A generalized biharmonic equation and its applications to hydrodynamic instability

MIHIR B BANERJEE, J R GUPTA and R G SHANDIL

Department of Mathematics, Himachal Pradesh University, Shimla 171 005, India

MS received 17 May 2000

Abstract. Problems concerning characterization of eigenvalues of some linear and homogenous differential systems by the Pellew and Southwell method of conjugate eigenfunctions in the domain of hydrodynamic instability are discussed and a general mathematical framework described.

In this general survey we look back on and rewrite this work almost in exactly the way it evolved out of a few naive looking calculations in hydrodynamic instability. We show in the process the close relationship that exists between mathematical analysis and its applications with due credit to intuition as the main source of mathematical activity.

Keywords. Generalized biharmonic equation; hydrodynamic instability.

1. Introduction

Ever since Osborne Reynolds performed his classic series of experiments on the disintegration of laminar flow to “sinuous” motion in a pipe and heralded an era of investigations concerning hydrodynamic instability and transition to turbulence by proposing the breakdown of stability of the laminar flow as the cause of this disintegration, it became fairly apparent that fundamental understanding of the subject matter of hydrodynamic instability, in the face of an intractable system of nonlinear partial differential equations, must come from an interplay of mathematical theory and experimental observations. Attention must however be primarily focussed on a small number of distinctive prototype instabilities presented by simplest types of flow as far as possible because of many technical difficulties. Subsequent mathematical investigations of these relatively simple prototype instabilities under the plausible hypothesis of linearization revealed that although each one of these instabilities is governed by linear eigenvalue problems which may in some cases be even self-adjoint with the ordinary linear differential equation having constant coefficients, the task of finding their explicit analytical solutions and thereby characterizing the critical conditions at the threshold of instability is not entirely trivial and certainly needs investigation.

In the absence of a clear cut general description of the state of transition from stability to instability through the analytical solution of the problem, the main source of arriving at a mathematical breakthrough is by having a feeling for the right result that may

have been suggested from anywhere or through the application of intuitive reasoning based on experience and observation. As a consequence, the subject matter during the last 120 years of its existence has seen the emergence of several ad hoc and heuristic methods of enquiry. These have been brought into application upon the respective eigenvalue problems describing the various fundamental instabilities by some of the most eminent hydrodynamicists of this century. These methods in turn have led to a rich repertoire of results concerning bounds to eigenvalues and other necessary restrictions in the parameter space of the problem. This has served the purpose of a foundation on which scores of experimental and theoretical investigations stand that would otherwise have been not very easy to retrieve.

Determination of bounds of eigenvalues of linear and homogenous differential systems are problems of mathematical interest in their own right. They assume added significance when these systems represent physical situations and their eigenvalues stand for some characteristic feature of the systems and are not exactly obtainable. The well-known problem of obtaining bounds of eigenvalues of the Helmholtz's equation in two dimensions with appropriate boundary conditions is, in fact, finding limitations on the natural frequency of transverse vibrations of a two-dimensional membrane of constant density that is clamped or otherwise at its edges.

The "method of definite integrals" also called the "method of conjugate eigenfunctions", is a familiar device in the literature on hydrodynamic instability which often succeeds in characterizing the eigenvalues and establishing the bounds, as well as determining other restrictions in the parameter space of the problem which must be necessarily true. The method, which has its roots in the theory of quadratic forms and is responsible for many of the crucially important results in the subject matter, was successfully brought to bear upon the much sought after problem of a resolution of Rayleigh's conjecture concerning the validity of a state of stationary motions at the initiation of Rayleigh-Benard convection (Rayleigh 1916) under general boundary conditions, in a seminal paper by Pellew & Southwell (1940). Since then it has become firmly established and celebrated in the field of hydrodynamic instability as Pellew and Southwell method though it had been used earlier on a few occasions such as when Rayleigh (1880) made use of it to derive his now famous point of inflexion criterion in the instability problem of inviscid incompressible homogenous parallel shear flows. A careful look at the various steps of the derivations leading to the results mentioned above shows that there is a basic similarity of approach in all these cases, namely, "multiplying the governing equation by the conjugate eigenfunction and integrating the resulting equation over the range of the independent variable for a suitable number of times with the help of the boundary conditions." Apart from this each case is treated as a particular problem and the various other steps taken in any two cases often appear ad hoc and unrelated. The question that naturally emerges is the following: Can a unified mathematical treatment be given to the above class of problems wherein the basic content of the method of quadratic forms is retained while its ad hoc nature, in the context of the above mentioned applications, is removed? Banerjee *et al* (1978) say it can and show that a generalized biharmonic equation with matrix coefficients and homogeneous boundary conditions provides the basis for this unified approach for a subclass of the above class of problems such that in any two cases the application differs only in the choice of the matrices involved.

The purpose of the present paper is two-fold, namely to display simultaneously, without loss of rigour and precision, the close relationship that exists between mathematical analysis and its applications, with due credit to intuition as the main source

of mathematical discovery, and to replace the separate discussions of individual cases by more and more general systematic methods, which on account of their generality and penetrating power give rise to a store of new results that might have been lying undiscovered otherwise.

In the following analysis we exhibit in a step-by-step manner the natural emergence of a biharmonic equation with appropriate boundary conditions from mathematical models, each of which consists of the prescription of a double eigenvalue problem, the statement of a consequent result and the demonstration of this result, that are associated with three central problems of hydrodynamic instability.

1.1 Rayleigh–Taylor instability model

Consider the double eigenvalue problem for n described by the following dimensional system of ordinary linear and homogeneous differential equation with linear and homogeneous boundary conditions, namely (Banerjee & Kalthia 1971),

$$n(D^2 - k^2)(D^2 - k^2(\rho_0 n/\mu))w = (gk^2/\mu)(D\rho)w, \quad (1)$$

$$w = 0 = Dw, \text{ at } z = 0 \text{ and } z = d, \quad (2)$$

where the mathematical meanings of the various dimensional symbols used in the above system are as follows: z is the real independent variable such that $0 \leq z \leq d$ and D stands for d/dz ; k^2 , ρ_0 , μ and g are positive constants; $n = n_r + in_i$ is, in general, a complex constant such that n_r and n_i are real constants; $\rho(z)$ with $\rho(0) = \rho_0$, is a positive valued continuously differentiable function of z in $[0, d]$ and $w(z) = w_r(z) + iw_i(z)$ is the dependent variable which is, in general, a complex valued function of z in $[0, d]$, such that $w_r(z)$ and $w_i(z)$ are real valued functions of z in $[0, d]$.

