{if } g_t(s_t^{i_1 i_2 \ldots i_t}) < \frac{1}{b-1} \sum_{\substack{i=1 \\ i \neq j}}^{b} L_{t+1}^{i_1 \ldots i_t i} \end{cases} \tag{27}$$

is lower biased. Since, any value out of $1, \ldots, b$ could have been chosen for j, Broadie & Glasserman (1997) suggest setting the lower estimate $L_t^{i_1 \ldots i_t}$ to

$$L_t^{i_1 \ldots i_t} = \frac{1}{b} \sum_{j=1}^{b} \tilde{L}_t^{i_1 \ldots i_t}(j), \tag{28}$$

for $t = T-1, 1, \ldots, 0$. Under reasonable regularity conditions, L too can be proved to be asymptotically unbiased as $b \to \infty$ and lower biased when b is finite. A rigorous proof for the same can be found in Broadie & Glasserman (1997).

8.3 *Importance sampling for RTA*

Suppose now that at each time $t \in \{0, 1, \ldots, T-1\}$ we have an approximation to the ideal zero-variance transition density (17), call it $\tilde{f}_t(\cdot,\cdot)$. This can be obtained, for instance, using approximation (22) to the value function to get the desired approximation (23) as in §7.3. Also, let $f_t(\cdot,\cdot)$ denote the original transition density. We generate the random tree using these approximate zero-variance transition densities. Then, for the high estimator (24) changes to

$$\begin{aligned} U_T^{i_1 i_2 \ldots i_T} &= g_T(s_T^{i_1 i_2 \ldots i_T}), \\ U_t^{i_1 i_2 \ldots i_t} &= \max\left\{ g_t(s_t^{i_1 i_2 \ldots i_t}), \frac{1}{b}\sum_{j=1}^{b} U_{t+1}^{i_1 \ldots i_t j}\mathcal{L}^{(j)} \right\}, \end{aligned} \tag{29}$$

and for the low estimator (27) changes to

$$\tilde{L}_t^{i_1 \ldots i_t}(j) = \begin{cases} g_t(s_t^{i_1 i_2 \ldots i_t}), & \text{if } g_t(s_t^{i_1 i_2 \ldots i_t}) \geq \frac{1}{b-1}\sum_{\substack{i=1 \\ i\neq j}}^{b} L_{t+1}^{i_1 \ldots i_t i}\mathcal{L}^{(j)}, \\ L_{t+1}^{i_1 \ldots i_t j}\mathcal{L}^{(j)}, & \text{if } g_t(s_t^{i_1 i_2 \ldots i_t}) < \frac{1}{b-1}\sum_{\substack{i=1 \\ i\neq j}}^{b} L_{t+1}^{i_1 \ldots i_t i}\mathcal{L}^{(j)}. \end{cases} \tag{30}$$

In both (29) and (30), $\mathcal{L}^{(j)}$ refers to the one-step likelihood ratio,

$$\frac{f_t(s_t^{i_1 i_2 \ldots i_t}, s_t^{i_1 i_2 \ldots i_t j})}{\tilde{f}_t(s_t^{i_1 i_2 \ldots i_t}, s_t^{i_1 i_2 \ldots i_t j})},$$

from node $s_t^{i_1 i_2 \ldots i_t}$ at time t to node $s_t^{i_1 i_2 \ldots i_t j}$ at time $t+1$ for $j = 1, 2, \ldots, b$. The use of $\mathcal{L}^{(j)}$ appropriately unbiases the estimates obtained from the random tree that has been generated using the approximate zero-variance change of measure. It is clear from arguments similar to those presented in §6.2 that if the value function $J_t(\cdot)$ is known exactly for all $t \leq T$, the high and low estimates both have variance zero. Empirically, we get improved performance even with the approximate value function (22) as reported in table 2. The results presented are for pricing an American put in single-dimension. Note that, in the importance sampling case, in addition to significant variance reduction, we also see a remarkable reduction in the difference between the upper and lower point estimates.

