Application of meshless EFG method in fluid flow problems

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Abstract. This paper deals with the solution of two-dimensional fluid flow problems using the meshless element-free Galerkin method. The unknown function of velocity $u(\mathbf{x})$ is approximated by moving least square approximants $u^h(\mathbf{x})$. These approximants are constructed by using a weight function, a monomial basis function and a set of non-constant coefficients. The variational method is used for the development of discrete equations. The Lagrange multiplier technique has been used to enforce the essential boundary conditions. A new exponential weight function has been proposed. The results are obtained for a two-dimensional model problem using different EFG weight functions and compared with the results of finite element and exact methods. The results obtained using proposed weight functions (exponential) are more promising as compared to those obtained using existing weight functions (quartic spline and Gaussian).

Keywords. Meshless method; element-free Galerkin method; steady state analysis; transient analysis; two-dimensional fluid flow.

1. Introduction

The finite element method (FEM) has been established as a very powerful numerical technique for the analysis of space domain problems having arbitrary shapes. It has been observed that in the finite element method, mesh generation is a far more time-consuming and expensive task than the assembly and solution of the finite element equations. Moreover, there are certain classes of problems for which FEM is difficult, or even impossible to apply. For such problems, it has become necessary to find the methods, which may be somewhat more expensive from the viewpoint of computer time but require less time in the preparation of data. Recently, a class of new methods, known as meshless methods, has been developed. Various meshless methods belonging to this family are: smooth particle hydrodynamics (Monaghan 1988, 1992), diffuse element method (Nayroles *et al* 1992), reproducing kernel particle method (Liu *et al* 1995), the method of finite spheres (De & Bathe 2000), free mesh method (Yagawa & Yamada 1996), local boundary integral equation (LBIE) method (Zhu *et al* 1998), the partition of unity

A list of symbols is given at the end of the paper

method (Babuska & Melenk 1997), element-free Galerkin (EFG) method (Belytschko *et al* 1994a, 1996; Lu *et al* 1994; Dolbow & Belytschko 1998; Liu & Gu 2000), meshless local Petrov–Galerkin (MLPG) method (Atluri & Zhu 1998), natural element method (Sukumar *et al* 1998) and natural neighbour Galerkin method (Sukumar *et al* 2001).

Among all these meshless methods, the element-free Galerkin method (EFG) method has been successfully used to solve the problems of plates and shells (Krysl & Belytschko 1996a, 1996b), fracture mechanics and wave propagation (Belytschko *et al* 1994b; Belytschko & Tabbara 1996; Lu *et al* 1995; Sukumar *et al* 1997), NDT (Xuan *et al* 2001), electromagnetic field (Cingoski *et al* 1998) and heat transfer (Singh *et al* 2002, 2003a, 2003b).

In this paper, we have used the meshless element-free Galerkin (EFG) method for solving two-dimensional fluid flow problems and have proposed a new exponential weight function. In the EFG method, function over the solution domain requires only a set of nodes, and does not require element connectivity. Integration over the solution domain requires only a simple integration cell to obtain the solution. Variational technique has been used for the discretization of the governing equations for two-dimensional fluid flow problems. Lagrange multiplier method is used to enforce the essential boundary conditions. FORTRAN programs have been developed for two-dimensional transient and steady state analysis of fluid flow problems. The results are obtained for a model problem and compared with the results of exact (Appel *et al* 1959) and finite element (Reddy 1993) methods.

2. The element-free Galerkin method

The discretization of the governing equations by element-free Galerkin method requires moving least square interpolation functions which are made up of three components: a weight function associated with each node, a monomial basis function and a set of non-constant coefficients. The weight function is non-zero over a small neighbourhood at a particular node, called support of the node.