We state the following result:

Result (a). If (n, w) with $n_i \neq 0$ is a solution of the double eigenvalue problem described by (1) and (2), then

$$|n|^2 < \max_{[0, d]} \{(-gD\rho/\rho_0)\} \quad (3)$$

Demonstration. Since n_i is nonzero we multiply (1) throughout by w^*/n (w^* being the complex conjugate of w) and integrate each term of the resulting equation over the range of z by parts a suitable number of times with the help of the boundary conditions (2), to obtain

$$\begin{aligned} \int_0^d [|D^2 w|^2 + 2k^2 |Dw|^2 + k^4 |w|^2] dz + \frac{\rho_0 n}{\mu} \int_0^d [|Dw|^2 + k^2 |w|^2] dz \\ = \frac{gk^2 n^*}{\mu |n|^2} \int_0^d D\rho |w|^2 dz, \end{aligned} \quad (4)$$

where n^* is the complex conjugate of n .

The imaginary part of (4) gives (since n_i is nonzero)

$$\frac{\rho_0}{\mu} \int_0^d |Dw|^2 dz + \frac{k^2}{\mu} \int_0^d [\rho_0 + (gD\rho/|n|^2)] |w|^2 dz = 0, \quad (5)$$

and it follows from (5) that

$$|n|^2 < \max_{[0,d]} \left\{ \frac{-gD\rho}{\rho_0} \right\}, \quad (6)$$

which establishes the desired result.

It may be noted here that (1) and (2) respectively represent the governing equation and boundary conditions of the instability problem of a Rayleigh–Taylor model of a Boussinesq liquid of constant viscosity wherein both the boundaries are rigid and the physical meanings of the various symbols used in these equations are as follows: z is the vertical coordinate and D stands for the differentiation along the vertical direction: $z = 0$ and $z = d$ are respectively the lower and the upper horizontal boundaries; k^2 is the square of the wave number, μ is the coefficient of viscosity and g is the acceleration due to gravity; n is the complex growth rate; $\rho(z)$ is the density whose value at the lower boundary is ρ_0 while $w(z)$ is the vertical component of the velocity.

1.2 Spiral flow instability model

Consider the double eigenvalue problem for σ and T_* described by the following partially nondimensional system of ordinary linear and homogeneous differential equations with linear and homogeneous boundary conditions, namely (Chandrasekhar 1961)

$$[(D^2 - a^2)\{(D^2 - a^2 - i(\sigma + R_*a)\} - 12iR_*a]u = v, \quad (7)$$

$$[D^2 - a^2 - i(\sigma + R_*a)]v = -T_*a^2u, \quad (8)$$

$$u = 0 = Du = v, \text{ at } \varsigma = \pm 1/2, \quad (9)$$

where the mathematical meanings of the various dimensional and nondimensional symbols (u and v are the dimensional ones) used in the above systems are as follows: ς is the real independent variable such that $-1/2 \leq \varsigma \leq +1/2$ and D stands for $d/d\varsigma$; a , R_* and T_* are positive constants while σ is a real constant; $u(\varsigma) = u_r(\varsigma) + iu_i(\varsigma)$ and $v(\varsigma) = v_r(\varsigma) + iv_i(\varsigma)$ are the dependent variables which are, in general, complex valued functions of ς in $[-1/2, +1/2]$ such that $u_r(\varsigma)$, $u_i(\varsigma)$, $v_r(\varsigma)$ and $v_i(\varsigma)$ are real valued functions of ς in $[-1/2, +1/2]$.

We state the following result.

Result (b). If (σ, T_*, u, v) is a solution of the double eigenvalue problem described by (7), (8), (9), then

$$-R_*a < \sigma < \frac{12R_*}{a} - R_*a. \quad (10)$$

Demonstration. We multiply (7) throughout by u^* (the complex conjugate of u) and integrate the resulting equation over the range of ς to obtain,

$$\begin{aligned} \int_{-1/2}^{+1/2} u^*(D^2 - a^2)^2 u d\varsigma - i(\sigma + R_*) \int_{-1/2}^{+1/2} u^*(D^2 - a^2) u d\varsigma \\ - 12iR_*a \int_{-1/2}^{+1/2} u^* u d\varsigma = \int_{-1/2}^{+1/2} u^* v d\varsigma. \end{aligned} \quad (11)$$

Now, we take the complex conjugate of both sides of (8), multiply the equation so obtained by v throughout and integrate the resulting equation over the range of ζ to obtain

$$\int_{-1/2}^{+1/2} u^* v d\zeta = -\frac{1}{T_* a^2} \int_{-1/2}^{+1/2} v (D^2 - a^2) v^* d\zeta - \frac{i(\sigma + R_* a)}{T_* a^2} \int_{-1/2}^{+1/2} v v^* d\zeta. \quad (12)$$

Equations (11) and (12) give

$$\begin{aligned} & \int_{-1/2}^{+1/2} u^* (D^2 - a^2)^2 u d\zeta - i(\sigma + R_* a) \int_{-1/2}^{+1/2} u^* (D^2 - a^2) u d\zeta - 12i R_* a \int_{-1/2}^{+1/2} u^* u d\zeta \\ &= -\frac{1}{2T_* a^2} \int_{-1/2}^{+1/2} v (D^2 - a^2) v^* d\zeta - \frac{i(\sigma + R_* a)}{T_* a^2} \int_{-1/2}^{+1/2} v v^* d\zeta. \end{aligned} \quad (13)$$

Integration of each term of (13) by parts an appropriate number of times with the help of the boundary conditions (9) yields

$$\begin{aligned} & \int_{-1/2}^{+1/2} (|D^2 u|^2 + 2a^2 |Du|^2 + a^4 |u|^2) d\zeta + i(\sigma + R_* a) \int_{-1/2}^{+1/2} (|Du|^2 + a^2 |u|^2) d\zeta \\ & - 12i R_* a \int_{-1/2}^{+1/2} |u|^2 d\zeta = \frac{1}{T_* a^2} \int_{-1/2}^{+1/2} (|Dv|^2 + a^2 |v|^2) d\zeta - \frac{i(\sigma + R_* a)}{T_* a^2} \int_{-1/2}^{+1/2} |v|^2 d\zeta. \end{aligned} \quad (14)$$

The imaginary part of (14) gives

$$\begin{aligned} & (\sigma + R_* a) \int_{-1/2}^{+1/2} |Du|^2 d\zeta + a^2 (\sigma + R_* a - 12R_*/a) \int_{-1/2}^{+1/2} |u|^2 d\zeta \\ & + \frac{i(\sigma + R_* a)}{T_* a^2} \int_{-1/2}^{+1/2} |v|^2 d\zeta = 0, \end{aligned} \quad (15)$$

and it follows from (15) that

$$-R_* a < \sigma < (12R_*/a) - R_* a, \quad (16)$$

which establishes the result.

