Modelling and control of broadband traffic using multiplicative multifractal cascades

P MURALI KRISHNA *, VIKRAM M GADRE and UDAY B DESAI

Signal Processing and Artificial Neural Networks Laboratory, Department of Electrical Engineering, Indian Institute of Technology, Mumbai 400 076, India
e-mail: mkrishna@ee.iitb.ac.in

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Abstract. We present the results on the modelling and synthesis of broadband traffic processes namely ethernet inter-arrival times using the VVGM (variable variance gaussian multiplier) multiplicative multifractal model. This model is shown to be more appropriate for modelling network traffic which possess time varying scaling/self-similarity and burstiness. The model gives a simple and efficient technique to synthesise Ethernet inter-arrival times. The results of the detailed statistical and multifractal analysis performed on the original and the synthesised traces are presented and the performance is compared with other models in the literature, such as the Poisson process, and the Multifractal Wavelet Model (MWM) process. It is also shown empirically that a single server queue preserves the multifractal character of the process by analysing its inter-departure process when fed with the multifractal traces. The result of the existence of a global-scaling exponent for multifractal cascades and its application in queueing theory are discussed. We propose tracking and control algorithms for controlling network congestion with bursty traffic modelled by multifractal cascade processes, characterised by the Holder exponents, the value of which at an interval indicates the burstiness in the traffic at that point. This value has to be estimated and used for the estimation of the congestion and predictive control of the traffic in broadband networks. The estimation can be done by employing wavelet transforms and a Kalman filter based predictor for predicting the burstiness of the traffic.

Keywords. Multifractal; Holder exponent; long-range dependence; bursty traffic; wavelets; Kalman filter.

1. Introduction

Modelling of broadband network traffic has emerged as an active area in the last decade. This has been primarily due to the discovery that broadband network traffic exhibits characteristics

*For correspondence

that cannot be explained in terms of traditional Markovian models. The seminal discovery that broadband network traffic exhibits statistical self-similarity triggered off an immense volume of research in this field. Various monofractal models like fractional Brownian motion have been proposed in the literature to model broadband traffic inter-arrival times (Leland *et al* 1994; Willinger *et al* 1997). In these approaches, it is assumed that the degree of self-similarity given by the Hurst parameter, H , is a constant for the process being considered. However in practical situations, the Hurst parameter varies during a process. Recent studies by Feldman *et al* (1999) along with those by Vehel & Riedi (1997) have shown that network traffic possesses multifractal scaling properties. The time varying burstiness present in the traffic processes is captured by multifractal process models. The parameter that is used to characterise burstiness is the Holder exponent α in a local neighbourhood. The multifractal processes are characterised by a spectrum of Holder exponents termed the multifractal spectrum or the $f(\alpha)$ curve. The analysis of multifractal processes is difficult as traditional signal processing techniques cannot be used to investigate their scaling behaviour. A multifractal model based on wavelets (MWM) was proposed by Riberio *et al* (1999). Other studies have also shown that the statistical properties exhibited by the measured traffic clearly show multifractal nature and new methods have to be evolved for the analysis and control of such processes (Gilbert 2001, Feldman *et al* 1999, Gilbert *et al* 1999). Keeping the above studies in mind, we have evolved a new model, namely, the variable variance gaussian multiplier (VVGM) model to model Ethernet traffic inter-arrival times. The VVGM model is based on the multiplicative multifractal cascades, which have received wide attention in the process modelling area in recent years. We illustrate the usefulness of the model through the comparison of the statistical and performance analysis done with traces generated using this model with the measured Bellcore Ethernet (BCAug89) traces and traces simulated using the MWM model.

The rest of the paper is organised as follows. Section 2 describes the basic theory of multiplicative multifractals and the method of moments used to evaluate the multifractal spectrum of a process. In § 3 we look at the problem of modelling broadband traffic inter-arrival times. We further parameterise the VVGM model in terms of two parameters to get the Parametric VVGM model for modelling the Ethernet inter-arrival times. It is shown that there is a good match between the statistical and multifractal properties of the synthesised traces and those of the original measured traces. In § 4 we present a comparison of the results obtained in queueing simulations for the VVGM, MWM and Poisson models. The implications for queueing analysis of multifractal processes is discussed in § 5. In § 6 we discuss the algorithm for the estimation of the Holder exponent using wavelets. We illustrate the results of the estimation for traffic traces having different levels of burstiness and also discuss the statistical properties of the estimated burst information time series. We present the results of Kalman filter based prediction of the traffic burstiness in § 7 and conclude the paper with a discussion of the results and future work in § 8.

2. Multifractal processes – an introduction

Research in fractal processes have been in progress since the late 1960s since the pioneering work of Mandelbrot & Van Ness (1968) in the field of fractional Brownian motion. A process $X(t)$ can be said to be of fractal or self-similar nature if

$$X(t) \stackrel{d}{=} a^{-H} X(at), \quad (1)$$

Figure 1. Self-similar/monofractal process.

where the equality is in distribution (Mandelbrot & Van Ness 1968). H is called the Hurst parameter which controls the degree of self-similarity of the process which lie in the range ($0 < H < 1$). Figure 1 illustrates the concept of statistical self-similarity.

In most physical situations, the degree or extent of self-similarity will vary with time, i.e. $H = H(t)$ is a function of time. Such a process when observed over a large time interval can be considered to be a “multifractal” in a limiting sense (Vehel & Riedi 1997). In the present work we are investigating a special model for non stationarity in the scaling property by using multiplicative cascades to model multifractal processes. The multifractal model gives more information regarding the distribution of data which a monofractal model cannot provide. As an example consider a set S supporting a measure. Let S be a fractal set so that it is possible to apply the box counting algorithm to estimate the fractal dimension of S . Let $N(\epsilon)$ be the number of boxes of side ϵ needed to cover the set S . Since the set S is a fractal,

$$N(\epsilon) \approx \epsilon^{-D}, \quad (2)$$

where D is the fractal dimension. Here D is a global *geometric* property. It does not provide any information regarding the distribution of ‘measure’ in the original set S . Figures 2a and b illustrate this concept. Both the figures are Cantor sets which have the same fractal dimension. The distribution of points in the set shown is not the same in both. The fractal dimension does not take this into account in analysis. There must be more parameters included rather than the fractional dimension on analysing such distributions.

