Free asymmetric transverse vibration of polar orthotropic annular sector plate with thickness varying parabolically in radial direction

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Abstract. In the present paper, free asymmetric transverse vibrations of a non-uniform polar orthotropic annular sector plate, with parabolically varying thickness in the radial direction, have been studied on the basis of classical plate theory. The circular edges of the plate are elastically restrained against translation and rotation while the straight (radial) edges are simply-supported. Fourth-order linear differential equations with variable coefficients governing the motion have been solved by using the collocation interpolation technique with Chebyshev points as the interpolating nodes. Frequency parameters of the plate with flexible boundary conditions at the circular edges for some typical values of the taper parameters, the rigidity ratios and the radii ratio have been presented.

Keywords. Asymmetric transverse vibration; polar orthotropic; annular sector plate; parabolically varying thickness; translational and rotational spring parameters; Chebyshev points.

1. Introduction

Plate-like structural elements of sectorial geometry have been widely used in many engineering applications, for example, in aircraft, missiles, naval structures, nuclear reactors etc. Transverse vibrations of uniform isotropic sector plates for symmetrical modes have earlier, been theoretically analysed by Rubin (1957), Ben-Amoz (1959) and Westmann (1962). Later on, Ramakrishnan & Kununnasseril (1973), Ramaiah & Vijaykumar (1974), Bhat-tacharya & Bhowmic (1975) and Wilson & Garg (1978) also studied isotropic sectorial plates with various circular edge conditions. Mizusawa (1991) applied spline elements in the study of annular sector plates. Recently, Liew & Liu (2000) studied flexural vibrations of shear deformable isotropic annular sector plates by a differential quadrature method while Houmat (2001) studied such plates by applying the Fourier p -version of the finite element method.

A list of symbols is given at the end of the paper

After the development of the high strength-to-weight ratio of fibre-reinforced composite materials exhibiting anisotropic characteristics, in the past few decades, much research has been done on the flexural vibration problems of polar orthotropic sectorial plates. To mention a few, Rubin (1975, 1978) studied free vibration and stability of polar orthotropic sector plates by a series solution. Irie *et al* (1979, 1988) treated the free vibrations of ring-shaped polar orthotropic sectorial plate by the Ritz method using a spline function as an admissible function for the deflection of plates. Mukhopadhyay (1979, 1982) also dealt with polar orthotropic sector plates by using a semi analytical method. Narita (1985) studied free vibrations of polar orthotropic sector plates with point support at an arbitrary location by the Lagrange multiplier method as an extension of the Ritz method.

Under these studies, free axisymmetric transverse vibration of isotropic/anisotropic sectorial plates of uniform thickness have been analysed mostly under classical boundary conditions (free, clamped, simply-supported). However, it is an accepted fact that the condition on a periphery often tends to be mid-way between classical boundary conditions and may correspond more closely to some form of elastic restraints. Many researchers have worked on flexible boundary conditions on the curved edges. To mention a few, Kim & Dickinson (1990) have studied flexural vibrations of isotropic and polar orthotropic annular and circular plates with elastically restrained peripheries. Mukhopadhyay (1982) has studied free vibrations of annular sector plates with edges possessing different degrees of rotational restraint.

The development of fibre-reinforced composite materials with high strength-to-weight ratios and exhibiting anisotropic characteristics and their increasing use in modern industries have encouraged researchers to study free vibrations of non-uniform sector plates. A lot of research work has been done on free vibrations of non-uniform annular plates. A few such as Lal & Gupta (1982) have studied axisymmetric vibrations of polar orthotropic annular plates of variable thickness, while Chen (1997) has studied axisymmetric vibrations of circular and annular plates with arbitrarily varying thicknesses. In the case of high speed turbo machinery, it is desirable to study free asymmetric transverse vibrations of such plates.

In an attempt to generalize the study of plate vibration, free asymmetric transverse vibrations of polar orthotropic non-uniform annular sector plates (parabolically varying thickness) along the radial direction have been studied. This has been done for straight (radial) edges when they are simply-supported while the curved edges are elastically restrained against translation and rotation. A simple collocation interpolation technique has been considered for computing the results and the effects of orthotropy, taper parameters and flexibility parameters on the frequency parameters of the plate. In our study on thin plate vibrations (Singh & Jain 2001), we have shown that, instead of equi-spaced nodes, using Chebyshev collocation points as the interpolating nodes gives faster convergence and better results. Hence, in the present research paper also, Chebyshev points have been taken as the interpolating nodes.