It may be noted that (7), (8) and (9) respectively represent the governing equations and boundary conditions of the instability problem of a spiral flow model at the marginal state under the narrow gap approximation, with averaged angular and axial velocities. The physical meanings of the various symbols used in the above equations are as follows: ζ is the radial coordinate, in cylindrical polar coordinates of a point in the flow domain consisting of the

annulus between coaxial corotating (with uniform angular velocities) infinitely long circular cylinders with constant pressure gradient acting in the axial direction, D the differentiation along the radial direction, $\varsigma = -1/2$ and $\varsigma = +1/2$ the inner and outer cylindrical boundaries respectively on which slip does not occur, a the wave number, σ the oscillation parameter, R_* is the Reynolds number defined with respect to the mean axial flow and T_* the Taylor number; $u(\varsigma)$ and $v(\varsigma)$ the radial and the azimuthal components respectively of velocity.

1.3 Rayleigh–Benard instability model

Consider the double eigenvalue problem for σ described by the following partially nondimensional system of ordinary linear and homogeneous differential equations with linear and homogeneous boundary conditions, namely (Chandrasekhar 1961),

$$n(D^2 - a^2)(D^2 - a^2 - \sigma)w = (g\alpha d^2/\nu)\Theta, \quad (17)$$

$$(D^2 - a^2 - p\sigma)\Theta = -(\beta d^2/x)w, \quad (18)$$

$$W = 0 = \Theta, \text{ at } z = 0 \text{ and } z = 1, \quad (19)$$

$$\text{and either } Dw = 0, \text{ at } z = 0 \text{ and } z = 1, \quad (20)$$

$$\text{or } D^2w = 0, \text{ at } z = 0 \text{ and } z = 1, \quad (21)$$

$$\text{or } Dw = 0, \text{ at } z = 0 \text{ and } D^2w = 0 \text{ at } z = 1, \quad (22)$$

$$\text{or } D^2w = 0, \text{ at } z = 0 \text{ and } Dw = 0 \text{ at } z = 1, \quad (23)$$

where the mathematical meanings of the various dimensional and nondimensional symbols (z , D , a^2 , p and σ are nondimensional) used in the above system are as follows: z is the real independent variable, such that $0 \leq z \leq 1$ and D stands for d/dz ; a^2 , p , g , α , d^2 , ν , β and x are positive constants; $\sigma = \sigma_r + i\sigma_i$ is, in general, a complex constant such that σ_r and σ_i are real constants; $w(z) = w_r(z) + iw_i(z)$ and $\Theta(z) = \Theta_r(z) + i\Theta_i(z)$ are the dependent variables which are, in general, complex valued functions of z in $[0, 1]$ such that $w_r(z)$, $w_i(z)$, $\Theta_r(z)$ and $\Theta_i(z)$ are real valued functions of z in $[0, 1]$.

We state and prove the following result.

Result (c). If (σ, w, Θ) is a solution of the double eigenvalue problem described by (17)–(23), then $\sigma_i = 0$.

Demonstration. We multiply (17) by w^* (the complex conjugate of w) and integrate the resulting equation over the range of z to obtain

$$\int_0^1 w^*(D^2 - a^2)^2 w dz - \sigma \int_0^1 w^*(D^2 - a^2)w dz = \frac{g\alpha d^2}{\nu} a^2 \int_0^1 w^* \Theta dz. \quad (24)$$

Now, we take the complex conjugate of both sides of (18), multiply the equation so obtained by Θ throughout and integrate the resulting equation over the range of z to obtain,

$$\int_0^1 w^* \Theta dz = -\frac{1}{\beta d^2/x} \int_0^1 \Theta (D^2 - a^2) \Theta^* dz + \frac{p\sigma^*}{\beta d^2/x} \int_0^1 \Theta \Theta^* dz. \quad (25)$$

Equations (24) and (25) give

$$\begin{aligned} & \int_0^1 w^*(D^2 - a^2)^2 w dz - \sigma \int_0^1 w^*(D^2 - a^2) w dz \\ &= -\frac{g\alpha d^2/\nu}{\beta d^2/x} \int_0^1 \Theta(D^2 - a^2) \Theta^* dz + \frac{g\alpha d^2/\nu}{\beta d^2/x} a^2 p \sigma^* \int_0^1 \Theta \Theta^* dz. \end{aligned} \quad (26)$$

Integration of each term of (26) by parts for an appropriate number of times with the help of the boundary conditions (19)–(23) yields

$$\begin{aligned} & \int_0^1 (|D^2 w|^2 + 2a^2 |Dw|^2 + a^4 |w|^2) dz + \sigma \int_0^1 (|Dw|^2 + a^2 |w|^2) dz \\ &= \frac{g\alpha d^2/\nu}{\beta d^2/x} a^2 p \sigma^* \int_0^1 |\Theta|^2 dz + \frac{g\alpha d^2/\nu}{\beta d^2/x} a^2 \int_0^1 (|D\Theta|^2 + a^2 |\Theta|^2) dz. \end{aligned} \quad (27)$$

The imaginary part of (27) gives that

$$\sigma_i \left[\int_0^1 (|Dw|^2 + a^2 |w|^2) dz + \frac{g\alpha d^2/\nu}{\beta d^2/x} a^2 p \int_0^1 |\Theta|^2 dz \right] = 0, \quad (28)$$

and it follows from (28) that

$$\sigma_i = 0, \quad (29)$$

which establishes the desired result.