2.1 Moving least square approximation

Using moving least square approximation, the unknown function $u(y, z)$ is approximated by $u^h(y, z)$ over the two-dimensional domain Ω (Belytschko *et al* 1994a)

$$u^h(y, z) = \sum_{j=1}^m p_j(y, z) a_j(y, z) \equiv \mathbf{p}^T(y, z) \mathbf{a}(y, z) \equiv \mathbf{p}^T(x) \mathbf{a}(\mathbf{x}) \quad (1)$$

where,

$$\begin{aligned} m &= \text{number of terms in the basis,} \\ p_j(y, z) &= \text{monomial basis function,} \\ a_j(y, z) &= \text{non-constant coefficients,} \\ \mathbf{x}^T &= [y \quad z], \mathbf{p}^T(x) = [1 \quad y \quad z]. \end{aligned}$$

The coefficients $a_j(\mathbf{x})$ are found by minimizing the quadratic functional $J(\mathbf{x})$ given by

$$J(\mathbf{x}) = \sum_{l=1}^n w(\mathbf{x} - \mathbf{x}_l) \left\{ \sum_{j=1}^m p_j(\mathbf{x}_l) a_j(\mathbf{x}) - u_l \right\}^2, \quad (2)$$

where $w(\mathbf{x} - \mathbf{x}_I)$ is a weight function which is non-zero over a small domain, called domain of influence. n is the number of nodes in the domain of influence.

The minimization of $J(\mathbf{x})$ w.r.t. $\mathbf{a}(\mathbf{x})$ leads to the following set of equations

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{T}, \quad (3)$$

where $\mathbf{A}$ and $\mathbf{B}$ are given as

$$\begin{aligned} \mathbf{A} &= \sum_{I=1}^n w(\mathbf{x} - \mathbf{x}_I) \mathbf{p}(\mathbf{x}_I) \mathbf{p}^T(\mathbf{x}_I) \\ &= w(\mathbf{x} - \mathbf{x}_1) \begin{bmatrix} 1 & y_1 & z_1 \\ y_1 & y_1^2 & y_1 z_1 \\ z_1 & y_1 z_1 & z_1^2 \end{bmatrix} + \dots w(\mathbf{x} - \mathbf{x}_n) \begin{bmatrix} 1 & y_n & z_n \\ y_n & y_n^2 & y_n z_n \\ z_n & y_n z_n & z_n^2 \end{bmatrix}, \end{aligned} \quad (4)$$

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= [w(\mathbf{x} - \mathbf{x}_1) \mathbf{p}(\mathbf{x}_1), \dots, w(\mathbf{x} - \mathbf{x}_n) \mathbf{p}(\mathbf{x}_n)] \\ &= \left[w(\mathbf{x} - \mathbf{x}_1) \begin{bmatrix} 1 \\ y_1 \\ z_1 \end{bmatrix}, \dots, w(\mathbf{x} - \mathbf{x}_n) \begin{bmatrix} 1 \\ y_n \\ z_n \end{bmatrix} \right], \end{aligned} \quad (5)$$

$$\mathbf{u}^T = [u_1, u_2, \dots, u_n]. \quad (6)$$

By substituting (3) in (1), the MLS approximants can be defined as

$$u^h(\mathbf{x}) = \sum_{I=1}^n \Phi_I(\mathbf{x}) u_I = \Phi(\mathbf{x}) \mathbf{u}, \quad (7)$$

where the shape function $\Phi_I(\mathbf{x})$ is defined by

$$\Phi_I(\mathbf{x}) = \sum_{j=0}^m p_j(\mathbf{x}) (\mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}))_{jI} = \mathbf{p}^T \mathbf{A}^{-1} \mathbf{B}_I. \quad (8)$$

The derivative of the shape function is given as

$$\Phi_{I,\mathbf{x}}(\mathbf{x}) = (\mathbf{p}^T \mathbf{A}^{-1} \mathbf{B}_I)_{,\mathbf{x}} = \mathbf{p}_{,\mathbf{x}}^T \mathbf{A}^{-1} \mathbf{B}_I + \mathbf{p}^T (\mathbf{A}^{-1})_{,\mathbf{x}} \mathbf{B}_I + \mathbf{p}^T \mathbf{A}^{-1} (\mathbf{B}_I)_{,\mathbf{x}}. \quad (9)$$