2. Analysis

2.1 Equation of motion

Consider a thin annular sector plate of opening angle Φ , inner radius b , outer radius a with parabolically varying thickness $h(r)$ and with its circular edges elastically restrained against translation and rotation. Its straight (radial) edges are simply-supported as shown in figure 1 (centrally thinner case). Let (r, θ) be the polar co-ordinates on any point in the neutral surface referred to the centre of the plate as origin. Without taking the rotary inertia and shear deformation into account, the equation of motion of a harmonically vibrating polar orthotropic

Figure 1. Non-uniform (centrally thinner) annular sector plate.

uniform thickness plate given by Lekhnitskii (1968) has been extended further for the non-uniform thickness case. After taking the transverse deflection $w(r, \theta) = a\Phi G(\xi)H(\eta)$ (after separation of variables and normalization), where $G(\xi)$ and $H(\eta)$ are functions of $\xi (= r/a)$ and $\eta (= \theta/\Phi)$ respectively and $H(\eta) = \sin n\pi\eta$ (Levy's problem), the equation of motion of polar orthotropic sectorial plate with radial edges as simply-supported has been expressed in non-dimensional form as:

$$F(\xi) \frac{d^4 G(\xi)}{d\xi^4} + \left(\frac{F_1}{\xi} \right) \frac{d^3 G(\xi)}{d\xi^3} + \left(\frac{F_2}{\xi^2} \right) \frac{d^2 G(\xi)}{d\xi^2} + \left(\frac{F_3}{\xi^3} \right) \frac{dG(\xi)}{d\xi} + \left(\frac{F_4}{\xi^4} \right) G(\xi) - \lambda^2 f(\xi)G(\xi) = 0, \quad (1)$$

where

$$\begin{aligned} F_1 &= 2 \left\{ \frac{\xi dF(\xi)}{d\xi + F(\xi)} \right\}, \\ F_2 &= \frac{\xi^2 d^2 F(\xi)}{d\xi^2 + (2 + v_\theta)} \frac{\xi dF(\xi)}{d\xi} - \left\{ k^2 + 2 \left(\frac{n\pi}{\Phi} \right)^2 (v_\theta + 2D_{kr}) \right\} F(\xi), \\ F_3 &= \xi^2 v_\theta \frac{d^2 F(\xi)}{d\xi^2 - \xi} \left\{ k^2 + 2 \left(\frac{n\pi}{\Phi} \right)^2 (v_\theta + 2D_{kr}) \right\} \frac{dF(\xi)}{d\xi} \\ &\quad + \left\{ k^2 + 2 \left(\frac{n\pi}{\Phi} \right)^2 (v_\theta + 2D_{kr}) \right\} F(\xi), \\ F_4 &= -\xi^2 v_\theta \left(\frac{n\pi}{\Phi} \right)^2 \frac{d^2 F(\xi)}{d\xi^2} + \xi \left(\frac{n\pi}{\Phi} \right)^2 \{ k^2 + 2(v_\theta + 2D_{kr}) \} \frac{dF(\xi)}{d\xi} \\ &\quad + \left(\frac{n\pi}{\Phi} \right)^2 \left\{ \left(\frac{kn\pi}{\Phi} \right)^2 - 2k^2 - 2(v_\theta + 2D_{kr}) \right\} F(\xi), \end{aligned}$$

where $f(\xi)$ is a thickness variation function. For a parabolically varying thickness (PVT), $f(\xi) = (1 + \alpha\xi^2)$ where α is a taper parameter, $F(\xi) = f^3(\xi)$, $D_{kr} = G_{r\theta}(1 - v_r v_\theta)/E_r$,

where $G_{r\theta}$ is shear modulus; ν_r, ν_θ are Poisson's ratios; E_r, E_θ are Young's modulus in radial and tangential direction, respectively and $k^2 = E_\theta/E_r$. The non-dimensional frequency parameter, $\Omega = \lambda^2 = \rho\omega^2 a^4 h_b/D_{rb}$, where ρ is mass per unit volume, ω is circular frequency, h_b is plate thickness at inner edge and $D_{rb} = E_r h_b^3/12(1 - \nu_r \nu_\theta)$, while n is a positive integer (1, 2, 3...) and $(n - 1)$ gives the number of radial nodal lines.

2.2 Boundary conditions

2.2a *When circular edges are elastically restrained against rotation:* Bending moment per unit length required to cause rotation (Lessia 1969),

$$M_r = -k_r(\partial w(r, \theta)/\partial r), \quad (2)$$

where k_r is rotational stiffness constant.

But from the moment-curvature relation (Lekhnitskii 1968)

$$M_r = -D_r \left[\frac{\partial^2 w(r, \theta)}{\partial r^2} + \nu_\theta \left\{ \left(\frac{1}{r} \right) \frac{\partial w(r, \theta)}{\partial r} + \left(\frac{1}{r^2} \right) \frac{\partial^2 w(r, \theta)}{\partial \theta^2} \right\} \right], \quad (3)$$

where $D_r = E_r h^3/12(1 - \nu_r \nu_\theta)$ and $h = h(r)$ is the local thickness of the plate.

After normalization and further simplifications, (2) and (3) give:

$$F(\varepsilon) \frac{d^2 G(\varepsilon)}{d\xi^2} + \left(\frac{(v_\theta F(\varepsilon) - K_{rb})}{\varepsilon} \right) \frac{dG(\varepsilon)}{d\xi} - \left(n^2 \pi^2 v_\theta \frac{F(\varepsilon)}{\Phi^2} \varepsilon^2 \right) G(\varepsilon) = 0, \quad (4)$$

for inner edge ($\xi = \varepsilon$),

and

$$F(1) \frac{d^2 G(1)}{d\xi^2} + (v_\theta F(1) + K_{ra}) \frac{dG(1)}{d\xi} - \left(\frac{n^2 \pi^2}{\Phi^2} \right) v_\theta F(1) G(1) = 0, \quad (5)$$

for outer edge ($\xi = 1$),

where $K_{rb} = bk_{rb}/D_{rb}$ and $K_{ra} = ak_{ra}/D_{rb}$. K_{rb}, K_{ra} are rotational flexibility parameters at inner and outer edge respectively, while k_{rb}, k_{ra} are rotational stiffness constants at inner and outer edge respectively.