9. Stochastic mesh algorithm

As mentioned earlier, the major drawback of the random tree algorithm is the exponential increase in computational time as a function of the number of early exercise opportunities. Broadie & Glasserman (2004) propose a new stochastic mesh method that attempts to circumvent this drawback. The essential idea may be seen as follows: Consider the random variables $(S_1, S_2, \ldots, S_T)$ where S_t denotes the asset prices at time t. Let $(S_{1,i}, S_{2,i}, \ldots, S_{T,i})$ for $i \leq b$ denote b independent copies of $(S_1, S_2, \ldots, S_T)$. Let $(s_{1,i}, s_{2,i}, \ldots, s_{T,i})$ for $i \leq b$ denote their realizations. Recall that $S_{0,i} = s_0$ for all i. Now suppose that $\hat{J}_{t+1}(S_{t+1,i})$ is an unbiased and an accurate estimator of $J_{t+1}(S_{t+1,i})$, the option price at time $t+1$ at state $S_{t+1,i}$ for each $i \leq b$. Suppose our interest is in estimating the continuation value $Q_t(S_{t,i})$ for each i. Then clearly, $\hat{J}_{t+1}(S_{t+1,i})$ provides an unbiased sample of $Q_t(S_{t,i})$ (again, as mentioned in Section 6.1, we may, without loss of generality, ignore the discounting factor). The interesting

Table 2. Importance sampling in random tree algorithm.

s_0	Naive Est (95% CI)	IS Est (95% CI)	VRF
Lower bound performance			
36	4·192 (4·089, 4·295)	4·312 (4·292, 4·332)	26·7
40	2·233 (2·182, 2·284)	2·246 (2·235, 2·257)	23·7
50	0·315 (0·301, 0·329)	0·315 (0·312, 0·318)	18·9
Upper bound performance			
36	4·392 (4·322, 4·462)	4·328 (4·308, 4·348)	12·4
40	2·292 (2·240, 2·344)	2·256 (2·245, 2·267)	23·7
50	0·324 (0·309, 0·339)	0·316 (0·312, 0·319)	19·4

Option parameters: $r = 6\%$ is assumed constant, $T = 3$, $b = 50$, strike price $= 40$, volatility $=$ 0·2. The results have been obtained for 100 independent replications of the random tree. s_0 indicates the starting state, the first row thus represents an in-the-money option, the second row an at-the-money option and the last row a deep out-of-money option. "Naive Est" indicates the estimate without using importance sampling, "IS Est" indicates the estimate with importance sampling, and variance reduction factor (VRF) is the ratio of naive variance to IS variance.

question considered by Broadie & Glasserman (2004) is to see if $\hat{J}_{t+1}(S_{t+1,j})$, $j \neq i$ provide any useful information for estimating $Q_t(S_{t,i})$. Clearly such a $\hat{J}_{t+1}(S_{t+1,j})$ is not an unbiased sample of $Q_t(S_{t,i})$. They observe that by multiplying $\hat{J}_{t+1}(S_{t+1,j})$ with an appropriate and an easily computable weight $W_t(S_{t,i}, S_{t+1,j})$, an unbiased sample of $Q_t(S_{t,i})$ may be obtained. Taking the average of the resultant samples for each $j \neq i$ and $\hat{J}_{t+1}(S_{t+1,i})$, an accurate estimate of $Q_t(S_{t,i})$ and hence of $J_t(S_{t,i})$ may be obtained. In practice, we typically do not have unbiased estimators of $J_{t+1}(S_{t+1,i})$ (although these are asymptotically unbiased as $b \to \infty$), however, the weights are chosen with the properties described above. As in the random tree algorithm, this procedure can be shown to give upper biased estimates. A similar modification may then be developed to obtain estimates with a lower bias. We first discuss how one arrives