One of the most popular models for analysing multifractals is the multiplicative cascade model. This was introduced in the context of turbulence analysis (Frisch & Parisi 1985). In this study we use this model to analyse and study network traffic data. A cascade is a process that fragments a given set into smaller and smaller sections according to a geometric rule and at the same time divides the measure of the set according to another (possibly random)

Figure 2. Cantor sets having same fractal dimensions but different distribution of measure.

Figure 3. Multiplicative process construction.

rule (Holley & Waymire 1992). The generator of the cascade specifies the fragmentation rule used. If the cascade preserves the measure of the initial set, it is called a *deterministic cascade* or if it does so in expectation, it is called a *micro canonical cascade* (Riedi & Scheuring 1997). The limiting process generated by the cascade describes the way in which the cascade redistributes the mass of the initial set.

2.1 Cascade construction

Consider the set I_0 denoted by the unit interval $[0, 1]$ such that $\mu(I_0) = 1$. This measure is divided into two sets $I_{0,0}$ from $[0, 1/2]$ and $I_{0,1}$ from $[1/2, 1]$ by allotting $m_0 = r$ to the first part and $m_1 = 1 - r$ to the second part respectively such that the initial measure is preserved. i.e. $\mu(I_{0,0}) + \mu(I_{0,1}) = m_0 + m_1 = 1$. At the next stage, this process is iterated on both the sets $I_{0,0}$ and $I_{0,1}$ to get four sets $I_{0,00}, I_{0,01}, I_{0,10}, I_{0,11}$ such that the measure is again preserved to 1. This process is iterated for k levels and at each stage it can be seen that the total measure is preserved at each stage. The cascade process is indicated in figure 3.

Consider the k^{th} stage of the cascade. The mass is fragmented over the dyadic intervals $[i2^{-k}, (i+1)2^{-k}]$ where $i = 0, \dots, 2^k - 1$. A rational point $x \in [0, 1]$ has the binary expansion $0.\beta_1\beta_2\dots\beta_k$; ($\beta_i = 0, 1$). An arbitrary interval $I_k = I_{0.\beta_1\dots\beta_k}$ of size 2^{-k} consists of points $x \in [0, 1]$ whose binary expansion begins with $0.\beta_1\dots\beta_k$. The measure of this interval equals

$$\mu(I_{0.\beta_1\dots\beta_k}) = \prod_{i=1}^k m_{\beta_i} = m_0^{n_0} m_1^{n_1}, \quad (3)$$

where n_0 is the number of zeros and n_1 is the number of ones in the binary address $0.\beta_1\dots\beta_k$. Note that $n_0 + n_1 = k$. An interesting observation can be made regarding the scaling property of the measure. Consider a point near $x = 0$ in the interval $[0, 1]$. The measure of the dyadic interval at stage k is

$$\mu([0, 2^{-k}]) = m_0^k = (2^{-k})^{v_0}, \quad (4)$$

where $v_0 = -\log_2 m_0$. In other words the measure in the neighbourhood of 0 scales as

$$\mu([0, \epsilon]) \sim \epsilon^\alpha. \quad (5)$$

This means that the density scales as follows

$$(\mu([0, \epsilon])/\epsilon) \sim \epsilon^{\alpha-1}. \quad (6)$$

If $\alpha \neq 1$, density $\rightarrow 0$ or ∞ . This means that the density of the measure will be singular in the limiting sense (Riedi 1999). When the measure of an ϵ neighbourhood of a point scales as a power law in the $\lim_{\epsilon \rightarrow 0}$, the scaling exponent indicates the Holder exponent or the singularity strength. Define

$$\alpha(x_0) = \lim_{\epsilon \rightarrow 0} \frac{\log \mu(B_{x_0}(\epsilon))}{\log \epsilon}, \quad (7)$$

where

$$B_{x_0}(\epsilon) = \{x : |x - x_0| < \epsilon\}, \quad (8)$$

is the set of points around x_0 which are at distance less than ϵ . For the dyadic interval I_k of size 2^{-k} , the above equation gets modified to

$$\begin{aligned} \alpha(0.\beta_1 \dots \beta_k) &= \frac{\log \mu(0.\beta_1 \dots \beta_k)}{\log (2^{-k})} \\ &= \frac{\log [m_0^{n_0} m_1^{n_1}]}{-k \log 2} = -\frac{n_0}{k} \log_2 m_0 - \frac{n_1}{k} \log_2 m_1. \end{aligned} \quad (9)$$

Let $-\log_2 m_0 = v_0$, $-\log_2 m_1 = v_1$ and substituting above we get

$$\alpha(0.\beta_1 \dots \beta_k) = \frac{n_0}{k} v_0 + \frac{n_1}{k} v_1. \quad (10)$$

Let φ_0 be the fraction of zeros in the binary address $0.\beta_1 \dots \beta_k = n_0/k$. Writing (10) in terms of φ_0 , we get

$$\alpha = \varphi_0 v_0 + (1 - \varphi_0) v_1. \quad (11)$$

If we take $v_0 = \alpha_{\min}$ and $v_1 = \alpha_{\max}$, then

$$\alpha = \varphi_0 \alpha_{\min} + (1 - \varphi_0) \alpha_{\max}. \quad (12)$$

It can be seen that α is a function of φ_0 alone.

2.2 Characterising multifractals through $f(\alpha)$ spectrum

In the earlier section, we have obtained the expressions for the Holder exponents of a random interval chosen in the resulting cascade. The cascade processes can be characterised by using the multifractal spectrum curve or the $f(\alpha)$ curve. The $f(\alpha)$ curve gives the fractal dimension of the set of intervals whose Holder exponent is α . The analytical derivation of the expression for the $f(\alpha)$ curve for a binomial cascade process is given next.