2.2b *When circular edges are elastically restrained against translation:* Shear force per unit length required to deflect translational spring (Lessia 1969),

$$V_r = k_t w(r, \theta). \quad (6)$$

From the Kirchhoff-Kelvin resultant (Lekhnitskii 1968)

$$V_r = N_r + (1/r) \partial M_{r\theta} / \partial \theta, \quad (7)$$

where N_r is transverse shear force per unit length and $M_{r\theta}$ is twisting moment per unit length.

Substituting for N_r and $M_{r\theta}$ from (Lekhnitskii 1968) in (7), the equations (6) and (7) give (after simplifications and normalization):

$$\begin{aligned}
 F(\varepsilon) \frac{d^3 G(\varepsilon)}{d\xi^3} + \left(\frac{\left(\frac{\varepsilon dF(\varepsilon)}{d\xi + F(\varepsilon)} \right)}{\varepsilon} \right) \frac{d^2 G(\varepsilon)}{d\xi^2} \\
 + \left(\frac{\left(v_\theta \varepsilon \frac{dF(\varepsilon)}{d\xi} - \left(k^2 + (v_\theta + 4D_{kr}) \left(\frac{n^2 \pi^2}{\Phi^2} \right) \right) F(\varepsilon) \right)^1}{\varepsilon^2} \right) \frac{dG(\varepsilon)}{d\xi} \\
 + \left(\frac{\left(\frac{-v_\theta \varepsilon dF(\varepsilon)}{d\xi} + (v_\theta + k^2 + 4D_{kr}) \left(\frac{n^2 \pi^2}{\Phi^2} \right)^1 F(\varepsilon) \right)}{\varepsilon^3 + K_{tb}} \right) G(\varepsilon) = 0
 \end{aligned}$$

for inner edge ($\xi = \varepsilon$), (8)

and

$$\begin{aligned}
 F(1) \frac{d^3 G(1)}{d\xi^3} + \left(\frac{dF(1)}{d\xi + F(1)} \right) \frac{d^2 G(1)}{d\xi^2} \\
 + \left(v_\theta \frac{dF(1)}{d\xi} + \left(k^2 + (v_\theta + 4D_{kr}) \left(\frac{n^2 \pi^2}{\Phi^2} \right) \right) \right) F(1) \frac{dG(1)}{d\xi} \\
 + \left(-v_\theta \frac{dF(1)}{d\xi} - (v_\theta + k^2 + 4D_{kr}) \left(\frac{n^2 \pi^2}{\Phi^2} \right) F(1) - K_{ta} \right) G(1) = 0,
 \end{aligned}$$

for outer edge ($\xi = 1$), (9)

where $K_{tb} = b^3 k_{tb} / D_{rb}$ and $K_{ta} = a^3 k_{ta} / D_{rb}$. K_{tb} , K_{ta} are translational flexibility parameters at inner and outer edge respectively, while k_{tb} , k_{ta} are translational stiffness constants at inner and outer edge respectively.

3. Method of solution

An exact solution of equation of motion (1) along with the equations derived for boundary conditions, namely, (4), (5), (8) and (9) does not seem possible due to the presence of variable coefficients. In view of this, an approximate solution has been obtained by applying a collocation technique. For this purpose, the non-dimensional displacement function $G(\xi)$ is approximated as:

$$G(\xi) = \sum_{i=1}^m A_i (1 + \alpha_i \xi + \beta_i \xi^2 + \gamma_i \xi^3 + \delta_i \xi^4) \xi^{a_i}, \quad (10)$$

where $a_i = i - 1$ and A_i , α_i , β_i , γ_i , and δ_i are unknown coefficients. The coefficients α_i , β_i , γ_i , and δ_i have been obtained after substituting for $G(\xi)$ and its derivatives from (10) in the four boundary condition equations (4), (5), (8) and (9). After substituting $G(\xi)$ and its derivatives (with known values of these coefficients) in (1), the required frequency equations have been

obtained. The frequency equations have been solved computationally taking Chebyshev points as the interpolating nodes in the range $(\varepsilon, 1)$ as given by Conte & Boor (1985):

$$\xi_i = ((\varepsilon + 1)/2) + ((\varepsilon - 1)/2) \cos[(2i - 1)/2m]\pi], \quad (11)$$

where $i = 1, \dots, m$ and $\varepsilon < \xi < 1$.