It may be noted that (17)–(18) and (19)–(23) respectively represent the governing equations and boundary conditions of the instability problem of the Rayleigh–Benard model of a Boussinesq liquid wherein both the boundaries are either rigid or dynamically free or any one of the boundaries is rigid while the other one is dynamically free. The physical meanings of the various symbols used in these equations are as follows: z is the vertical coordinate and D stands for differentiation along the vertical direction; $z = 0$ and $z = 1$ are respectively the lower and the upper horizontal boundaries; a^2 is the square of the wave number; σ is, in general, the complex growth rate while σ_i is the oscillation parameter, g is the acceleration due to gravity, α is the coefficient of volume expansion, d^2 is the square of the depth of the layer, ν is the coefficient of kinematic viscosity, p is the thermal Prandtl number, β is the maintained uniform adverse temperature gradient and x is the coefficient of thermometric conductivity; $w(z)$ is the vertical component of velocity and $\Theta(z)$ is the temperature.

1.4 Veronis's thermohaline instability model wherein diffusion of salt is much slower than the diffusion of heat

Consider the double eigenvalue problem for p described by the following nondimensional system of ordinary linear and homogeneous differential equations with linear and homogeneous

boundary conditions namely

$$p(D^2 - \pi^2\alpha^2 - (p/\sigma))(D^2 - \pi^2\alpha^2)\psi = -R\pi\alpha pT - R_s\pi^2\alpha^2\psi, \quad (30)$$

$$(D^2 - \pi^2\alpha^2 - p)T = \pi\alpha\psi, \quad (31)$$

$$\psi = 0 = T, \text{ at } z = 0 \text{ and } z = 1, \quad (32)$$

$$\text{and either, } D\psi = 0, \text{ at } z = 0 \text{ and } z = 1, \quad (33)$$

$$\text{or } D^2\psi = 0, \text{ at } z = 0 \text{ and } z = 1, \quad (34)$$

$$\text{or } D\psi = 0, \text{ at } z = 0 \text{ and } D^2\psi = 0, \text{ at } z = 1, \quad (35)$$

$$\text{or } D^2\psi = 0, \text{ at } z = 0 \text{ and } D\psi = 0, \text{ at } z = 1, \quad (36)$$

where the various nondimensional symbols are as follows. z is a real independent variable such that $0 \leq z \leq 1$, D stands for d/dz ; α , σ , R and R_s are positive constants; $p = p_r + ip_i$ is, in general, a complex constant such that p_r and p_i are real constants, $\psi(z) = \psi_r(z) + i\psi_i(z)$ and $T(z) = T_r(z) + iT_i(z)$ are independent variables which are complex valued functions of z in $[0, 1]$ such that $\psi_r(z)$, $\psi_i(z)$, $T_r(z)$ and $T_i(z)$ are real valued functions of z in $[0, 1]$.

We state and prove the following result.

Result (d). If (p, ψ, T) with $p_i \neq 0$ is a solution of the double eigenvalue problem described by (30)–(36), then

$$|p|^2 < \sigma R_s. \quad (37)$$

Demonstration. Since p_i is nonzero we multiply (30) throughout by ψ^*/p and integrate the resulting equation over the range of z to obtain

$$\begin{aligned} \int_0^1 \psi^* (D^2 - \pi^2\alpha^2)^2 \psi \, dz - \frac{p}{\sigma} \int_0^1 \psi^* (D^2 - \pi^2\alpha^2) \psi \, dz \\ = -R\pi\alpha \int_0^1 \psi^* T \, dz - \frac{R_s\pi^2\alpha^2 p^*}{|p|^2} \int_0^1 \psi^* \psi \, dz. \end{aligned} \quad (38)$$

Now, we take the complex conjugate of both sides of (31), multiply the equation so obtained by T throughout and integrate the resulting equation over the range of z to obtain

$$\int_0^1 \psi^* T \, dz = \frac{1}{\pi\alpha} \int_0^1 T (D^2 - \pi^2\alpha^2 - p^*) T^* \, dz. \quad (39)$$

Equations (38) and (39) give

$$\begin{aligned} \int_0^1 \psi^* (D^2 - \pi^2\alpha^2)^2 \psi \, dz - \frac{p}{\sigma} \int_0^1 \psi^* (D^2 - \pi^2\alpha^2) \psi \, dz \\ = -R \int_0^1 T (D^2 - \pi^2\alpha^2 - p^*) T^* \, dz - \frac{R_s\pi^2\alpha^2 p^*}{|p|^2} \int_0^1 \psi^* \psi \, dz. \end{aligned} \quad (40)$$

Integration of each term of (40) by parts an appropriate number of times with the help of boundary conditions (32)–(36) yields

$$\begin{aligned} & \int_0^1 (|D^2\psi|^2 + 2\pi^2\alpha^2|D\psi|^2 + \pi^4\alpha^4|\psi|^2)dz + \frac{p}{\sigma} \int_0^1 (|D\psi|^2 + \pi^2\alpha^2|\psi|^2)dz \\ &= R \int_0^1 (|DT|^2 + \pi^2\alpha^2|T|^2 + p^*|T|^2)dz - R_s\pi^2\alpha^2 \frac{p^*}{|p|^2} \int_0^1 |\psi|^2 dz. \end{aligned} \quad (41)$$

The imaginary part of (41) gives (since p_i is nonzero)

$$\frac{1}{\sigma} \int_0^1 |D\psi|^2 dz + \pi^2\alpha^2 \int_0^1 \left(\frac{1}{\sigma} - \frac{R_s}{|p|^2} \right) |\psi|^2 dz = -R \int_0^1 |T|^2 dz. \quad (42)$$

and it follows from (42) that

$$|p|^2 < \sigma R_s, \quad (43)$$

which is the desired result.

It may be noted that (30)–(31) and (32)–(36) respectively represent the governing equations and boundary conditions of the instability problem of Veronis's (1965) thermohaline model of a Boussinesq liquid, wherein the diffusion of salt is much slower than the diffusion of heat, and either both the boundaries are rigid or dynamically free or anyone of the boundaries is rigid while the other one is dynamically free. The physical meanings of the various symbols used in these equations are as follows: z is vertical coordinate and D stands for differentiation along the vertical direction; $z = 0$ and $z = 1$ are respectively the lower and the upper horizontal boundaries; $\pi\alpha$ is the wave number, p is, in general, the complex growth rate while p_i is the oscillation parameter; σ is the thermal Prandtl number; R is the Rayleigh number while R_s is the salinity Rayleigh number; $\psi(z)$ is the stream function and $T(z)$ is the temperature. A detailed look at the mathematical manner in which the above systems are represented and at the various steps of the mathematical method through which results (a)–(d) are demonstrated shows that there is an element of ad hocism and unrelatedness in both these aspects of investigation, namely, those of 'representation' and 'demonstration'. For instance, while for the first system which describes the Rayleigh–Taylor instability model the governing equations and the boundary conditions are written in terms of the single dependent variable w alone by eliminating all other dependent variables from the relevant equations and boundary conditions of the problem, the corresponding systems describing the spiral flow instability model, the Rayleigh–Benard instability model and the Veronis's thermohaline instability model, are written in terms of coupled systems in two dependent variables (by choice) rather than in terms of uncoupled systems in one dependent variable alone. Further, while deriving bounds for the oscillation parameter n_i in the case of the first system the governing equation is divided by the complex growth rate factor n which, of course, is nonzero (similar remarks apply in the case of the last system), no such division is effected in the case of the third system while treating the corresponding problem. Several such systems can be mentioned in this context in hydrodynamic stability wherein important results have been derived through