Let $N_k(\alpha)$ be the number of intervals of size 2^{-k} with Holder exponent α . We have seen in the earlier section that α is a function of φ_0 which is the fraction of zeros in the binary address $0.\beta_1 \dots \beta_k$. Thus the number of intervals with Holder exponent α is the same as the number of ways to distribute $n_0 = \varphi_0 k$ zeros among k positions.

$$N_k(\alpha) = \binom{k}{\varphi_0 k}. \quad (13)$$

For the sake of simplicity we will denote φ_0 as z . Using the relations

$$\binom{k}{zk} = \frac{k!}{(zk)!(k-zk)!}, \quad k! = (2\pi k)^{1/2} (k/e)^k \quad (14)$$

in the asymptotic sense, we have

$$\begin{aligned} N_k(\alpha) &= \frac{(2\pi k)^{1/2} (k/e)^k}{(2\pi kz)^{1/2} (kz/e)^k [2\pi(k-zk)]^{1/2} [(k-zk)/e]^{k-zk}} \\ &\approx \frac{[z^z(1-z)^{1-z}]^{-k}}{[zk(1-z)]^{1/2}}. \end{aligned} \quad (15)$$

Consider $k \gg 1$ and $0 < z \ll 1$, so that $kz \approx 1$, $(1-z) \approx 1$

$$N_k(\alpha) \approx [z^z(1-z)^{1-z}]^{-k} = [\varphi_0^{\varphi_0}(1-\varphi_0)^{1-\varphi_0}]^{-k}. \quad (16)$$

Let

$$g(z) = -\log_2 [z^z(1-z)^{1-z}], \quad (17)$$

so that

$$N_k(\alpha) \sim [2^{-k}]^{-g(z)}. \quad (18)$$

This has to be expressed as a function of α . We have from (12) that

$$\alpha = z \alpha_{\min} + (1-z) \alpha_{\max}, \quad (19)$$

where $z = \varphi_0$. From this relation we get

$$z = \left(\frac{\alpha_{\max} - \alpha}{\alpha_{\max} - \alpha_{\min}} \right). \quad (20)$$

Substituting the value of z in $g(z)$, we get

$$\begin{aligned} f(\alpha) &= - \left(\frac{\alpha_{\max} - \alpha}{\alpha_{\max} - \alpha_{\min}} \right) \log_2 \left(\frac{\alpha_{\max} - \alpha}{\alpha_{\max} - \alpha_{\min}} \right) - \left(\frac{\alpha - \alpha_{\min}}{\alpha_{\max} - \alpha_{\min}} \right) \\ &\quad \log_2 \left(\frac{\alpha - \alpha_{\min}}{\alpha_{\max} - \alpha_{\min}} \right). \end{aligned} \quad (21)$$

Expanding (21) around $\alpha_0 = (\alpha_{\min} + \alpha_{\max})/2$ using Taylor series expansion for $\log(x)$, we get for $(\alpha - \alpha_0) \ll 1$

$$f(\alpha) = 1 - \frac{2}{\ln 2} \left(\frac{\alpha - \alpha_0}{\alpha_{\max} - \alpha_{\min}} \right)^2. \quad (22)$$

The following figure 4 illustrates a typical $f(\alpha)$ curve where the Holder exponents vary from 0.7 to 1.2. Note that (18) can be generalised as

$$N_k(\alpha) \sim \epsilon^{-f(\alpha)}. \quad (23)$$

We use the method of moments to estimate the $f(\alpha)$ spectrum for the multifractal measure. The details of this method are given in the following section.

2.3 Method of moments

This method uses mathematical principles borrowed from statistical physics and thermodynamics (Beck & Schlogl 1997). We define a partition function for the distributed measure as a result of the cascade process as follows.

$$\chi_q(\epsilon) = \sum_{i=1}^{N(\epsilon)} (\mu_i)^q, \quad (24)$$

where μ_i is the measure of the interval $B_i(\epsilon)$ of size ϵ and q is a real number. This can be compared analogously to the partition function in statistical physics. For the binomial cascade, we have the scaling relationship of the measure as $\mu_i = \epsilon^{\alpha_i}$ by (5). Substituting this in (24), we get

$$\chi_q(\epsilon) = \sum_{i=1}^{N(\epsilon)} (\epsilon^{\alpha_i})^q. \quad (25)$$

Figure 4. Typical multifractal spectrum/ $f(\alpha)$ curve.

The multiplicative cascade process possesses all Holder exponents between $\alpha_{\min}$ and $\alpha_{\max}$. In order to characterise the distribution of the Holder exponents in the cascade, we need to introduce a function $N_\epsilon(\alpha)$. Let $N_\epsilon(\alpha)$ be the number density so that $N_\epsilon(\alpha)d\alpha$ is the number of boxes for which the Holder exponent lies between α and $\alpha + d\alpha$. The contribution of the subset of boxes with Holder exponent α_i lying in the range $\alpha < \alpha_i < \alpha + d\alpha$ to $\chi_q(\epsilon)$ is $N_\epsilon(\alpha_i) (\epsilon^{\alpha_i})^q d\alpha$. Considering the contribution from all α in the limiting sense as $\epsilon \rightarrow 0$, we have

$$\chi_q(\epsilon) = \int N_\epsilon(\alpha) (\epsilon^\alpha)^q d\alpha. \quad (26)$$