4. Results and discussions

Herein, a computer program has been developed. The frequency parameters of a polar orthotropic annular sector plate of parabolically varying thickness (PVT) when the straight (radial) edges are simply-supported and the curved edges are elastically restrained against translation and rotation have been evaluated for the first three modes of vibration considering a typical combination of plate parameters. Considering Poisson's ratio $\nu_\theta (= 0.3)$ and shear rigidity modulus $D_{kr} (= 0.35)$, all the results have been computed upto four decimal places taking the Chebyshev points as the interpolating nodes with total subintervals m equal to 12.

The results for free transverse vibrations of polar orthotropic PVT annular sector plates are not available in the literature. Hence, to ascertain the accuracy of the present results, the frequency parameters computed for a uniform thickness polar orthotropic annular sector plate with all edges simply-supported have been compared with the results obtained by Rubin (1975), Irie *et al* (1979), Ramaiah (1980) for radii ratio $\varepsilon (= 0.05)$, opening angle $\Phi (= 100$ degree) and Poisson's ratio $\nu_\theta (= 0.3)$ and shown in table 1.

Table 1. Comparison of frequency parameter Ω of polar orthotropic sector plate when radial and circular edges are simply-supported; $\Phi = 100$ degree, $\varepsilon = 0.05$, $\nu_\theta = 0.3$.

P	D_{kr}	Estimates from	Number of nodal diameters and circles (n, s)			
			(1,0)	(2,0)	(3,0)	(1,1)
0.1	0.35	Present	20.1350	35.4957	54.5953	61.4873
		Ramaiah (1980)	{20.148}	{35.503}	{54.614}	{61.968}
			(20.142)	(35.497)	(54.595)	(61.514)
		Irie <i>et al</i> (1979)	20.3	35.9	55.0	61.6
		Rubin (1975)	20	36	55	62
1.0	0.35	Present	23.2171	49.7877	84.6303	66.1827
		Ramaiah (1980)	{23.226}	{49.847}	{84.730}	{66.849}
			(23.220)	(49.788)	(84.631)	(66.203)
		Irie <i>et al</i> (1979)	23.5	49.9	84.6	66.5
		Rubin (1975)	23	50	85	66
10	4.85	Present	56.2754	136.5492	243.7693	129.6886
		Ramaiah (1980)	{56.289}	{136.58}	{244.4}2	{130.21}
			(56.276)	(136.55)	(243.77)	(129.71)
		Irie <i>et al</i> (1979)	56.8	136.7	243.8	130.9
		Rubin (1975)	56	137	244	128

Notes: 1. $n - 1$ = number of radial nodal diameters;

2. s = number of nodal circles;

3. Estimates in Irie *et al* (1979) are obtained by the Rayleigh–Ritz method with quintic spline polynomials as admissible functions, in Rubin (1975) by Forbenius method and in Ramaiah (1980) by the Rayleigh–Ritz method with simple polynomials as admissible functions taking number of terms 5 in parentheses { } and 8 in parentheses ()

Frequency parameters Ω of polar orthotropic PVT annular sector plate with the radial edges simply supported and the circular edges elastically restrained against translation and rotation have been computed for the first three modes of vibration considering two separate sets of plate parameters. In the first set, taking radii ratio $\varepsilon (= 0.5)$, the opening angle $\Phi (= 30^\circ)$ and the taper parameters $\alpha (= \pm 0.3)$, the results have been computed for the rigidity ratios $P (= 0.5, 1.0 \text{ and } 3.0)$ in order to consider the three cases of a stiffened plate – radially, isotropically and circumferentially stiffened respectively. In the second set, taking the radial ratio $\varepsilon (= 0.3)$, the opening angle $\Phi (= 60^\circ)$ and the taper parameters $\alpha (= \pm 0.5)$, the results have been computed only for the rigidity ratios $P (= 0.5)$.

In first set, the four values, i.e. (0, 1, 100 and 10^{15}) for each of the flexibility parameters (K_{tb} , K_{ta} , K_{rb} , and K_{ra}) have been considered. In the second set, the three values, i.e. (0, 1, and 10^{15}) for each of the translational flexibility parameters and the five values, i.e. (0, 1, 10, 100, 1000) for each of the rotational flexibility parameters have been considered. In both the cases, for simplicity, ($K_{tb} = K_{ta}$) and ($K_{rb} = K_{ra}$) have been considered and the numerical value 10^{15} has been assumed to be infinity (**).

During computations, in each set, for each given value of the translational flexibility parameters, all the given values of rotational flexibility parameters have been considered separately. The results computed for the first set have been shown in tables 2–4. The results computed for the second set have been presented as plots in figures 2–4, wherein the frequency

Figure 2. Frequency parameter Ω versus rotational flexibility parameters for a radially stiffened plate (a) when ($K_{rb} = K_{ra} = K_r$) and (b) when $K_{tb} = K_{ta} = 0$ ($P = 0.5$, $\varepsilon = 0.3$, $\Psi = 60^\circ$).

Table 2. Frequency parameter Ω of polar orthotropic radially and parabolically varying thickness sector plate when radial edges are simply-supported and circular edges are restrained elastically against translation and rotation; $\varepsilon = 0.5$, $\Phi = 30^\circ$, first mode.