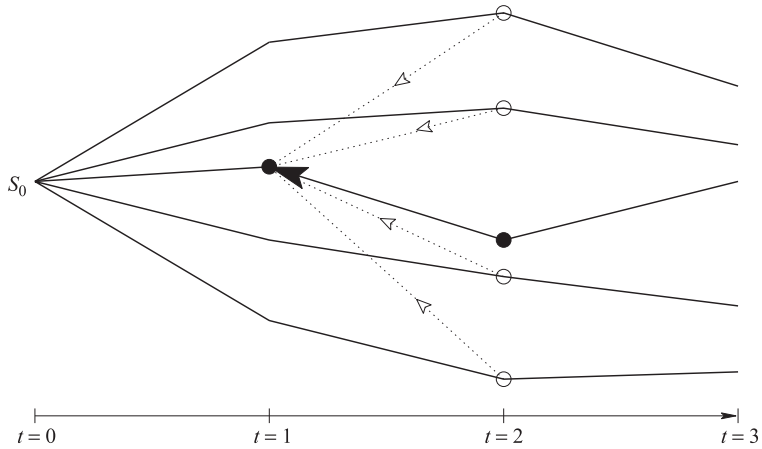

Figure 3. Motivation for the stochastic mesh.

at the unbiasing weights $W_t(S_{t,i}, S_{t+1,j})$. For this purpose let $\hat{J}_{t+1}(S_{t+1,j}) = J_{t+1}(S_{t+1,j})$ for each j. Let $h_{t+1}(\cdot)$ denote the unconditioned density of S_{t+1}. Let $f_t(x, y)$ denote the pdf of S_{t+1} evaluated at y when $S_t = x$. For $j = i$ the corresponding weight is set to 1. When $j \neq i$, then, for the resulting estimator $W_t(S_{t,i}, S_{t+1,j})J_{t+1}(S_{t+1,j})$ to be unbiased we need

$$\int_{s\in\mathcal{S}} W_t(S_{t,i}, y)J_{t+1}(y)h_{t+1}(y)\mathrm{d}y$$

to equal $Q_t(S_{t,i})$. This can be achieved when

$$W_t(x, y) = \frac{f_t(x, y)}{h_{t+1}(y)},$$

as $Q_t(x) = \int f_t(x, y)J_{t+1}(y)\mathrm{d}y$. Since this weight is a ratio of two densities, hereafter we also refer to it as the "likelihood ratio" weight. Note that in this case we could have replaced the pdf $h(\cdot)$ by the joint pdf of observing the path $(s_{1,j}, s_{2,j}, \ldots, s_{t+1,j})$. Then, the weights may be analogously defined. Broadie & Glasserman (2004) show that in many cases, the resulting estimator may have variance that increases exponentially with time T. Later in §9.3 we discuss more sophisticated ways of selecting weights and conducting simulation (proposed by Broadie & Glasserman 2004) so that this exponential growth is avoided. We now discuss how the upper and lower biased estimates may be obtained, when the selected weights possess the above unbiasing property.

9.1 *Upper biased estimator*

For each t and i, let $U(t, S_{t,i})$ denote the upper biased estimator of $J_t(S_{t,i})$. These are obtained recursively as:

$$\begin{aligned} U(T, S_{T,i}) &= g_T(S_{T,i}), \\ U(t, S_{t,i}) &= \max\left\{g_t(S_{t,i}), \frac{1}{b}\sum_{j=1}^{b} W_t(S_{t,i}, S_{t+1,j})\, U(t+1, S_{t+1,j})\right\}, \end{aligned} \tag{31}$$

for $i = 1, \ldots, b$ and $t = 1, \ldots, T-1$. Finally, if exercise is allowed at time 0,

$$U(0, s_0) = \max\left\{g_0(s_0), \frac{1}{b}\sum_{j=1}^{b} U(1, S_{1,j})\right\},$$

gives the desired estimate of the continuation value at time 0. Under some regularity conditions, it can be proved that $U(0, s_0)$ is asymptotically unbiased, i.e.,

$$E[U(0, s_0)] \to Q_0(s_0),$$

as $b \to \infty$. Applying Jensen's inequality recursively in a manner similar to §8.2a, it can be proved that for finite b,

$$E[U(0, s_0)] \geq Q_0(s_0).$$

For rigorous proofs, we refer the reader to Broadie & Glasserman (2004).