2.2 Weight function description

The weight function is non-zero over a small neighbourhood of $\mathbf{x}_I$, called the domain of influence of node I . The shape of this domain is arbitrary but in the present analysis rectangular domain is preferred (Belytschko *et al* 1994a). The choice of weight function $w(\mathbf{x} - \mathbf{x}_I)$ affects the resulting approximation $u^h(\mathbf{x}_I)$. In this paper, a new weight function namely exponential weight function has been proposed. The different weight functions used in the present analysis can be written as a function of normalized radius r as the quartic-spline weight function (Belytschko *et al* 1996),

$$w(\mathbf{x} - \mathbf{x}_I) = w(r) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4, & 0 \leq r \leq 1 \\ 0 & r > 1 \end{cases}, \quad (10a)$$

Gaussian weight function (Belytschko *et al* 1995),

$$w(\mathbf{x} - \mathbf{x}_I) = w(r) = \begin{cases} e^{-(2.5r)^2}, & 0 \leq r \leq 1 \\ 0 & r > 1 \end{cases}, \quad (10b)$$

the exponential weight function,

$$w(\mathbf{x} - \mathbf{x}_I) = w(r) = \begin{cases} \frac{1}{100^r}, & 0 \leq r \leq 1 \\ 0 & r > 1 \end{cases}, \quad (10c)$$

where

$$(r_y)_I = \frac{\|y - y_I\|}{d_{myI}} \quad \text{and} \quad (r_z)_I = \frac{\|z - z_I\|}{d_{mzI}},$$

$$d_{myI} = d_{\max} c_{yI} \quad \text{and} \quad d_{mzI} = d_{\max} c_{zI}, \quad d_{\max} = \text{scaling parameter},$$

c_{yI} and c_{zI} at node I , are the distances to the nearest neighbours. d_{myI} and d_{mzI} are chosen such that the matrix is non-singular everywhere in the domain.

The weight function at any given point can be calculated as

$$w(\mathbf{x} - \mathbf{x}_I) = w(r_y)w(r_z) = w_y w_z, \quad (11a)$$

where $w(r_y)$ or $w(r_z)$ can be calculated by replacing r by r_y or r_z in the expression of $w(r)$.

The derivatives of the weight function are obtained as

$$w_{,y} = \frac{dw_y}{dy} w_z, \quad w_{,z} = \frac{dw_z}{dz} w_y. \quad (11b)$$

3. Discretization of the governing equations

The momentum equation for incompressible fluid flowing through the long and uniform duct is given as

$$\mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{dp}{dx} = \rho \frac{\partial u}{\partial t}. \quad (12a)$$

The Initial conditions are

$$\text{at } t = 0, \quad u = u_{ini}, \quad \text{in } \Omega. \quad (12b)$$

The essential boundary conditions are

$$\begin{aligned} \text{at the surface } S_1, \quad u &= u_{S_1} = 0, \\ \text{at the surface } S_2, \quad u &= u_{S_2} = 0, \\ \text{at the surface } S_3, \quad u &= u_{S_3} = 0, \\ \text{at the surface } S_4, \quad u &= u_{S_4} = 0. \end{aligned} \quad (12c)$$

The weak form of (12a) with the boundary conditions is obtained as

$$-\int_{\Omega} [\mu (w_{,y} u_{,y} + w_{,z} u_{,z})] d\Omega + \int_{\Omega} w M d\Omega - \int_{\Omega} \rho w \dot{u} d\Omega = 0. \quad (13)$$

The functional $\Pi(u)$ can be written as

$$\Pi(u) = \int_{\Omega} \frac{\mu}{2} [u_{,y}^2 + u_{,z}^2] d\Omega + \int_{\Omega} \rho \dot{u} u d\Omega - \int_{\Omega} M u d\Omega. \quad (14)$$