K_{tb} & K_{ta}	K_{rb} & K_{ra}	$P = 0.5 (n, s)$			$P = 1.0 (n, s)$			$P = 3.0 (n, s)$		
		(1, 0)	(2, 0)	(3, 0)	(1, 0)	(2, 0)	(3, 0)	(1, 0)	(2, 0)	(3, 0)
$\alpha = -0.3$										
0	0	29.3337	101.0490	211.2755	37.7936	133.4186	283.0663	56.2657	210.0011	455.7887
	1	30.5477	102.8819	213.3953	39.3422	135.4062	285.3214	58.2807	212.4908	458.5622
	100	33.5899	111.3641	227.1340	43.5368	144.9691	300.4341	64.4222	225.3073	478.3181
	***	33.6988	111.8644	228.2325	43.6948	145.5513	301.6782	64.6790	226.1312	480.0322
1	0	29.1248	100.9674	211.2266	37.6223	133.3471	283.0235	56.1236	209.9421	455.7536
	1	30.3607	102.8067	213.3491	39.1919	135.3401	285.2809	58.1560	212.4360	458.5289
	100	33.4563	111.3159	227.1043	43.4361	144.9267	300.4078	64.3445	225.2729	478.2968
	***	33.5671	111.8177	228.2039	43.5958	145.5103	301.6530	64.6030	226.0979	480.0119
100	0	115.9543	92.1539	206.2377	59.9252	125.8169	278.6930	37.6899	203.8854	452.2317
	1	8.4171	94.7460	208.6482	15.0149	128.4098	281.1866	42.6996	206.8271	455.1884
	100	11.0353	106.3360	224.1098	31.1356	140.6022	297.7806	55.7781	221.7993	476.1737
	***	12.2306	106.9987	225.3332	31.5964	141.3280	299.1360	56.2665	222.7496	477.9864
***	0	75.8310	180.6149	334.0266	83.9318	220.2940	419.1723	108.7769	317.4534	624.1257
	1	78.9840	182.7024	336.0370	86.8613	222.5295	421.5119	111.3657	320.1716	627.1145
	100	101.1235	201.0846	357.9517	107.5997	242.1355	447.9128	129.5574	345.6539	663.7543
	***	103.0968	203.1002	360.8365	109.4580	244.2104	451.4673	131.1643	348.4510	669.0667
$\alpha = 0.3$										
0	0	42.3896	160.6617	348.6429	57.2423	218.8635	478.7061	90.2032	357.6280	793.4758
	1	42.8239	161.2021	349.2198	57.7896	219.4292	479.2945	90.9257	358.3229	794.1798
	100	48.4790	172.6356	365.6119	64.6108	231.7236	496.4201	100.6296	374.2479	815.5562
	***	49.3240	175.8328	372.3514	65.6447	235.2804	503.6838	102.2270	379.1373	825.1193
1	0	42.3027	160.6354	348.6282	57.1803	218.8415	478.6937	90.1553	357.6105	793.4658
	1	42.7391	161.1763	349.2054	57.7292	219.4076	479.2823	90.8792	358.3056	794.1700
	100	48.4176	172.6183	365.6016	64.5695	231.7090	496.4114	100.6014	374.2363	815.5492
	***	49.2654	175.8176	372.3428	65.6059	235.2677	503.6765	102.2015	379.1273	825.1135
100	0	28.8335	157.9849	347.1746	50.3798	216.6388	477.4602	85.1573	355.8591	792.4803
	1	30.5564	158.5712	347.7676	51.1265	217.2396	478.0615	86.0435	356.5803	793.1941
	100	41.2749	170.8806	364.5904	60.2043	230.2521	495.5435	97.7350	373.0851	814.8569
	***	42.5186	174.2945	371.4913	61.5192	233.9982	502.9467	99.6178	378.1394	824.5396
***	0	110.3404	270.9235	514.7065	122.5846	334.8887	658.0670	161.1270	497.1762	1009.6384
	1	111.8314	271.5523	515.1727	123.9046	335.4826	658.5960	162.1112	497.8486	1010.3130
	100	142.1982	291.5219	533.8854	152.1102	355.3162	680.4216	185.8753	521.8274	1039.7339
	***	151.3720	301.2387	546.6065	160.9365	365.3743	695.7841	193.9854	534.9201	1062.1994

Table 3. Frequency parameter Ω of polar orthotropic radially and parabolically varying thickness sector plate when radial edges are simply-supported and circular edges are elastically restrained against translation and rotation; $\varepsilon = 0.5$, $F = 30^\circ$, second mode.