9.2 *Lower biased estimator*

Note that if we know the continuation value $Q_t(s)$, $\forall s$ and $\forall t \leq T$, then, along the set of random variables $\{S_t : t = 1, \ldots, T\}$, the optimal stopping time τ^* corresponds to $\tau^* = \min\{t : g_t(S_t) \geq Q_t(S_t)\}$ (see, e.g., Duffie 1996). Broadie & Glasserman (2004) compute an estimate of the lower bound by first developing an estimator of $Q_t(s)$, call it, $\hat{Q}_t(s)$ for all s, using the generated mesh as follows:

$$\begin{aligned} \hat{Q}_t(s) &= \sum_{j=1}^{b} W_t(s, s_{t+1,j})\, U(t+1, s_{t+1,j}) \\ &\equiv \sum_{j=1}^{b} \frac{f_t(s, s_{t+1,j})}{h_{t+1}(s_{t+1,j})} U(t+1, s_{t+1,j}). \end{aligned} \tag{32}$$

Then, their method corresponds to generating independent samples of $S = \{S_t : t = 0, 1, \ldots, T\}$. Setting the approximately optimal time to

$$\tau_b(S) = \min\{t : g_t(S_t) \geq \hat{Q}_t(S_t)\}, \tag{33}$$

where subscript b in τ_b is used to indicate the dependence of the approximate policy on the mesh parameter b. The r.v.

$$L_b = g_{\tau_b}(S(\tau_b)), \tag{34}$$

then denotes an estimator of the option price with a lower bias. The sample mean of i.i.d. samples of L_b provides a lower bias (since it is based on a sub-optimal stopping time τ_b). Under some regularity conditions, L_b can be proved to be asymptotically unbiased (i.e., as $b \to \infty$). We refer the reader to Broadie & Glasserman (2004) for a rigorous proof.

9.3 *Average mesh density functions*

For the estimates $U(0, s_0)$ and L_b to have low variance, it is desirable that the weights $W_t(\cdot, \cdot)$ $\forall t \leq T$ do not have high variance. Broadie & Glasserman (2004) suggest using the mesh density $f_0(s_0, u)$ to generate b samples $(s_{1,i} : i \leq b)$ and given that $(s_{t,i} : i \leq b)$ have been generated, to generate samples $(s_{t+1,i} : i \leq b)$ using density,

$$\frac{1}{b} \sum_{j=1}^{b} f_t(s_{t,j}, u),$$

for $t = 1, \ldots, T-1$. With this as the mesh density, the weight $W_t(s_{t,i}, s_{t+1,j})$ is given by

$$W_t(s_{t,i}, s_{t+1,j}) = \frac{f_t(s_{t,i}, s_{t+1,j})}{\frac{1}{b}\sum_{k=1}^{b} f_t(s_{t,k}, s_{t+1,j})}.$$

This choice of mesh density functions is termed as the *average density functions*. In §9.3a, we discuss how generating from these density functions through stratification, reduces to conducting routine simulation involving generation of independent paths of asset prices. Here, we note that this density has a desirable property that the average weight into a node $s_{t+1,j}$ is one, i.e.,

$$\frac{1}{b} \sum_{l=1}^{b} W(s_{t,l}, s_{t+1,j}) = 1. \tag{35}$$

As mentioned earlier, our estimate $U(0, S_0)$ can potentially have variance that increases exponentially in the number of time periods T. It can be shown (Broadie & Glasserman 2004) that as a consequence of (35), the use of average density functions gives us estimators that do not exhibit exponential increase in variance. Another desirable characteristic of the weights in this case is that they are bounded from above by b and hence cannot be too large.