the application of this method of quadratic forms in which the various steps often appear ad hoc and unrelated (see for example Howard's (1961) semicircle theorem, Kochar & Jain's (1979) semiellipse theorem and Banerjee *et al*'s (1981) semicircle theorem). It is our attempt in the subsequent section to bring the mathematical treatments of such cases under a unified framework wherein the ad hocism and unrelatedness that is associated with the various steps of the individual treatments could be reduced to operations which can be handled in a more systematic mathematical manner.

From the systems of equations and boundary conditions represented by (1)–(2), (7)–(9), (17)–(23), (30)–(36), it follows easily that these systems can be represented in the matrix form as follows: For the system representing the Rayleigh–Taylor instability model, we have

$$\left\{ [1]_{1 \times 1} D^4 + [-(2k^2 + \rho_0 n / \mu)]_{1 \times 1} D^2 + [k^2(k^2 + \rho_0 n / \mu) - \frac{gk^2}{\mu n} (D\rho)]_{1 \times 1} \right\} [w(z)]_{1 \times 1} = 0, \quad (44)$$

$$[w]_{1 \times 1} = [0]_{1 \times 1} = [1]_{1 \times 1} [Dw]_{1 \times 1}, \text{ at } z = 0 \text{ and } z = d, \quad (45)$$

wherein and subsequently we follow the usual notation to denote a matrix.

Similarly, for the systems representing the spiral flow instability model, the Rayleigh–Benard instability model and the Veronis' thermohaline instability model wherein the diffusion of salt is much slower than the diffusion of heat, we respectively have

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2} D^4 + \begin{bmatrix} -2a^2 - i(\sigma + R_* a) & 0 \\ 0 & -1/(T a^2) \end{bmatrix}_{2 \times 2} D^2 + \begin{bmatrix} a^2(a^2 + i(\sigma + R_* a)) - 12i R_* a & -1 \\ -1 & \frac{a^2 + i(\sigma + R_* a)}{T_* a^2} \end{bmatrix}_{2 \times 2} \right\} \begin{Bmatrix} u(\zeta) \\ v(\zeta) \end{Bmatrix}_{2 \times 1} = [0]_{2 \times 1}, \quad (46)$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix}_{2 \times 1} = [0]_{2 \times 1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} Du \\ Dv \end{Bmatrix}_{2 \times 1}, \text{ at } \zeta = -1/2 \text{ and } \zeta = +1/2, \quad (47)$$

and

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2} D^4 + \begin{bmatrix} -2a^2 - \sigma & 0 \\ 0 & -xg\alpha a^2/(\beta v) \end{bmatrix}_{2 \times 2} D^2 + \begin{bmatrix} a^2(a^2 + \sigma) & -g\alpha d^2 a^2/v \\ -g\alpha d^2 a^2/v & xg\alpha a^2(a^2 + p\sigma)/\beta v \end{bmatrix}_{2 \times 2} \right\} \begin{Bmatrix} w(z) \\ \Theta(z) \end{Bmatrix}_{2 \times 1} = [0]_{2 \times 1}, \quad (48)$$

$$\begin{pmatrix} w \\ \Theta \end{pmatrix}_{2 \times 1} = [0]_{2 \times 1}, \text{ at } z = 0 \text{ and } z = 1, \quad (49)$$

either

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{2 \times 2} \begin{pmatrix} Dw \\ D\Theta \end{pmatrix}_{2 \times 1} = [0]_{2 \times 1}, \text{ at } z = 0 \text{ and } z = 1, \quad (50)$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{2 \times 2} \begin{pmatrix} D^2 w \\ D^2 \Theta \end{pmatrix}_{2 \times 1} = [0]_{2 \times 1}, \text{ at } z = 0 \text{ and } z = 1, \quad (51)$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{2 \times 2} \begin{pmatrix} Dw \\ D\Theta \end{pmatrix}_{2 \times 1} = [0]_{2 \times 1}, \text{ at } z = 0 \text{ and } z = 1, \quad (52)$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{2 \times 2} \begin{pmatrix} D^2 w \\ D^2 \Theta \end{pmatrix}_{2 \times 1} = [0]_{2 \times 1}, \text{ at } z = 0 \text{ and } z = 1, \quad (53)$$

and

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{2 \times 2} D^4 + \begin{pmatrix} -2\pi^2 \alpha^2 - \frac{p}{\sigma} & 0 \\ 0 & -R \end{pmatrix}_{2 \times 2} D^2 + \\ \begin{pmatrix} \pi^2 \alpha^2 (\pi^2 \alpha^2 + \frac{p}{\sigma}) + R_s \pi \alpha^2 / p^2 & R \pi \alpha \\ R \pi \alpha & (\pi^2 \alpha^2 + p) \end{pmatrix}_{2 \times 2} \end{pmatrix} \begin{pmatrix} \varphi(z) \\ T(z) \end{pmatrix}_{2 \times 1} = [0]_{2 \times 1} \quad (54)$$

$$\begin{pmatrix} \varphi \\ T \end{pmatrix}_{2 \times 1} = [0]_{2 \times 1}, \text{ at } z = 0 \text{ and } z = 1, \quad (55)$$

either

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{2 \times 2} \begin{pmatrix} D\varphi \\ DT \end{pmatrix}_{2 \times 1} = [0]_{2 \times 1}, \text{ at } z = 0 \text{ and } z = 1, \quad (56)$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{2 \times 2} \begin{pmatrix} D^2 \varphi \\ D^2 T \end{pmatrix}_{2 \times 1} = [0]_{2 \times 1}, \text{ at } z = 0 \text{ and } z = 1, \quad (57)$$

or

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2} \begin{bmatrix} D\varphi \\ DT \end{bmatrix}_{2 \times 1} &= [0]_{2 \times 1}, \text{ at } z = 0 \text{ and} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2} \begin{bmatrix} D^2\varphi \\ D^2T \end{bmatrix}_{2 \times 1} &= [0]_{2 \times 1}, \text{ at } z = 1, \end{aligned} \quad (58)$$

or

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2} \begin{bmatrix} D^2\varphi \\ D^2T \end{bmatrix}_{2 \times 1} &= [0]_{2 \times 1}, \text{ at } z = 0 \text{ and} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2} \begin{bmatrix} D\varphi \\ DT \end{bmatrix}_{2 \times 1} &= [0]_{2 \times 1}, \text{ at } z = 1. \end{aligned} \quad (59)$$