K_{tb} & K_{ta}	K_{rb} & K_{ra}	$P = 0.5 (n, s)$			$P = 1.0 (n, s)$			$P = 3.0 (n, s)$		
		(1, 2)	(2, 2)	(3, 2)	(1, 2)	(2, 2)	(3, 2)	(1, 2)	(2, 2)	(3, 2)
$\alpha = -0.3$										
0	0	78.3438	219.5960	402.6536	97.8717	266.0546	491.9040	132.5476	367.1829	703.5882
	1	81.0980	221.8815	405.0374	99.8310	268.8397	494.7510	135.7398	370.8485	707.4191
	100	91.4116	234.0347	421.9223	107.3115	283.8066	515.5372	147.2146	392.4252	737.4127
	***	91.9107	234.8295	423.3635	107.6788	284.7814	517.3516	147.7279	393.9372	740.1758
1	0	78.0007	219.5535	402.6392	97.7127	266.0344	491.8920	132.5099	367.1677	703.5787
	1	80.8000	221.8420	405.0237	99.6765	268.8207	494.7395	135.7027	370.8343	707.4100
	100	91.2450	234.0114	421.9142	107.1936	283.7950	515.5307	147.1761	392.4177	737.4078
	***	91.7493	234.8073	423.3558	107.5633	284.7703	517.3455	147.6891	393.9302	740.1715
100	0	257.1012	214.6461	401.2149	126.5835	264.0565	490.7031	128.7338	365.6843	702.6428
	1	126.6470	217.3873	403.6729	69.3760	266.9608	493.6125	131.8223	369.4464	706.5205
	100	69.6973	231.6187	421.1075	92.5146	282.6496	514.8822	142.9050	391.6813	736.9245
	***	71.2816	232.5355	422.5925	93.3972	283.6731	516.7380	143.4026	393.2398	739.7245
***	0	183.1088	319.4701	521.8988	188.3521	360.7725	621.0736	208.2310	477.1539	862.6432
	1	188.2758	322.4957	524.2082	193.3263	363.4325	623.4006	212.5355	479.5832	865.2437
	100	236.3626	358.0473	555.3204	240.1360	395.1013	655.3073	254.7706	509.1313	904.5691
	***	241.7752	363.0998	560.3857	245.4432	399.5965	660.4311	259.6920	513.2269	911.4183
$\alpha = 0.3$										
0	0	105.8689	309.9285	597.2560	131.9379	389.5167	747.6496	186.7532	556.6228	1106.0771
	1	107.7570	310.9679	597.8451	132.8453	390.1416	748.2752	187.6448	557.5399	1106.9336
	100	125.2852	326.7122	614.0824	145.6658	405.6341	767.5868	202.2387	580.7247	1134.6227
	***	127.8878	330.8106	621.0190	147.9814	410.5445	776.2811	205.0300	588.5008	1147.7707
1	0	105.6284	309.8850	597.2511	131.8055	389.5082	747.6462	186.7312	556.6184	1106.0744
	1	107.5399	310.9280	597.8393	132.7204	390.1331	748.2719	187.6219	557.5356	1106.9310
	100	125.1705	326.6927	614.0793	145.5790	405.6274	767.5845	202.2068	580.7221	1134.6209
	***	127.7792	330.7926	621.0159	147.8973	410.5381	776.2792	204.9968	588.4987	1147.7695
100	0	68.5859	304.9074	596.5912	112.5149	388.6428	747.3099	184.2293	556.1884	1105.8159
	1	75.7541	306.4325	597.2055	115.2942	389.2758	747.9387	185.0197	557.1110	1106.6748
	100	112.2620	324.6884	613.7005	135.8844	404.9469	767.3545	198.7447	580.4585	1134.4508
	***	115.7582	328.9568	620.6938	138.6334	409.9017	776.0960	201.4155	588.2920	1147.6419
***	0	260.7057	456.0746	753.5494	267.8601	514.3525	906.7019	295.0429	690.4410	1297.3498
	1	263.4513	457.3869	754.2541	270.5022	515.3965	907.2870	297.3165	691.1040	1297.9492
	100	325.1900	499.0036	783.8511	330.6183	551.2268	934.0335	351.6167	718.7510	1327.7769
	***	348.4480	521.5205	806.3611	353.5679	572.1153	956.1242	373.4471	737.7812	1355.4043

Table 4. Frequency parameter Ω of polar orthotropic radially and parabolically varying thickness sector plate when radial edges are simply-supported and circular edges are elastically restrained against translation and rotation; $\varepsilon = 0.5$, $F = 30^\circ$, third mode.