Since $N_\epsilon(\alpha) \sim \epsilon^{-f(\alpha)}$, substituting above we get

$$\chi_q(\epsilon) = \int \epsilon^{\alpha q - f(\alpha)} d\alpha. \quad (27)$$

In the $\lim_{\epsilon \rightarrow 0}$, the main contribution to the integral comes from the value of α that minimizes the exponent. Assuming that $f(\alpha)$ is differentiable, the condition for a minimum is

$$\frac{d}{d\alpha} (q\alpha - f(\alpha)) = 0 \quad (28)$$

and for a given value of 'q', the minimum occurs for $\alpha = \alpha(q)$ that satisfies the following conditions

$$\frac{d}{d\alpha} f(\alpha) |_{\alpha=\alpha(q)} = q \quad (29)$$

and

$$\frac{d^2}{d\alpha^2} f(\alpha) |_{\alpha=\alpha(q)} < 0. \quad (30)$$

Keeping only the dominant part of the exponent in the integral and introducing the notation

$$\tau(q) = \min_{\alpha} (q\alpha - f(\alpha)), \quad (31)$$

we have

$$\chi_q(\epsilon) \sim \epsilon^{\tau(q)}. \quad (32)$$

A formal relationship can be obtained between $\tau(q)$ and $f(\alpha)$ as follows

$$\tau(q) = \min_{\alpha} \{q\alpha - f(\alpha)\} \quad \& \quad f(\alpha) = \min_q \{q\alpha - \tau(q)\} \quad (33)$$

The above relation is called the Legendre Transform which plays an important role in several thermodynamical contexts (Bucklew 1990; Beck & Schlögl 1997). Multifractality of a process is associated with irregular behaviour of its scaling property. The pointwise regularity of such processes keeps on changing. An alternative definition of the Holder exponent can be obtained as follows from principles of real analysis (Rudin 1976). The local smoothness or strength of singularity of a function or a process $X(t)$ is characterized by the Holder exponent. Let $X(t)$ be a function, then the real number given by,

$$\alpha(t) = \sup (\beta \geq 0 : |X(t + \Delta t) - X(t)| = O(|\Delta t|^\beta) \text{ as } \Delta t \rightarrow 0), \quad (34)$$

is called the Holder exponent of $X(t)$ at t . In other words

$$|X(t + \Delta t) - X(t)| \sim (\Delta t)^{\alpha(t)}, \quad (35)$$

where $\alpha(t)$ is the order of the polynomial that approximates $X(t + \Delta t) - X(t)$ as $\Delta t \rightarrow 0$.

The Holder exponent of a signal or random process is related to its regularity. The Taylor series expansion relates the differentiability of a signal to its polynomial approximation. Let $X(t)$ be m times differentiable in the interval $[v - h, v + h]$. Let $p_v(t)$ be the Taylor series expansion of $X(t)$ in the neighbourhood of v .

$$p_v(t) = \sum_{k=0}^{m-1} \frac{X^{(k)}(v)}{k!} (t - v)^k \quad (36)$$

The existence of Taylor series formula proves that the approximation error $e_v(t) = X(t) - p_v(t)$ satisfies the following $\forall t \in [v - h, v + h]$,

$$e_v(t) \leq \frac{|t - v|^m}{m!} \sup_{u \in [v-h, v+h]} |X^{(m)}(u)|. \quad (37)$$

A function $X(t)$ is said to be point wise Holder $\alpha \geq 0$ at v if $\exists$ a constant $K > 0$ and a polynomial $p_v(t)$ of degree $m = \lfloor \alpha \rfloor$ such that

$$\forall t \in R, |X(t) - p_v(t)| \leq K |t - v|^\alpha. \quad (38)$$

At each point v , the polynomial is uniquely defined which means that the Holder exponents can vary from point to point. An example of such a process is given in figure 5 which illustrates an inter-arrival trace. Such a process can be characterised by the multifractal spectrum or the $f(\alpha)$ curve we discussed earlier. The $f(\alpha)$ curve gives the fractal dimension of the set of points in the process whose Holder exponent is α . The burstiness of the process can be parametrised based on the nature of the $f(\alpha)$ curve. The region of the curve for $\alpha < 1$ indicates very high burstiness in the data (Riedi 1999).

3. Multifractal modelling of inter-arrival process

The empirical observation of broadband Ethernet traffic has revealed that it is highly correlated in nature (Beran 1994). It is also observed that the degree of burstiness in the traffic is varying in nature. A realistic model for the broadband traffic must also take this factor into account in addition to the long range dependence characteristics. The main factors considered for modelling are: (i) data must always be positive, and (ii) self-similarity varies with time. The first assumption is validated from the very nature of data. The second assumption is now a relaxation from the self-similar models which requires the same Hurst parameter through out the process.

3.1 VVGM multifractal process

In the present study, we are proposing a new approach to model cascade processes in which the multiplier values are samples of a Gaussian distribution $f_{R_i}(r)$ at each stage i . We are modelling the multipliers as normally distributed with mean 0.5 and variances that change

Figure 5. Ethernet interarrival trace.

at each stage of the cascade generation. The name VVGM (variable variance gaussian multiplier) multiplicative multifractals is used to denote the process obtained. The synthesis of multiplicative cascades with multiplier distributions varying at each stage of the cascade had shown to yield better approximations to the statistical properties of the generated process than the case where they are invariant with stage (Murali *et al* 2001). The model is simple to analyse and the synthesis of data based on this model is very fast.

3.1a Estimating model parameters: For developing the model, the main parameters to be estimated are multiplier distribution $f_{R_i}(r)$ at each stage i of the cascade construction. We assume $f_{R_i}(r)$ is centred around $r = 1/2$ so that both r and $(1 - r)$ have the same probability distribution. Let x_i^N , ($i = 1, \dots, 2^N$) denote the process obtained as a result of the cascade construction at stage N . Every point in the sequence x_i^N can be written as the product of several random variables $U_1 U_2 \dots U_N$, where U_j , ($j = 1, \dots, N$) is either r or $(1 - r)$ which is the multiplier at stage j . The algorithm to estimate the parameters of the multiplier distribution is as follows:

1. Given the data at stage N , x_i^N ($i = 1, \dots, 2^N$) (with time resolution of 2^{-N}), the data at stage $N - 1$ is obtained by aggregating the consecutive values at stage N over non overlapping blocks of size two. Similarly, given the data at a coarser scale $N - j$, x_i^{N-j} ($i = 1, \dots, 2^{N-j}$), we obtain the data at stage $N - (j + 1)$ (lesser resolution) by adding consecutive values at stage j over non overlapping blocks of size two, i.e.,

$$x_i^{N-j-1} = x_{2i-1}^{N-j} + x_{2i}^{N-j}, \quad (39)$$

for $i = 1, \dots, 2^{N-j}$.