9.3a *Stratified implementation:* We first describe how sample paths may be generated using average density functions. Suppose that using the average density functions, up to time t we have generated samples $\{s_{0,i}, s_{1,i}, \ldots, s_{t,i} : i = 1, \ldots, b\}$. Then at time $t+1$, b samples of S_{t+1} may be generated as follows: Select any of the b samples $(s_{t,i}, i \leq b)$ with equal probability $1/b$ and then using the selected sample say $s_{t,k}$, generate the next sample using the density $f_t(s_{t,k}, \cdot)$. Repeat this b times independently to generate b such samples. The average of the b samples multiplied with the suitable weights then gives an estimate for each node $(s_{t,i}, i \leq b)$.

Fortunately, by applying stratification this process simplifies. Using stratification, instead of selecting each of the b samples $s_{t,k}, k \leq b$ with equal probability $1/b$ and then generating the next sample and doing this b times independently, we may simply select each sample $s_{t,k}$ once for $k = 1, \ldots, b$ and then generate the next sample using the density $f_t(s_{t,k}, \cdot)$. Thus, to estimate the continuation value at state $s_{t,i}$, we take the average of the option price estimate $U(t+1, s_{t+1,j})$ at each successor node $s_{t+1,j}$, multiplied by the corresponding likelihood ratio weight

$$W_t(s_{t,i}, s_{t+1,j}) = f_t(s_{t,i}, s_{t+1,j}) \Bigg/ \left[\frac{1}{b} \sum_{k=1}^{b} f(t, s_{t,k}, s_{t+1,j}) \right],$$

call it $C_t(j)$. Average of these for all $j \leq b$,

$$\sum_{j=1}^{b} \left(\frac{1}{b} \right) C_t(j),$$

or

$$\sum_{j=1}^{b} \left(\frac{1}{b} \right) \frac{f_t(s_{t,i}, s_{t+1,j})}{\frac{1}{b} \sum_{k=1}^{b} f(t, s_{t,k}, s_{t+1,j})} U(t+1, s_{t+1,j}),$$

gives an estimate of the continuation value at state $s_{t,i}$.

Thus, the stratified implementation of the complete mesh involves generating b independent samples of $(s_0, S_1, \ldots, S_T)$, namely $\{s_0(i), s_{1,i}, \ldots, s_{T,i} : i = 1, \ldots, b\}$ using the transition densities $(f_t(\cdot, \cdot) : t \leq T-1)$, just as done in routine simulation. Equation (31) then changes to (for each i)

$$\begin{aligned} U(T, s_{T,i}) &= g_T(s_{T,i}), \\ U(t, s_{t,i}) &= \max \left\{ g_t(s_{t,i}), \frac{1}{b} \sum_{j=1}^{b} \frac{f_t(s_{t,i}, s_{t+1,j})}{(1/b) \sum_{k=1}^{b} f(t, s_{t,k}, s_{t+1,j})} U(t+1, s_{t+1,j}) \right\}, \end{aligned} \tag{36}$$

and (32) to

$$\hat{Q}_t(s_t) = \sum_{j=1}^{b} \frac{f_t(s_t, s_{t+1,j})}{(1/b) \sum_{k=1}^{b} f(t, s_{t,k}, s_{t+1,j})} U(t+1, s_{t+1,j}). \tag{37}$$

9.4 *Importance sampling for stochastic mesh algorithm*

Now we generalize the stochastic mesh method in two ways: we use importance sampling densities $\tilde{f}_t(\cdot,\cdot), t \le T$ to generate the sample paths and instead of the average mesh density we use a weighted average mesh density. We note that when the importance sampling densities correspond to the approximate zero variance densities, then a suitable choice of weighted average mesh density has some desirable properties.