Enforcing essential boundary conditions using Lagrange multipliers, the functional $\Pi^*(u)$ is obtained as

$$\begin{aligned} \Pi^*(u) = & \int_{\Omega} \frac{\mu}{2} [u_{,y}^2 + u_{,z}^2] d\Omega + \int_{\Omega} \rho \dot{u} u d\Omega - \int_{\Omega} u M d\Omega + \int_{\Gamma_1} \lambda(u - u_{S_1}) d\Gamma \\ & + \int_{\Gamma_2} \lambda(u - u_{S_2}) d\Gamma + \int_{\Gamma_3} \lambda(u - u_{S_3}) d\Gamma + \int_{\Gamma_4} \lambda(u - u_{S_4}) d\Gamma \end{aligned} \quad (15)$$

Using the variational principle, (15) reduces to

$$\begin{aligned} \delta \Pi^*(u) = & \int_{\Omega} \mu [u_{,y}^T \delta u_{,y} + u_{,z}^T \delta u_{,z}] d\Omega + \int_{\Omega} \rho \dot{u} \delta u d\Omega - \int_{\Omega} M \delta u d\Omega \\ & + \int_{\Gamma_1} \delta \lambda(u - u_{S_1}) d\Gamma + \int_{\Gamma_1} \lambda \delta u d\Gamma + \int_{\Gamma_2} \delta \lambda(u - u_{S_2}) d\Gamma + \int_{\Gamma_2} \lambda \delta u d\Gamma \\ & + \int_{\Gamma_3} \delta \lambda(u - u_{S_3}) d\Gamma + \int_{\Gamma_3} \lambda \delta u d\Gamma + \int_{\Gamma_4} \delta \lambda(u - u_{S_4}) d\Gamma + \int_{\Gamma_4} \lambda \delta u d\Gamma. \end{aligned} \quad (16)$$

Since δu and $\delta \lambda$ are arbitrary in preceding equation, the following relations are obtained using (7),

$$[\mathbf{K}]\{\mathbf{u}\} + [\mathbf{C}]\{\dot{\mathbf{u}}\} + [\mathbf{G}]\{\boldsymbol{\lambda}\} = \{\mathbf{f}\}, \quad (17a)$$

$$[\mathbf{G}^T]\{\dot{\mathbf{u}}\} = \{\mathbf{q}\}, \quad (17b)$$

where

$$K_{IJ} = \int_{\Omega} \begin{bmatrix} \Phi_{I,y} \\ \Phi_{I,z} \end{bmatrix}^T \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} \Phi_{J,y} \\ \Phi_{J,z} \end{bmatrix} d\Omega, \quad (18a)$$

$$C_{IJ} = \int_{\Omega} \Phi_I^T \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix} \Phi_J d\Omega, \quad (18b)$$

$$G_{IK} = \int_{\Gamma_1} \Phi_I N_K d\Gamma_1 + \int_{\Gamma_2} \Phi_I N_K d\Gamma_2 + \int_{\Gamma_3} \Phi_I N_K d\Gamma_3 + \int_{\Gamma_4} \Phi_I N_K d\Gamma_4, \quad (18c)$$

$$f_I = \int_{\Omega} M \Phi_I d\Omega, \quad (18d)$$

$$q_K = \int_{\Gamma_1} u_{S_1} N_K d\Gamma_1 + \int_{\Gamma_2} u_{S_2} N_K d\Gamma_2 + \int_{\Gamma_3} u_{S_3} N_K d\Gamma_3 + \int_{\Gamma_4} u_{S_4} N_K d\Gamma_4. \quad (18e)$$

Figure 1. Cross-section of two-dimensional model for fluid flow problem.