Figure 6. (a) Histogram of multipliers, (b) variance of multipliers.

2. An estimate of the multipliers for proceeding from stage j to stage $j + 1$ can be obtained as follows:

$$r_i^{(j)} = x_i^{N-j} / x_{2i-1}^{N-j-1}, \quad (40)$$

for $i = 1, \dots, 2^{N-j-1}$.

3. We view $\{r_i^{(j)}, i = 1, \dots, 2^{N-j-1}\}$ as samples of the multiplier distribution $f_{R_j}(r)$ at scale j . Now $f_{R_j}(r)$ can be approximated by the histogram of $r_i(j)$
4. The variance of the multipliers σ_j^2 at each stage j is estimated from the histogram obtained by the previous step. Figure 6a illustrates the multiplier histogram obtained at an intermediate stage in the estimation process.

The process is repeated till the aggregates converge to a single point which can be looked upon as the starting value. The variances at each stage obtained for the traces are plotted in figure 6b.

3.1b Synthesis of VVGM process: For the purpose of synthesis, we start from the coarsest value of the aggregate and multiply it at each stage using multiplier values chosen from gaussian distributions with variances that we estimated in the estimation phase. The algorithm for synthesis of the traces is as follows:

1. Begin with starting value of the aggregate obtained during the estimation phase.
2. Generate random numbers from $N(0.5, \sigma_i^2)$ where σ_i^2 is the variance at stage i .
3. Multiply the starting aggregate value by multipliers generated at each stage from the distributions mentioned earlier to obtain the multiplicative cascade.

3.1c Parameterisation of the model: The variances at each scale obtained for the BCAug89 trace are plotted in figure 6b. The change in variances can be parameterised as:

$$\sigma(i) = \exp(-i0.1285 - 1.3378). \quad (41)$$

The model can thus be parametrised in terms of two parameters a and b to obtain the parametrised VVGM model, in which the variance at each stage is

$$\sigma(i) = \exp(ai + b). \quad (42)$$

Figure 7. VVGM synthesized trace.

The original and parameterised variances are given in figure 6b. The availability of a parametric model is very important for fast synthesis of traces. Moreover it also facilitates the study to test the robustness of the parametric model. Figure 7 illustrates an inter arrival trace synthesised using the VVGM multiplicative multifractal model. It can be verified by observation that the VVGM model captures the burstiness present in the original data. We verify the usefulness of the model through various statistical and performance comparisons in the following sections.

3.2 Testing for robustness

Once the parametric model was available, the robustness of the model performance with change in the variances has to be tested. For this we kept an upper and lower bound of ten percent change in the individual variances. The new variances obtained are plotted in figure 8a. The inter-arrival traces were synthesized using the original parametric model and the two bounds. The multifractal spectrum was evaluated as before. Figure 8b compares the multifractal spectra obtained in each case. It can be seen that the lower bound variation has not affected the spectra much, whereas the upper bound clearly shows a much greater variation in the spectrum in the region $\alpha > 1$. This is particularly significant as the range of values of α in the region (< 1) contribute to the burstiness encountered in the process.

3.3 Statistical comparison tests

In order to validate the model, we have done a series of statistical comparison tests of the simulated traces with the measured data (Bellcore) and the MWM model that has also been shown to possess multifractal properties. We are comparing the autocorrelation function and the higher order statistics of the aggregated data.

Figure 8. (a) Variance perturbation bounds for robustness testing. (b) $f(\alpha)$ robustness of the VVGM model.

3.3a Higher order moments of aggregate data: In practical networks, there is aggregation of data occurring at various nodes like routers, switches etc. Study of the statistics of the aggregated data is important to determine the nature of the data at these points. Define

$$\mu^{(m)}(q) = E \left[\left| X_k^{(m)} \right|^q \right], \quad (43)$$

where

$$X_k^{(m)} = \frac{1}{m} \sum_{i=(k-1)m+1}^{km} X_i^N; \quad (i = 1, \dots, 2^N), \quad (k = 1, \dots, L). \quad (44)$$

$L = 2^N/m$ is the total number of aggregations made for a fixed value of m . The q^{th} moment of the aggregated data was estimated as

$$\hat{\mu}^{(m)}(q) = \frac{1}{L} \sum_{k=1}^L \left| X_k^{(m)} \right|^q. \quad (45)$$

The plot of $\log_{10}(\hat{\mu}^{(m)}(q))$ against $\log_{10} m$ is shown in figures 9a and b for all the traces for $q = 3$ and $q = 4$ respectively. The variation of $\log_{10}(\hat{\mu}^{(m)}(q))$ against $\log_{10} m$ has an increasing nonlinear trend with increasing q . The results indicate the VVGM synthesized data traces closely approximate the higher order statistics of the data.

3.3b Multifractal spectrum: In the development of the multifractal model for the inter-arrival times, a key test is a comparison of the multifractal spectrum or the $f(\alpha)$ curve of the process. We used the method of moments for estimating the multifractal spectrum of the processes. The data $\{X_i\}_{i=1}^{2^N}$ are considered to be sampling on the measure μ on the interval $[0,1]$ at a scale $1/2^N$. The partition function is defined as (Riedi 1999),

$$\chi_m(q) := \sum_{k=1}^{N/m} \left(\overline{X_k^{(m)}} \right)^q, \quad (46)$$

Figure 9. (a) and (b) Comparison of moments of aggregate data $q = 3, 4$.