Thus, we recommend that the mesh density $\tilde{f}_0(s_0, u)$ be used to generate b samples $(s_{1,i} : i \le b)$ and given that $(s_{t,i} : i \le b)$ have been generated, generate samples $(s_{t+1,i} : i \le b)$ using weighted density

$$\sum_{k=1}^{b} v_t(k)\, \tilde{f}_t(s_{t,k}, \cdot), \tag{38}$$

for $t = 1, \ldots, T-1$. Here, the weights $(v_t(k) : k \le b)$ are non-negative and sum to 1. With these mesh densities, the likelihood ratio

$$\tilde{W}_t(s_{t,i}, s_{t+1,j}) = f_t(s_{t,i}, s_{t+1,j}) \Big/ \sum_{k=1}^{b} v_t(k)\, \tilde{f}_t(s_{t,k}, s_{t+1,j}). \tag{39}$$

Again, rather then generating samples using the density (38), we may use stratification so that, instead of selecting each of the b samples $s_{t,k}, k \le b$ with probability $v_t(k)$ and then generating the next sample using the associated density and repeating this process independently b times, we may simply select each sample $s_{t,k}$ once for $k = 1, \ldots, b$ and then generate the next sample using the density $f_t(s_{t,k}, \cdot)$ and assign weight $v_t(k)$ to it. Thus, our procedure for developing an upper biased estimate changes to generating b independent samples of $(s_0, S_1, \ldots, S_T)$, namely $(s_0(i), s_{1,i}, \ldots, s_{T,i} : i = 1, \ldots, b)$ using the transition densities $(\tilde{f}_t(\cdot,\cdot) : t \le T-1)$

$$\begin{aligned} U(T, s_{T,i}) &= g_T(s_{T,i}), \\ U(t, s_{t,i}) &= \max\left\{ g_t(s_{t,i}), \sum_{j=1}^{b} v_t(j)\, \tilde{W}_t(s_{t,i}, s_{t+1,j})\, U(t+1, s_{t+1,j}) \right\}, \end{aligned} \tag{40}$$

for $t = T-1, \ldots, 1, i \le b$.

It is important to note that in the above computations, the weights $(v_t(j) : j \le b)$ used in computing $U(t, s_{t,i})$ may depend upon i and may be different for different i. If the density $\tilde{f}_t(\cdot,\cdot)$ corresponds to the zero variance density, i.e.,

$$\tilde{f}_t(s, u) = [f_t(s, u) J_{t+1}(u)]/Q_t(s),$$

then the best choice of weights for a given i is clearly $v_t(i) = 1$ and the rest are zero, i.e., revert to straightforward simulation. If instead approximate zero variance densities are known then again a larger weight to $v_t(i)$ may be desirable. We note that there exists another choice of weights with some desirable properties.

Suppose that we have approximations $\hat{J}_t(\cdot)$ to the option value $J_t(\cdot)$ for $t \le T$. Suppose also that our approximate zero variance density for transition from state s at time t to state u at time $t+1$ corresponds to

$$[f_t(s, u) \hat{J}_t(u)]/P_t \hat{J}_t(x),$$

where, recall that, $P_t \hat{J}_t(\cdot)$ is defined as

$$(P_t \hat{J}_t)(s) \stackrel{\Delta}{=} E[\hat{J}_{t+1}(S_{t+1})|S_t = s] = \int_{\mathcal{S}} \hat{J}_{t+1}(u) f_t(s, u) \mathrm{d}u.$$

Then the stratification weights $v_t(j)$ defined as

$$v_t(j) = P_t \hat{J}_t(s_{t,j}) \Big/ \sum_{l=1}^{b} P_t \hat{J}_t(s_{t,l}).$$

lead to the estimate of continuation value at $s_{t,i}$ given by

$$\sum_{j=1}^{b} \frac{P \hat{J}_t(s_{t,j}) f_t(s_{t,i}, s_{t+1,j})}{\sum_{k=1}^{b} f_t(s_{t,k}, s_{t+1,j})} \cdot \frac{U(t+1, s_{t+1,j})}{\hat{J}_{t+1}(s_{t+1,j})},$$

which is bounded if $P \hat{J}_t(s_{t,j})$ and the ratio $[U(t+1, s_{t+1,j})]/[\hat{J}_{t+1}(s_{t+1,j})]$ is bounded. Therefore, if we have a good approximation $\hat{J}(\cdot)$ to the option price, we may expect improved variance characteristics of the importance sampling estimator. The empirical performance of this method will be analysed separately.