Using Crank–Nicolson method for time approximation, (17) can be written as

$$\left[\begin{array}{c|c} \mathbf{K}^* + \mathbf{C} & \mathbf{G} \\ \hline \mathbf{G}^T & 0 \end{array} \right] \begin{Bmatrix} \mathbf{u}_{n'} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_{n'} \\ \mathbf{q} \end{Bmatrix}, \quad (19)$$

where

$$\mathbf{R}_{n'} = ([\mathbf{C}] - (1 - \alpha)\Delta t[\mathbf{K}])\{\mathbf{u}\}_{n'-1} + \alpha\Delta t\{\mathbf{f}\}_{n'} + (1 - \alpha)\Delta t\{\mathbf{f}\}_{n'-1}, \quad (20a)$$

$$\mathbf{K}^* = \alpha\Delta t[\mathbf{K}]. \quad (20b)$$

4. Numerical results and discussion

The parameters considered for the transient and steady analysis of the model shown in figure 1 are tabulated in table 1. Table 2 shows a comparison of velocity obtained by EFG method for different weight functions with that obtained by finite element method at the location ($y = D/4$, $y = H/4$) of the duct cross-section for 121 nodes. A similar comparison of

Table 1. Data for the 2-D transient and steady state fluid flow.

Parameters	Value of parameters
Depth (D)	0.10 m
Height (H)	0.10 m
Pressure gradient ($-\partial P/\partial x$)	4000.0 (N/m ² /m)
Density of the fluid (ρ)	1000 kg/m ³
Dynamic viscosity (μ)	2.5 Ns/m ²
All surface velocities (u_s)	0.0 m/s
Initial velocity (u_{ini})	0.0 m/s
Time step (Δt)	0.01 s

Table 2. Comparison of transient EFG results with FEM at the location ($y = D/4, z = H/4$) of the duct cross-section.

Time(s)	Velocity (m/s)			
	EFG			FEM
	Quartic spline	Gaussian	Exponential	
0-00	0-0000	0-0000	0-0000	0-0000
0-10	0-3256	0-3235	0-3257	0-2994
0-20	0-4895	0-4855	0-4896	0-4702
0-30	0-5859	0-5805	0-5861	0-5711
0-40	0-6442	0-6375	0-6444	0-6332
0-50	0-6795	0-6719	0-6797	0-6718
0-60	0-7009	0-6925	0-7012	0-6958
0-70	0-7139	0-7050	0-7141	0-7107
0-80	0-7217	0-7125	0-7220	0-7200
0-90	0-7265	0-7170	0-7268	0-7258
1-00	0-7294	0-7197	0-7297	0-7293

velocity has also been shown in figure 2 at the centre ($y = 0, z = 0$) of the duct cross-section for the same number of nodes.

The steady state analysis has been carried out for a model problem by using different EFG weight functions. Table 3 shows a comparison of the results obtained using EFG

Figure 2. Comparison of transient EFG results with FEM at the centre ($y = 0, z = 0$) of the duct cross-section.

Table 3. Comparison of steady state EFG results for different weight functions with FEM and exact methods at the centre ($y = 0, z = 0$) of the duct cross-section.

No. of nodes	Velocity (m/s)				
	EFG			FEM	Exact
	Quartic spline	Gaussian	Exponential		
25	1.2338	1.1753	1.2358	0.9839	1.1802
81	1.1916	1.1776	1.1920	1.1619	1.1802
121	1.1869	1.1780	1.1872	1.1696	1.1802
441	1.1808	1.1786	1.1808	1.1764	1.1802

method with FEM and exact methods. The L_2 -error has been calculated for different EFG weight functions and FEM with exact solution and is given in table 4. The L_2 -error has also been plotted in figure 3 for the calculation of rate of convergence. The rate of convergence for quarticspline, Gaussian, exponential and FEM are obtained as 1.0521, 0.9305, 1.0630 and 1.3709 respectively. From table 4 and figure 3, it is observed that the convergence of exponential weight function is better than those obtained using quartic spline and Gaussian weight functions. It can be also be noted that the accuracy in EFG results is more as compared with FEM results but the convergence of FEM results is better than EFG results.