where

$$\overline{X_k^{(m)}} := \sum_{i=1}^m X_{(k-1)m+i}^m. \quad (47)$$

For multifractal data the partition function exhibits a scaling nature depending on the value of m (Riedi 1999). It is of the form

$$\chi_m(q) \sim m^{\tau(q)}. \quad (48)$$

Taking logarithms we can write the above as

$$\log \chi_m(q) = \tau(q) \log m + K, \quad (49)$$

where K is a constant. If $\log \chi_m^X(q)$ exhibits linearity with $\log m$, it indicates that the data is multifractal in nature. The parameter $\tau(q)$ is the slope of the log - log regression curve. To compute the multifractal spectrum, we obtained the values of $\tau(q)$ for different q by performing a least square fitting between $\log \chi_m(q)$ against $\log m$. The fitting is done for values of $\log m$ where the partition function is linear. The range of values for which linearity holds is obtained by visual inspection of the quality of linear approximation of the graph of $\log \chi_m(q)$ v/s $\log m$. The parameter $\tau(q)$ is an increasing function of q . The multifractal spectrum $f(\alpha)$ of the process $X(t)$ can be obtained by finding the Legendre Transform of $\tau(q)$ defined as (Riedi 1999)

$$f(\alpha) = \min_q \{q\alpha - \tau(q)\}. \quad (50)$$

The original and simulated traces showed multifractal scaling of the partition function for $q \in [-9.5, 10]$ and $m = 2, 4, \dots, 2^{18}$. The partition function for the original measured data and the VVGM synthesised traces are shown in figures 10a and b respectively. Figure 11a gives the $\tau(q)$ plots and figure 11b shows the $f(\alpha)$ plots obtained for the original and synthesised traces. This results have clearly proved that the inter-arrival times in Ethernet do possess multifractal nature and that the parametric VVGM multifractal model can be used as an excellent approximation of the same. The range of values for the Holder exponent α decreases as we progress to more coarser scales. This is in agreement with the expected result as the amount of burstiness present in the data also decreases at coarser scales.

Figure 10. Partition function for (a) Bellcore, and (b) VVGM data.

Figures 12a and b give the variation of the multifractal spectrum across the scales for the Bellcore data and the synthesized VVGM traces. This also shows that a pure fractal (monofractal) process is a very bad approximation of the traffic process as its burstiness and self-similar parameter is invariant with scale. At coarser scales, the $f(\alpha)$ curve becomes more centered in nature and the range of the scaling exponents α also decreases. Both the original and the VVGM simulated traces showed the same behaviour. The match between the two was particularly close in the range ($\alpha \leq 1$), which characterizes high degree of burstiness.

4. Comparison of queueing results

4.1 Queue length distribution

The queueing simulations were done for different values of the utilization factor λ of the buffer. The utilization factor λ is defined as the ratio of the total service time for all the packets to the total time for which the buffer was in use. It is given by

$$\lambda = \sum_n X_n k / \sum_n a_n, \quad (51)$$

Figure 11. (a) $\tau(q)$ comparison, (b) $f(\alpha)$ comparison.

Figure 12. Variation of $f(\alpha)$ over scales (a) Bellcore and (b) VVGM data.

where X_n = length of packet n , k = 1/Bit rate and a_n = inter-arrival time for packet n . In real networks, a buffer utilization ratio of 0.4 - 0.7 is considered optimal. The value of the link rate for a given λ can be found using 11. The queue length distributions for $\lambda = 0.5$ and $\lambda = 0.7$ are shown in figures 13a and b respectively. The Poisson trace was generated so as to have the same first order statistic as the original inter-arrival trace (BCAug89). From the queueing simulations, it is evident that the Poisson arrival traces do not give a good match with the actual arrival process. It can be observed from the queue length distribution plots that the VVGM multifractal process closely approximates the queueing behaviour of the original inter-arrival times.

4.2 Comparison of packet loss

Finite buffer queueing simulations were performed with original packet lengths and exponentially distributed packet lengths. Figures 14a and b show the packet loss as a function of the

Figure 13. Queue length distribution: Original packet length (a) $\lambda = 0.5$, (b) $\lambda = 0.7$.

Figure 14. Comparison of packet loss. (a) Buffer size = 32K. (b) Buffer size = 64K.

buffer utilization factor for buffer sizes of 32kB and 48 kB respectively. The VVGM model approximates the packet loss better than the MWM model, while the Poisson model underestimates it. This is because for a Poisson process the packets do not arrive in bursts of a large number of packets over a small interval of time.

4.3 Comparison of delay

Simulations were done with both the original packets lengths and constant packet lengths. Figures 15a and b show the variation of the mean delay with λ for original and constant packet lengths respectively. Again the Poisson process underestimates the delay, while the VVGM model best approximates it. The Poisson process again underestimates it because the queue never builds up rapidly as the packets do not arrive in bursts.

4.4 Inter-departure process

In order to analyse the effect of a queue on the multifractality of the process, the inter departure process was analysed. This process can be considered the output of the system to which the

Figure 15. Delay comparison (a), original packet length, (b) constant packet length.

Figure 16. Variation of $f(\alpha)$ with output link capacity. (a) Bellcore, (b) VVCM.

input is the inter-arrival process. We found that the inter-departure process also possesses multifractal nature. We conducted the simulations with different output link capacities. The nature of the variation in the range of α was as expected behaviour. It is found that as the link capacity is increased, the range of α also increased. Physically it implies that as the capacity to handle more number of bytes at the output increases, the more bursty can be the output traffic. The $f(\alpha)$ curves for the inter departure process for the original Bellcore traces and the VVGM synthesised traces are as shown in Figures 16a and b. The following table 1 gives the variation of the range of α with the output link capacity. The fact that a FIFO queue maintains the multifractal nature of the incoming process has implications, for the analysis of a network of queues fed with multifractal traffic as input.

5. Applications to queueing

Data traffic models are used to capture and summarize important characteristics of data traffic (Frost & Melamed 1994). The analysis of a storage model based on a self-similar input was done by (Norros 1994). The model used for the analysis was fBm which is a monofractal process characterised by a single global-scaling exponent. The model proposed by Norros models the total traffic arriving up to time t by

$$Z(t) = \lambda t + (a \lambda)^{1/2} X_H(t), \quad (52)$$

Table 1. Variation of α with output link capacity for inter-departure process.