10. Duality

As discussed earlier, an estimate for the lower bound of the price of the Bermudan option may be obtained by developing and evaluating an approximation to the optimal exercise policy. In recent literature, two methods: 'additive duality' and 'multiplicative duality' have been proposed to develop upper bounds to the option price. We now briefly review these methods.

10.1 *Additive duality*

Note that $J_t(S_t) = \max(g_t(S_t), E(J_{t+1}(S_{t+1})|S_t))$. Thus $J_t(S_t) \geq E(J_{t+1}(S_{t+1})|S_t))$, i.e., on the average the option price decreases with time and hence is a super-martingale. Also note that $J_t(S_t) \geq g_t(S_t)$. In fact, it can be shown that $(J_t(S_t) : t \leq T)$ is the smallest super-martingale with this property (see, e.g., Haugh & Kogan 2001).

Let $(M_t : t \leq T)$ be an arbitrary mean zero martingale w.r.t. process $(S_t : t \leq T)$ and let τ be any stopping time w.r.t. this process. Note that due to the martingale stopping time theorem (see, e.g., Williams 1991) $EM_\tau = 0$. Haugh & Kogan (2001) make the following interesting observation:

$$E(g_\tau(S_\tau)) = E(g_\tau(S_\tau) - M_\tau) \leq E \max_{t \leq T}(g_t(S_t) - M_t).$$

Since the above inequality holds for all stopping times τ, it follows that

$$J_0(s_0) = \sup_\tau E(g_\tau(S_\tau)) = \sup_\tau E(g_\tau(S_\tau) - M_\tau) \leq E\left[\max_{t \leq T}(g_t(S_t) - M_t)\right]. \tag{41}$$

Therefore, an average of samples of $\max_{t\leq T}(g_t(S_t) - M_t)$ estimates an upper bound to the option value. Since, (41) is true for all zero mean martingales $M = (M_t : t \leq T)$,

$$J_0(s_0) \leq \inf_M E\left[\max_{t\leq T}(g_t(S_t) - M_t)\right].$$

Furthermore, Haugh & Kogan (2001) note that if M_t corresponds to the martingale obtained from $(J_t(S_t) : t \leq T)$ through Doob's decomposition, then the above inequality holds as an equality. That is, for

$$\tilde{M}_t = \sum_{k\leq t}[J_k(S_k) - E(J_k(S_k)|S_{k-1})],$$

$$J_0(s_0) = E\left[\max_{t\leq T}(g_t(S_t) - \tilde{M}_t)\right],$$

i.e., $J_0(s_0)$ is a solution to the dual problem

$$\inf_M E\left[\max_{t\leq T}(g_t(S_t) - M_t)\right],$$

where the infimum is over all zero mean martingales $M = (M_t : t \leq T)$.

In practice, $(\tilde{M}_t : t \leq T)$ is not known but may be approximated. However, care has to be taken that the approximation be a martingale. Haugh & Kogan (2001) and Andersen & Broadie (2001) develop several methods to come up with martingale processes that approximate $(\tilde{M}_t : t \leq T)$ and provide good upper bounds to the option price.

10.2 *Multiplicative duality*

The computation of an upper bound to the option price using additive duality relied on identifying a martingale process $M = (M_t : t \leq T)$. Bolia *et al* (2004) develop an upper bound using *multiplicative duality* based on importance sampling. As in §6.2, let $\tilde{L}_t$ denote the t step likelihood ratio of P w.r.t. $\tilde{P}$ (again, assume that P is absolutely continuous w.r.t. $\tilde{P}$), i.e.,

$$\tilde{L}_t = \frac{f_0(s_0, S_1)}{\tilde{f}_0(s_0, S_1)}\frac{f_1(S_1, S_2)}{\tilde{f}_1(S_1, S_2)}\cdots\frac{f_{t-1}(S_{t-1}, S_t)}{\tilde{f}_{t-1}(S_{t-1}, S_t)}.$$