It emerges as a natural consequence that the above systems of equations and boundary conditions are special cases of the more general mathematical system

$$[AD^4 + BD^2 + C(x)]\chi(x) = 0, \quad (60)$$

$$\chi(x) = 0, \text{ at } x = a \text{ and } x = b, \quad (61)$$

$$\text{and either } AD\chi = 0, \text{ at } x = a \text{ and } x = b, \quad (62)$$

$$\text{or } AD^2\chi = 0, \text{ at } x = a \text{ and } x = b, \quad (63)$$

$$\text{or } AD\chi = 0, \text{ at } x = a \text{ and } AD^2\chi = 0 \text{ at } x = b, \quad (64)$$

$$\text{or } AD^2\chi = 0, \text{ at } x = a \text{ and } AD\chi = 0 \text{ at } x = b, \quad (65)$$

where A is an appropriate symmetric $n \times n$ matrix with real constants as entries; while B is an appropriate $n \times n$ matrix with complex constants as entries; $C(x)$ is an appropriate $n \times n$ matrix with complex valued functions on $[a, b]$ as entries; $\chi(x)$ is the column vector $[\chi_i(x)]_{n \times 1}$, $\chi_i(x)$ being an appropriate complex valued function on $[a, b]$; $D^{2k}\chi(x) = [D^{2k}\chi_i(x)]_{n \times 1}$, $k = 1, 2$;

$$A D\chi = \left[\sum_{j=1}^n A_{ij} D\chi_j \right]_{n \times 1} \text{ and } A D^2\chi = \left[\sum_{j=1}^n A_{ij} D^2\chi_j \right]_{n \times 1}$$

where $A = [A_{ij}]_{n \times n}$.

This more general mathematical system described by (60)–(65) in one dimension can further be extended to m dimensions; wherein the point x now refers to the point $(x_1, x_2, \dots, x_m)$ in m dimensions; the open interval (a, b) to the open simply connected subset V in m dimensions; the end points $x = a$ and $x = b$ of (a, b) to the surface S of V ; the operator D^{2k} to the operator

$$\nabla^{2k} = \left\{ \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} \right\}^k, k = 1, 2;$$

the operator D to the operator $\partial/\partial n$, which denotes the differentiation along the positive outward-drawn normal at any point on S and the matrix A to the Hermitian $n \times n$ matrix A with complex constants as entries; such that the four system of equations and boundary conditions mentioned earlier can still be obtained as special cases of this extended system for particular choices of m , A , B and C . Thus, we are led to the considerations of a boundary value problem in m dimensions that consists of a generalized biharmonic equation with matrix coefficients and homogeneous boundary conditions and our aim in this paper is to derive a necessary condition for the existence of a nontrivial solution of the present generalized biharmonic system so that known as well as unknown results could be derived from it by choosing the matrices A , B , and C appropriately.

The work of Banerjee *et al* (1978) is precisely in this direction and they have shown that such a necessary condition could be derived in terms of an integral constraint (that involves the solution $\chi(x)$) which leads to an inequality between the eigenvalues of the matrices involved and has no explicit dependence on the solution $\chi(x)$. We present this work in the following sections.

2. Formulation and mathematical analysis

Consider the coupled system of n linear homogenous partial differential equations

$$[A\nabla^4 + B\nabla^2 + C(x)]\chi(x) = 0, \quad (66)$$

in a simply connected open subset V of the Euclidean space R^m ; x refers to the point $(x_1, x_2, \dots, x_m)$ of V ; A and B are $n \times n$ matrices with entries, that are complex constants A being Hermitian; $C(x)$ is an $n \times n$ matrix with complex valued functions on V as entries; ∇^{2k} stands for the operator $\{\sum_{i=1}^m \partial^2/\partial x_i^2\}^k$, $k = 1, 2$; $\chi(x)$ is the column vector $[\chi_i(x)]_{n \times 1}$, $\chi_i(x)$ being a complex valued function on V and $\nabla^{2k}\chi(x) = [\nabla^{2k}\chi_i(x)]_{n \times 1}$.

We consider (66) together with homogeneous boundary conditions

$$\chi(x) = 0, \text{ and either } A \partial\chi(x)/\partial n = 0 \text{ or } A \nabla^2\chi(x) = 0, \quad (67)$$

on the boundary S of V ; here $A \partial\chi/\partial n$ and $A \nabla^2\chi$ stand for the vectors

$$\left[\sum_{j=1}^n A_{ij} \frac{\partial \chi_j}{\partial n} \right]_{n \times 1} \quad \text{and} \quad \left[\sum_{j=1}^n A_{ij} \nabla^2 \chi_j \right]_{n \times 1} \quad \text{respectively;}$$

$\partial/\partial n$ denotes the differentiation along the positive outward drawn normal at any point on S . Henceforth, we shall use the summation convention for repeated indices unless stated otherwise.