Link cap $\times 10^3$ kbps	Bellcore		VVGM	
	$\alpha_{\min}$	$\alpha_{\max}$	$\alpha_{\min}$	$\alpha_{\max}$
400	0.8085	1.0839	0.7679	1.3042
600	0.8049	1.0974	0.7645	1.3094
800	0.8035	1.1062	0.7630	1.3128
1000	0.8029	0.9572	0.7622	1.3149

where $X_H(t)$ is fBm with Hurst exponent H , a and λ are constants. The success of the self-similar models have been mainly because of their ability to capture the long range dependence while at the same time permitting tractable theoretical analysis. Multifractals on the other hand can characterise time varying burstiness based on the Holder exponents. The results of higher order statistical analysis of the aggregated traffic also have shown that the multifractal processes model performs better than self-similar models (Murali *et al* 2001). The main problem with the multifractal based models are that analytically tractable queueing models are difficult to arrive at. A global-scaling exponent similar to the Hurst parameter for self-similar processes can be shown to exist for multiplicative cascade processes (Murali *et al* 2001). Based on the results obtained, results of the queueing analysis for self-similar models can be generalised.

Classical queueing results with Markov models show that the tail of the distribution of the queue length in a single server queue satisfies

$$Pr[Q > x] \sim e^{-\delta x}, \quad (53)$$

where the constant δ depends on the service rate of the queue and the statistical properties of the arrival process. In the case where the input process is long range dependent with Hurst parameter H , the queue lengths exhibit Weibull tail distributions of the form (Norros 1995)

$$Pr[Q > x] \sim e^{-\delta x^{2-2H}}. \quad (54)$$

The queueing simulations also have indicated that the tail queue probability of the traffic decays at a slower rate than expected by the Markov models. The global scaling exponent for the VVGM multifractal process makes it possible to arrive at analytical expressions similar to the self-similar models for multifractal processes. This is a crude approximation that does not reveal the complex scaling structure that is inherently present in a multifractal process, but it can be used to evaluate closed form expressions for the various QoS requirements like buffer requirements, queue length distributions, service rate etc (Eramilli *et al* 2000; Taralp *et al* 1998). In the next section, we discuss the results of estimation and prediction of the burstiness in traffic inter-arrival times and its application in broadband traffic control.

6. Wavelet based estimation of local Holder exponents

To measure the local regularity of a signal, the vanishing moments of the wavelet used is crucial. If the wavelet has N vanishing moments, then the wavelet transform can be interpreted as a multi scale differential operator of order N (Mallat & Hwang 1992). If a function $X(t)$, is Holder α around the neighbourhood of a point v , then it can be written as

$$X(t) = p_v(t) + e_v(t), \quad (55)$$

where $p_v(t)$ is the Taylor series polynomial and $e_v(t)$ is the error in the approximation. Let the wavelet used $\psi(t)$ have $N > \alpha$ vanishing moments, namely

$$\int_{-\infty}^{\infty} t^k \psi(t) dt = 0, \quad k = 0, 1, \dots, N. \quad (56)$$

An interesting property is that a wavelet with N vanishing moments is orthogonal to all polynomials of degree up to $N - 1$. Since $N > \alpha$, the polynomial $p_v(t)$ has degrees at most

$N - 1$. Let $T_X(a, u)$ denote the wavelet transform of the process $X(t)$ at time instant u and scale a . Due to the vanishing moments property, we have

$$T_{p_v}(a, u) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} p_v(t) \psi\left(\frac{t-u}{a}\right) dt = 0. \quad (57)$$

Since $X(t) = p_v(t) + e_v(t)$ and the wavelet transform is linear, we have

$$T_{X(t)}(a, u) = T_{p_v}(a, u) + T_{e_v}(a, u). \quad (58)$$

By (55) and (58), we can write the above as

$$T_{X(t)}(a, u) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} [X(t) - p_v(t)] \psi\left(\frac{t-u}{a}\right) dt, \quad (59)$$

$$\begin{aligned} |T_{X(t)}(v, a)| &= \left| \frac{1}{\sqrt{a}} \int (X(t) - p_v(t)) \psi\left(\frac{t-v}{a}\right) dt \right| \\ &\leq \frac{1}{\sqrt{a}} \int K |t-v|^\alpha \left| \psi\left(\frac{t-v}{a}\right) \right| dt, \end{aligned} \quad (60)$$

let $x = (t - v)/a$, then

$$\begin{aligned} |T_{X(t)}(u, a)| &\leq K \int |ax|^\alpha \sqrt{a} |\psi(x)| dx \\ &\leq K a^{\alpha+1/2} \int |x|^\alpha |\psi(x)| dx \\ &= K a^{\alpha+1/2}, \end{aligned} \quad (61)$$

with the condition that $|x|^\alpha |\psi(x)|$ is integrable. Taking logarithms on both sides, we get

$$\log(|T_{X(t)}(u, a)|) = (\alpha + 1/2) \log a + \log K. \quad (62)$$

Steps for estimating α : (a) Evaluate the wavelet transform at various scales of the trace over the interval chosen. (b) Find the slope of the regression curve between the magnitude of the wavelet transform and the scale. (c) From the measured slope, estimate the value of the Holder exponent α .

6.1 Results of estimation of traffic burstiness

Figure 17a shows the measured inter-arrival times in a LAN from the BellCore traffic archive. To estimate the local Holder exponent, the inter-arrival traces are divided into blocks of length 1024 points each with an overlap of 256 points. The wavelet transform is then obtained using the filter bank method. The local Holder exponent $\alpha[k]$ (where k denotes the k^{th} interval) for the data set is obtained by using the steps (a)–(c) mentioned earlier. The estimated local Holder exponents $\hat{\alpha}[k]$ for each block is given in Figure 17b. The advantage of having the burstiness parameter $\hat{\alpha}[k]$ is that we can now take some control action on the traffic. For example, the queue buffer size can be modulated based on the estimated $\hat{\alpha}[k]$.