(Define L_t^* similarly for the zero variance measure P^* corresponding to (17)). Note that for any measure $\tilde{P}$ and all stopping times τ,

$$E_{\tilde{P}}[g_\tau \tilde{L}_\tau] \leq E_{\tilde{P}}\left[\max_{t\leq T} g_t \tilde{L}_t\right],$$

where, as usual, $E_{\tilde{P}}$ denotes expectation under the measure $\tilde{P}$. It therefore follows that,

$$J_0(s_0) = \sup_\tau E_{\tilde{P}}[g_\tau \tilde{L}_\tau] \leq E_{\tilde{P}}\left[\max_{t\leq T} g_t \tilde{L}_t\right],$$

for all $\tilde{P}$. Therefore,

$$J_0(s_0) \leq \inf_{\tilde{P}} E_{\tilde{P}}\left[\max_{t\leq T} g_t \tilde{L}_t\right].$$

Furthermore, Bolia *et al* (2004) also prove that when $\tilde{P} = P^*$, the upper bound on the samples is constant and tight, i.e., under P^*,

$$\max_{t \le T} L_t^* g_t(S_t) = J_0(s_0),$$

i.e., $J_0(s_0)$ is a solution to the dual problem

$$\inf_{\tilde{P}} E\left[\max_{t \le T} \tilde{L}_t g_t(S_t)\right].$$

The average of independent samples of

$$\max_{t \le T} g_t \tilde{L}_t,$$

provides an unbiased estimator for an upper bound on $J_0(s_0)$. Thus, the upper estimate of the option price gets better as our estimate of the zero variance measure P^* improves. Using the approximate zero-variance measure as in §7.3, Bolia *et al* (2004) find that the upper bound is within 2–20% of the lower bound.

Also, note that multiplicative duality was developed independently by Jamshidian (2003). However, he did not explore its connection to importance sampling and to the zero variance measure.

11. Derivatives market in India

Equity derivative trading on exchange started in India in June 2000. Prior to this, India's primary market had experience with derivatives in the form of convertible bonds and warrants, a variation of call options (see Shah & Thomas 2000, 2003 for an overview of the evolution of the derivatives market in India). Since these warrants are listed and traded, a very limited options market existed even prior to June 2000. The derivatives market in India took off after the Securities and Exchange Board of India (SEBI) gave permission to Bombay Stock Exchange (BSE) and National Stock Exchange (NSE) to trade index futures in 2000. Trading on Index Options on NSE (Options on S&P CNX Nifty) started on June 4, 2001 and options on individual stocks in the same year on July 2. Finally, the futures trading on individual stocks on NSE started on Nov 9, 2001. Interest rate derivatives in the form of interest rate futures on notional 91 day T-bill, notional 10-year coupon bearing bond and notional 10-year zero coupon bond were launched on June 26, 2003, but the volumes have always been very low, in fact, at a flat level of zero, for most of the days.

The currently active stable of products on the NSE, thus, is made up of index futures, stock futures, index options and stock options (on individual securities). The Nifty (index) options are of two types: European-style vanilla call and put. The options on individual stocks currently have 52 securities (stipulated by the SEBI) as underlyings and have an American-style exercise. The market thus, is pretty nascent in terms of the range of derivatives available for trading. However, India's experience with the launch of equity derivatives markets has been quite positive in terms of the turnover. In the recent times, the volumes of trade have seen an impressive growth to come up to a figure of more than Rs. 9500 crore (daily turnover) on the NSE. The ratio of spot market to derivative market turnover (for NSE) has seen a continuous growth from 0·029% in June 2000 (when the first derivative products were introduced), to having crossed the 100% mark by Feb 2003. Currently, the total volume of derivatives (including options and futures) is almost twice the volumes seen in the spot market.

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