We remark that the governing equations and boundary conditions of the earlier mentioned mathematical models can be obtained as special cases of the present general mathematical model for appropriately chosen m , A , B and C . For example, for the Rayleigh–Taylor instability model, we have

$$m = 1$$

$$A = [1]_{1 \times 1},$$

$$B = \left[-2k^2 - \frac{\rho_0 n}{\mu} \right]_{1 \times 1},$$

$$C = \left[k^2(k^2 + \frac{\rho_0 n}{\mu}) - gk^2 \frac{(D\rho)}{\mu n} \right]_{1 \times 1} \quad (68)$$

while for the spiral flow instability model, the Rayleigh–Benard instability model and the thermohaline instability model, wherein the diffusion of salt is much slower than the diffusion of heat, we have respectively

$$\begin{aligned} m &= 1 \\ A &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}, \\ B &= \begin{bmatrix} -2a^2 - i(\sigma + R_*a) & 0 \\ 0 & -1/(T_*a^2) \end{bmatrix}_{2 \times 2}, \\ C &= \begin{bmatrix} a^2(a^2 + i(\sigma + R_*a)) - 12iR_*a & -1 \\ -1 & a^2 + i(\sigma + R_*a)/(T_*a^2) \end{bmatrix}_{2 \times 2}, \end{aligned} \quad (69)$$

$$\begin{aligned} m &= 1 \\ A &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}, \\ B &= \begin{bmatrix} -2a^2 - \sigma & 0 \\ 0 & -xg\alpha a^2/(\beta\nu) \end{bmatrix}_{2 \times 2}, \\ C &= \begin{bmatrix} a^2(a^2 + \sigma) & -g\alpha d^2 a^2/\nu \\ -g\alpha d^2 a^2/\nu & xg\alpha a^2(a^2 + p\sigma)/\beta\nu \end{bmatrix}_{2 \times 2}, \end{aligned} \quad (70)$$

and

$$\begin{aligned} m &= 1 \\ A &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}, \\ B &= \begin{bmatrix} -2\pi^2\alpha^2 p/\sigma & 0 \\ 0 & -R \end{bmatrix}_{2 \times 2}, \\ C &= \begin{bmatrix} \pi^2\alpha^2(\pi^2\alpha^2 + p/\sigma) + R_s\pi^2\alpha^2/p & R\pi\alpha \\ R\pi\alpha & R(\pi^2\alpha^2 + p) \end{bmatrix}_{2 \times 2} \end{aligned} \quad (71)$$

We prove the following lemma.

Lemma 1. If a solution χ satisfying (66) and (67) exists then we have

$$\int_V (\text{grad} \chi)^\dagger B_1 (\text{grad} \chi) dv = \int_V \chi^\dagger C_1(x) \chi dv, \quad (72)$$

where $B_1 = (B - B^\dagger)/2i$, $C_1 = (C(x) - C^\dagger(x))/2i$, $\text{grad} \chi = [\text{grad} \chi_i]_{n \times 1}$, and the grad operator is in R^m and the symbols $\dagger$ and $*$ stand for complex conjugate transpose and the complex conjugate respectively so that the left hand side of (72), for example, stands for

$$\int_V (\partial \chi_i / \partial x_j)^* (B_1)_{ik} (\partial \chi_k / \partial x_j) dv.$$

Proof. Multiplication of (66) to the left by $\chi^\dagger$ and integration over the domain V gives

$$\int_V \chi^\dagger A \nabla^4 \chi dv + \int_V \chi^\dagger B \nabla^2 \chi dv + \int_V \chi^\dagger C(x) \chi dv = 0. \quad (73)$$

Now

$$\int_V \chi^\dagger A \nabla^4 \chi dv = \int_V \chi_i^* A_{ij} \nabla^4 \chi_j dv = A_{ij} \int_V \chi_i^* \nabla^2 (\nabla^2 \chi)_j dv, \quad (74)$$

and making repeated use of Gauss's theorem in R^m and boundary conditions (67), we get

$$\begin{aligned} A_{ij} \int_V \chi_i^* \nabla^2 (\nabla^2 \chi)_j dv &= A_{ij} \int_S \chi_i^* \frac{\partial}{\partial n} (\nabla^2 \chi)_j ds - \\ &\quad A_{ij} \int_V \text{grad} \chi_i^* \cdot \text{grad} (\nabla^2 \chi)_j dv \\ &= -A_{ij} \int_V \text{grad} \chi_i^* \cdot \text{grad} (\nabla^2 \chi)_j dv \\ &= -A_{ij} \int_S \nabla^2 \chi_j \frac{\partial \chi_i^*}{\partial n} ds + A_{ij} \int_V \nabla^2 \chi_j \nabla^2 \chi_i^* dv \\ &= A_{ij} \int_V \nabla^2 \chi_j \nabla^2 \chi_i^* dv \\ &= \int_V (\nabla^2 \chi)^\dagger A (\nabla^2 \chi) dv, \end{aligned} \quad (75)$$

where the symbol ' $\cdot$ ' is the dot product.

Similarly, we have

$$\int_V \chi^\dagger B \nabla^2 \chi dv = - \int_V (\text{grad} \chi)^\dagger B (\text{grad} \chi) dv. \quad (76)$$

Using (74)–(76) we have from (73)

$$\int_V (\nabla^2 \chi)^\dagger A (\nabla^2 \chi) dv - \int_V (\text{grad} \chi)^\dagger B (\text{grad} \chi) dv + \int_V \chi^\dagger C(x) \chi dv = 0. \quad (77)$$

The imaginary part of (77) gives

$$\int_V (\text{grad} \chi)^\dagger B_1 (\text{grad} \chi) dv = \int_V \chi^\dagger C(x) \chi dv, \quad (78)$$

and this completes the proof of the lemma.

Theorem 1. (A fundamental theorem) Under the hypothesis of lemma 1 if $B_1 = -lB_2$, $C_1(x) = -C_2(x)$, then

$$\text{Sup}\{\text{eigenvalues of } C_2(x)\} > 0, \quad (79)$$

where l is a non-zero real number, B_2 is a positive definite Hermitian matrix, $C_2(x)$ is a Hermitian matrix and ‘Sup’ is taken over all the eigenvalues of $C_2(x)$ over all x in V .

Proof. Since B_2 is positive definite, we have

$$\int_V (\text{grad} \chi)^\dagger B_2 (\text{grad} \chi) dv > 0. \quad (80)$$

Lemma 1 then gives

$$\int_V \chi^\dagger C_2(x) \chi dv > 0. \quad (81)$$

Let $u_1(x), u_2(x), \dots, u_n(x)$ be the n eigenvectors of $C_2(x)$. Here it is to be noted that for each $i = 1, 2, \dots, n$ and x in V , $u_i(x)$ is an $n \times 1$ matrix. Further, let $\lambda_i(x)$ be the eigenvalue of $C_2(x)$ corresponding to $u_i(x)$ so that $C_2(x)u_i(x) = \lambda_i(x)u_i(x)$ (no summation implied). The matrix $C_2(x)$ being Hermitian in a finite dimensional space, the spectral theorem gives that these form an orthonormal basis to C^n for each x in V i.e., $u_i^\dagger(x)u_j(x) = \delta_{ij}$. We thus have an expression

$$\chi(x) = \sum_{i=1}^n f_i(x)u_i(x), \quad (82)$$

where $f_i(x)$ ’s are complex valued functions on V .