Figure 18a shows the estimated burstiness from two inter-arrival time traces obtained by multiplexing several LAN traces having multifractal character. For illustrative purposes, we

Figure 17. (a) Measured LAN Inter-arrival times, (b) local Holder exponents for the LAN trace.

had generated one inter-arrival trace by multiplexing six LAN inter-arrival time traces that showed multifractal character and another by multiplexing twelve LAN inter-arrival time traces having a similar nature. The traces were synthesised by using the VVGM multifractal model (Murali *et al* 2001). As one might expect, the burstiness present in the inter-arrival time trace obtained by multiplexing twelve LAN traces is seen to be more than in the one obtained by multiplexing six LAN traces. The value of $\hat{\alpha}[k]$ for the former one is mostly less than unity which implies more burstiness.

The measured burstiness time series corresponding to $\hat{\alpha}[k]$ also showed interesting features. Its autocorrelation function indicates that it does not possess long range dependence unlike the inter-arrival traces from which it is derived (Leland *et al* 1994). This indicates the possibility of using classical time series techniques to model the burstiness information in broadband traces. The autocorrelation function of the estimated burst trace is given in figure 18b. The estimated values of the local Holder exponents $\hat{\alpha}[k]$, are then stored for the prediction purpose which is done in the next stage.

Figure 18. (a) Estimated burstiness comparison, (b) autocorrelation of local Holder exponents/burstiness time series.

7. Burstiness prediction using Kalman filter

The estimated local Holder exponents are used to predict the burstiness in the traffic for future time epochs. The Kalman filter provides an efficient recursive algorithm for this prediction problem. It is a very powerful technique in that it can give good predictive performance even when the precise nature of the modeled system is not known (Anderson & Moore 1979). We implemented a system in which a Kalman filter is used to predict the burstiness in various traces at the same time. This kind of situation will be found in network routers where traces coming from various links will be converging to a node. This calls for vector Kalman filter implementation where each of the entries in $\hat{\alpha}[k]$ is taken from a different inter-arrival trace. The level of burstiness present in the inter-arrival traces showed variation from medium to heavy bursty traffic conditions. The basic equations governing the Kalman filter are as follows. The process equation is given by

$$\hat{\alpha}[k] = \Phi_{k,k-1}\hat{\alpha}[k-1] + \Gamma_{k-1}w[k-1], \quad (63)$$

where $\Phi_{k,k-1}$ is the process transition matrix from instant k to $(k-1)$ which is non-singular and Γ_k is the input matrix. w_k is modeled as white gaussian noise with zero mean and covariance $E[w_k w_k^T] = Q_k$. The observation process is given by

$$\hat{\alpha}[k] = H_k \alpha[k] + v[k], \quad (64)$$

where $\alpha[k]$ is the actual value of the local Holder exponent, H_k is a non-random weighting matrix and v_k is modeled as white gaussian noise with zero mean and covariance $E[v_k v_k^T] = R_k$. The processes w_k and v_k are assumed to be orthogonal, i.e. $E[w_k v_k^T] = 0$. The predicted value for the Holder exponent $\hat{\alpha}[k]$ is given by

$$\hat{\alpha}[k] = \hat{\alpha}[k-1] + K_k (\hat{\alpha}[k] - H_k \hat{\alpha}[k-1]), \quad (65)$$

where $\hat{\alpha}[k]$ is the estimated value of the local Holder exponent of the inter-arrival process under consideration. The Kalman gain K_k is given by

$$K_k = P_{k-1} H_k^T (H_k P_{k-1} H_k^T + R_k^{-1}). \quad (66)$$

Figure 19. Original and predicted burstiness. (a) Trace 1, (b) trace 2.

Figure 20. Control strategy based on burstiness estimation.

The experiments conducted with the measured and simulated traces have shown that the Kalman filter based predictor is able to track the change in the measured burstiness of the inter-arrival traces. The results were valid for all the traces where the degree of burstiness varied from one to another. Figures 19a–b show the results of the tracking and prediction of the burstiness using the Kalman filter based algorithm. In the case of the original BellCore data, the variance of the error in the predicted data was measured to be 0.0151 which indicates that the prediction is valid.

Figure 20 shows a typical configuration of a broadband communication network that is controlled by using DSPs which implements the algorithm stated in this paper. The traffic burstiness in the links will be monitored and the value of local Holder exponents will be estimated at key nodes in the network. Using the estimated value of Holder exponents, the traffic burstiness at a future epoch will be predicted using Kalman filter. The proposed method can be coupled with CAC control techniques and adaptive prediction of delay for bandwidth management in broadband networks to ensure system performance guarantees (Qui & Knightly 2001).

8. Conclusions

In this paper, we have proposed a new model for the synthesis of broadband Ethernet inter-arrival times using the VVGM multiplicative multifractal model. Multifractal processes having the property of local scaling characterized by Holder exponents are more suited to represent traffic that shows varying scaling and burstiness nature. These processes are associated with a spectrum of Holder exponents also known as the $f(\alpha)$ or the multifractal spectrum. In this paper we have introduced a new model for traffic inter-arrival time series based on the theory of multiplicative multifractal cascades. The VVGM model introduced in this paper is shown to match the statistical, multifractal and performance parameters of the original measured data very accurately. The algorithm for the synthesis is very efficient computationally. The extensions to the model developed include the following.

- (1) Use the model to analyse WAN traffic and explain the increased burstiness present in WAN traffic compared to the LAN environment
- (2) Try to use the model to explain the emergence of multifractal scaling in traffic environment
- (3) Apply the model to other broadband traffic like variable bit rate (VBR) video traces and if possible in designing strategies in controlling QoS like call admission control.

The field of traffic modelling is fast evolving into an area of intense research wherein insights from other areas like signal processing, statistical physics and non linear systems can provide valuable hints for broadband traffic analysis and control.

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