K_{tb} & K_{ta}	K_{rb} & K_{ra}	$P = 0.5 (n, s)$			$P = 1.0 (n, s)$			$P = 3.0 (n, s)$		
		(1, 1)	(2, 1)	(3, 1)	(1, 1)	(2, 1)	(3, 1)	(1, 1)	(2, 1)	(3, 1)
$\alpha = -0.3$										
0	0	141.8672	313.4680	573.9766	157.3277	395.0329	689.7453	211.4125	527.0686	937.8766
	1	150.6436	319.2671	577.3227	164.4570	397.8356	692.9860	214.5029	530.8819	941.8296
	100	187.4506	351.2300	602.4563	196.2352	416.1724	719.1697	231.6887	557.2840	977.0614
	***	189.4481	353.4293	604.7283	198.0297	417.5786	721.6316	232.8033	559.3714	980.6645
1	0	141.5603	313.3152	573.9472	157.0239	394.9923	689.7376	211.2634	527.0599	937.8710
	1	150.3811	319.1347	577.2966	164.1985	397.7965	692.9788	214.3604	530.8736	941.8243
	100	187.3353	351.1789	602.4450	196.1196	416.1463	719.1656	231.5910	557.2791	977.0581
	***	189.3390	353.3821	604.7182	197.9201	417.5537	721.6278	232.7086	559.3668	980.6637
100	0	267.8780	486.5965	837.6115	277.7830	568.6542	1024.2709	317.5288	787.5105	1438.7894
	1	269.7211	304.2930	574.5259	137.6293	393.4423	692.2568	196.5045	530.0572	941.2902
	100	175.9806	345.8927	601.3280	184.4982	413.4502	718.7393	221.1963	556.7959	976.7764
	***	178.6191	348.5321	603.7086	186.9459	414.9908	721.2313	222.7262	558.9098	980.3936
***	0	348.5886	500.7179	731.7919	351.7713	533.9273	833.3447	364.3972	649.8608	1105.4564
	1	355.0249	504.9913	734.7197	358.1332	537.7554	835.8766	370.4637	652.6184	1107.9313
	100	430.5224	564.5678	781.8899	432.9450	592.2819	877.1721	442.5623	694.0185	1150.5048
	***	440.9745	574.4819	791.1164	443.3246	601.4840	885.1838	452.6559	701.2197	1158.9214
$\alpha = 0.3$										
0	0	199.0685	421.7312	782.4540	215.3880	532.4090	989.2613	278.2072	750.0019	1389.3485
	1	204.4516	426.3621	785.8402	220.0493	534.0227	989.9530	280.1546	750.9271	1390.2545
	100	258.9290	476.6289	826.6404	269.2607	560.8552	1013.0899	310.6296	777.4413	1422.1061
	***	267.9451	486.6316	837.6228	277.8508	568.6791	1024.2731	317.5938	787.5139	1438.7900
1	0	198.8882	421.6194	782.3971	215.2031	532.3520	989.2567	278.0785	749.9986	1389.3470
	1	204.2840	426.2589	785.7935	219.8793	533.9696	989.9487	280.0333	750.9239	1390.2529
	100	258.8543	476.5887	826.6248	269.1854	560.8282	1013.0866	310.5596	777.4379	1422.1050
	***	182.1527	410.1309	777.2907	196.9525	526.2673	988.8062	263.7105	749.6792	1389.1840
100	0	456.4678	295.9619	570.8274	263.0411	390.4035	688.9862	191.7641	526.2078	937.3163
	1	188.3946	415.6277	781.1873	202.9324	528.3167	989.5064	266.6280	750.6006	1390.0911
	100	251.4913	472.5192	825.2713	261.7133	558.0792	1012.7687	303.4227	777.0919	1421.9955
	***	261.2820	483.0619	836.4782	271.0786	566.1409	1023.9826	310.9577	787.1650	1438.7111
***	0	496.2645	710.1623	1036.7315	500.4841	752.9790	1177.0833	517.1508	908.2153	1590.4836
	1	499.7494	712.3002	1037.9578	503.9329	754.8953	1177.9893	520.4577	909.4289	1591.0599
	100	590.5311	780.7075	1087.2415	593.9248	817.8310	1218.0312	607.3667	955.3159	1622.5335
	***	632.5488	821.2272	1125.9937	635.7391	856.5303	1253.0114	648.3900	988.3367	1655.8002

Figure 3. Frequency parameter Ω versus rotational parameters for a radially stiffened plate (a) when ($K_{rb} = K_{ra} = K_r$) and (b) when $K_{tb} = K_{ta} = 1$ ($P = 0.5$, $\varepsilon = 0.3$, $\Psi = 60^\circ$).

parameters versus rotational flexibility parameters curves have been plotted considering a reduced scale for the rotational flexibility parameters by taking their logarithmic values after unity.

From the tables and the curves plotted in the figures, it can be noted that for each given value of the translational flexibility parameters the numerical value of a frequency parameter computed for a centrally thinner plate ($\alpha = 0.3, 0.5$), in most of the cases, is higher than its corresponding value computed for a centrally thicker plate ($\alpha = -0.3, -0.5$) in each mode of vibration of the all considered rigidity ratios except a few results where the trend is reverse.

See, for examples, in tables 2 to 4 when $K_{tb} = K_{ta} = 100$ and

$$K_{rb} = K_{ra} = 0; n = 1, \text{ for } P = 0.5 \text{ and } 1.0, \text{ (first mode)}$$

$$K_{rb} = K_{ra} = 0; n = 1 \text{ for } P = 0.5 \text{ and } 1.0, \text{ (second mode)}$$

$$K_{rb} = K_{ra} = 1; n = 1 \text{ for } P = 0.5, \text{ (second mode)}$$

$$K_{rb} = K_{ra} = 0; n = 1, 2, 3 \text{ for } P = 1.0 \text{ and } 3.0, \text{ (third mode)}$$

$$K_{rb} = K_{ra} = 1; n = 1 \text{ for } P = 0.5, \text{ (third mode).}$$

It should also be noted that for each given value of translational flexibility parameter, a frequency parameter, in most of the cases, increases with the increase of rotational flexibility

Figure 4. Frequency parameter Ω versus rotational flexibility parameters for a radially stiffened plate (a) when ($K_{rb} = K_{ra} = K_r$) and (b) when $K_{tb} = K_{ta} = \infty$ ($P = 0.5$, $\varepsilon = 0.3$, $\Psi = 60^\circ$).

parameters in each mode of vibration both for the centrally thinner as well as for the centrally thicker plates except a few results for $K_{tb} = K_{ta} = 100$ (see tables 2 to 4).