Table 5 shows the effect of scaling parameter ($d_{\max}$) on EFG results at the location ($y = D/4, z = H/4$) of the duct cross-section. A similar comparison of results has also been shown in figure 4 at the centre ($y = 0, z = 0$) of the duct cross-section. From these results, it is clear that the variation in EFG results with scaling parameter for exponential weight function is less as compared quartic spline and Gaussian weight functions. The scaling parameter ($d_{\max}$) gives acceptable results for the range $1 < d_{\max} \leq 1.5$.

It can be noted from the tables and figures that the EFG results are well converged and in good agreement with the results of finite element and exact methods. From the analysis, it is also clear that the proposed exponential weight function is more promising as compared to quatric spline and Gaussian weight functions. For higher accuracy, large number of nodes

Table 4. L_2 -error in velocity for different weight functions of EFG and FEM.

Elemental area/cell area (m ²)	EFG			
	Quartic spline ($\times 10^{-4}$)	Gaussian ($\times 10^{-4}$)	Exponential ($\times 10^{-4}$)	FEM ($\times 10^{-4}$)
0.000625	30.77780	32.20820	31.95243	214.93360
0.000156	6.56262	2.21061	6.81253	39.13477
0.000100	3.82998	2.07301	3.97280	6.80464
0.000025	1.06004	1.76356	1.06089	2.88473

Figure 3. A plot of L_2 -error for different weight functions of EFG and FEM.

Figure 4. Effect of scaling parameter ($d_{\max}$) on EFG results at the centre ($y = 0, z = 0$) of the duct cross-section.

Table 5. Effect of scaling parameter ($d_{\max}$) on EFG results for different weight functions at the location ($y = D/4, z = H/4$) of the duct cross-section.

Scaling parameter	Velocity (m/s)				
	EFG			FEM	Exact
	Quartic spline	Gaussian	Exponential		
1.01	0.7338	0.7299	0.7341	0.7139	0.7260
1.1	0.7333	0.7334	0.7412	0.7139	0.7260
1.2	0.7369	0.7368	0.7417	0.7139	0.7260
1.3	0.7428	0.7395	0.7420	0.7139	0.7260
1.4	0.7461	0.7432	0.7336	0.7139	0.7260
1.5	0.7499	0.7457	0.7389	0.7139	0.7260
1.6	0.7517	0.7480	0.7402	0.7139	0.7260
1.7	0.7516	0.7502	0.7456	0.7139	0.7260
1.8	0.7520	0.7512	0.7473	0.7139	0.7260
1.9	0.7517	0.7508	0.7489	0.7139	0.7260
2.0	0.7498	0.7485	0.7549	0.7139	0.7260

can be easily generated with least effort. This work can be extended for the three-dimensional analysis of complex fluid flow problems using EFG method.

List of symbols

a	elemental (cell) area;
$a_j(y, z)$	non constant coefficients;
d_{myI} & d_{mzI}	size of the domain of influence at I th node in y and z direction;
$d_{\max}$	scaling parameter;
M	pressure gradient ($-\partial P/\partial x$);
n	number of nodes in the domain of influence;
n'	number of iterations;
$p_j(y, z)$	polynomial basis function;
S	boundary of the domain;
t	time;
$u^h(y, z)$	moving least square approximant;
$u_{,y}, u_{,z}$	$\partial u/\partial y, \partial u/\partial z$;
$u_{,yy}, u_{,zz}$	$\partial^2 u/\partial y^2, \partial^2 u/\partial z^2$;
$w(\mathbf{x} - \mathbf{x}_I)$	weight function;
N_K	Lagrange interpolant;
δ	variational parameter;
λ	Lagrangian multiplier;
μ	dynamic viscosity;
Ω	domain of the problem;
ρ	density of the fluid;
$\Phi(\mathbf{x})$	shape function.

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