From figures 2–4, it is seen that, when the value of the rotational flexibility parameters is more than 100, the variation in the value of the frequency parameter, in each mode of vibration, seems to be greater for a centrally thinner plate in comparison to that for a centrally thicker plate but it is just the opposite when the value of the rotational flexibility parameters is in between 0 and 1.

5. Conclusions

Comprehensive sets of numerical results of asymmetric vibration of polar orthotropic annular sector plate of parabolically varying thickness have been obtained. The straight (radial) edges are simply-supported while the curved edges are elastically restrained against translation and rotation and the first three modes of vibration have been considered for both centrally thinner and thicker cases of each type of stiffened plate. The results obtained in this note are expected to be useful for the designers and the researchers working in this field.

List of symbols

$G_{r\theta}$	shear modulus;
h_b	plate thickness at inner edge;
P	rigidity ratio ($P = E_\theta/E_r$);
$w(r, \theta)$	transverse deflection;
α	taper parameter;
ξ	non-dimensional radius;
Φ	opening angle of sector plate;
ρ	mass per unit volume;
Ω	non-dimensional frequency parameter, $\Omega = \lambda^2$;
ν_r, ν_θ	Poisson's ratio.

References

- Ben-Amoz M 1959 Note on deflections and flexural vibrations of clamped sectorial plates. *J. Appl. Mech.* 26: 136–137
- Bhattacharya A P, Bhowmic K N 1975 Free vibration of a sectorial plate. *J. Sound Vibr.* 41: 503–505
- Chen D Y 1997 Axisymmetric vibration of circular and annular plates with arbitrarily varying thickness. *J. Sound Vibr.* 206: 114–121
- Conte, Boor 1985 *Elementary numerical analysis (an algorithmic approach)* 3rd edn (New York: McGraw-Hill)
- Houmat A 2001 A sector Fourier p -element applied to free vibration analysis of sectorial plates. *J. Sound Vibr.* 243: 269–282
- Irie T, Yamada G, Ito F 1979 Free vibration of polar orthotropic sector plates. *J. Sound Vibr.* 67: 89–100
- Irie T, Tanaka K, Yamada G 1988 Free vibration of a cantilever annular sector plate with curved radial edges. *J. Sound Vibr.* 122: 69–78
- Kim C S, Dickinson S M 1990 The flexural vibration of the isotropic and polar orthotropic annular and circular plates with elastically restrained peripheries. *J. Sound Vibr.* 143: 171–179
- Lekhnitskii S G 1968 *Anisotropic plates* (New York: Gordon & Breach)
- Lessia A W 1969 Vibration of plates. NASA SP 160
- Lal R, Gupta U S 1982 Axisymmetric vibration of polar orthotropic annular plates of variable thickness. *J. Sound Vibr.* 83: 229–240
- Liew K M, Liu F L 2000 Differential quadrature method for vibration analysis of shear deformable annular sector plates. *J. Sound Vibr.* 230: 335–356
- Mukhopadhyay M 1979 A semi-analytic solution for free vibration of annular sector plates. *J. Sound Vibr.* 63: 87
- Mukhopadhyay M 1982 Free vibration of annular sector plates with edges possessing different degrees of rotational restraint. *J. Sound Vibr.* 80: 275–279
- Mizusawa T 1991 Application of spline element method to analyze vibration of annular sector plates. *J. Sound Vibr.* 149: 461–470
- Narita Y 1985 Free vibration of polar-orthotropic sector plates resting on point supports. *Trans. ASME*
- Ramaiah G K 1980 Flexural vibrations of polar orthotropic sector plates with simply supported straight edges. *J. Sound Vibr.* 70: 589–596
- Ramaiah G K, Vijaykumar K 1974 Natural frequencies of circumferentially truncated sector plates with simply supported straight edges. *J. Sound Vibr.* 34: 53–61
- Ramakrishnan R, Kununnasseril V X 1973 Free vibration of annular sector plates. *J. Sound Vibr.* 30: 127–129

- Rubin C 1957 Nodal circles and natural frequencies for the isotropic wedge. *J. Sound Vibr.* 39: 523–526
- Rubin C 1975 Vibrating modes for simply supported polar orthotropic sector plates. *J. Acoust. Soc. Am.* 58: 841–845
- Rubin C 1978 Stability of polar orthotropic sector plates. *J. Appl. Mech.* 45: 448–450
- Singh R P, Jain S K 2001 Asymmetric vibration of non-uniform circular annular plate by quintic spline collocation technique. *Proc. 2nd Int. Conf. Theoretical, Applied, Computational and Experimental Mechanics*
- Westmann R A 1962 A note on free vibrations of triangular and sector plates. *J. Aerosp. Sci.* 29: 1139–1140
- Wilson J F, Garg D P 1978 Frequencies of annular plate and curved beams elements. *AIAA J.* 16: 270–272