Substitution of $\chi_i(x)$ from (82) in the left hand expression of inequality (81) yields

$$\begin{aligned}
 \int_V \chi^\dagger C_2(x) \chi \, dv &= \int_V \left[\sum_{i=1}^n f_i(x) u_i(x) \right]^\dagger C_2(x) \left[\sum_{i=1}^n f_i(x) u_i(x) \right] \, dv \\
 &= \int_V \left[\sum_{i=1}^n f_i^*(x) u_i^\dagger(x) \right] \left[\sum_{i=1}^n \lambda_i(x) f_i(x) u_i(x) \right] \, dv \\
 &= \int_V \left[\sum_{i=1}^n |f_i(x)|^2 \cdot 1 \cdot \lambda_i(x) \right] \, dv \\
 &\leq \left[\sup_{1 \leq i \leq n} \sup_{x \in V} \{\lambda_i(x)\} \right] \int_V \left[\sum_{i=1}^n |f_i(x)|^2 \right] \, dv \\
 &\leq \left[\sup_{1 \leq i \leq n} \sup_{x \in V} \{\lambda_i(x)\} \right] \int_V \chi^\dagger \chi \, dv.
 \end{aligned} \tag{83}$$

It now follows from inequalities (81) and (83) that

$$\sup_{1 \leq i \leq n} \sup_{x \in V} \{\lambda_i(x)\} > 0. \tag{84}$$

This completes the proof of the theorem.

COROLLARY 1

Under the hypothesis of theorem 1 if $C_2(x) = C_3(x) - C_4(x)$, then

$$\text{Sup}[\text{eigenvalues of } C_3(x)] > \text{Inf}[\text{eigenvalues of } C_4(x)], \tag{85}$$

where ‘Sup’ and ‘Inf’ are taken as in inequality (79).

The proof readily follows from theorem 1.

COROLLARY 2

If (66) is replaced by

$$[A \nabla^4 + B \nabla^2 + C(x) + i \sum_{j=1}^m \gamma_j \frac{\partial}{\partial x_j}] \chi(x) = 0, \tag{86}$$

γ_j ’s being real constants, then the conclusions of lemma 1 and theorem 1 are still valid.

Proof. We exactly follow the proof of lemma 1 and note that the only additional term that we now have to consider on the left hand side of (73) is

$$i \int_V \chi^\dagger \left[\sum_{j=1}^m \gamma_j \frac{\partial \chi}{\partial x_j} \right] \, dv.$$

Now

$$i \int_V \chi^\dagger \left[\sum_{j=1}^m \gamma_j \frac{\partial \chi}{\partial x_j} \right] dv = i \sum_{j=1}^m \int_V \chi_r^* \gamma_j \frac{\partial \chi_r}{\partial x_j} dv \quad (87)$$

Further, let $\chi_r = u_r + i v_r$. We then have

$$\begin{aligned} i \int_V \chi^\dagger \left[\sum_{j=1}^m \gamma_j \frac{\partial \chi}{\partial x_j} \right] dv &= i \sum_{j=1}^m \gamma_j \int_V (u_r - i v_r) \frac{\partial}{\partial x_j} (u_r + i v_r) dv \\ &= i \sum_{j=1}^m \gamma_j \int_V \frac{1}{2} \frac{\partial}{\partial x_j} (u_r^2 + v_r^2) dv - \\ &\quad \sum_{j=1}^m \gamma_j \int_V (u_r \frac{\partial}{\partial x_j} v_r - v_r \frac{\partial}{\partial x_j} u_r) dv \\ &= - \sum_{j=1}^m \gamma_j \int_V (u_r \frac{\partial}{\partial x_j} v_r - v_r \frac{\partial}{\partial x_j} u_r) dv, \end{aligned} \quad (88)$$

which is purely real. Conclusions of lemma 1 and theorem 1 therefore remain unchanged.

This completes the proof of the corollary.

In the following we apply the fundamental theorem (theorem 1) to the systems of equations and boundary conditions that correspond to the various instability problems as mentioned earlier and derive thereupon in a unified manner the already established results that are associated with these systems, i.e. results (a)–(d).

For the Rayleigh–Taylor instability problem with $n_i \neq 0$, the matrices B and $C(z)$ are given by (68) and therefore from the definitions of B_1 and $C_1(z)$, we have

$$B_1 = -n_i \left[\frac{\rho_0}{\mu} \right]_{1 \times 1}, \quad (89)$$

$$C_1(z) = -n_i \left[-\frac{k^2 \rho_0}{\mu} - g k^2 \frac{D\rho}{\mu |n|^2} \right]_{1 \times 1}. \quad (90)$$

It is easily seen that conditions of the fundamental theorem are satisfied with $l = n_i$ and hence it follows that

$$|n|^2 < \max_{[0, d]} \left[\frac{-g D\rho}{\rho_0} \right], \quad (91)$$

which establishes result (a).

For the spiral flow instability problem we first note that $(\sigma + R_* a)$ cannot be zero since it will then lead to a trivial solution for u and v as a consequence of the validity of (7)–(8) and boundary conditions (9). The matrices B and $C(\zeta)$, in this case, are given by (69) and therefore from the definitions of B_1 and $C_1(\zeta)$, we have

$$B_1 = -(\sigma + R_* a) \text{dg}(1, 0), \quad (92)$$

$$C_1(\zeta) = -(\sigma + R_* a) \text{dg} \left[-a^2 + \frac{12 R_* a}{(\sigma + R_* a)}, -\frac{1}{T_* a^2} \right], \quad (93)$$

where $\text{dg}(a_1, a_2, \dots, a_n)$ denotes an $n \times n$ diagonal matrix with diagonal entries $a_1, a_2, \dots, a_n$.

It is easily seen that conditions of the fundamental theorem are satisfied with $l = (\sigma + R_*a)$ and hence it follows that

$$-R_*a < \sigma < \frac{12R_*}{a} - R_*a, \quad (94)$$

which establishes result (b). This result is in agreement with the numerical calculations of Chandrasekhar (1961).

Remark. Almost identical reasoning shows that if $\sigma = \sigma_r + i\sigma_i$, σ_r and σ_i real, then

$$-R_*a < \sigma_r < \frac{12R_*}{a} - R_*a. \quad (95)$$

Similarly, for the Rayleigh–Benard instability problem, the matrices B and $C(z)$ are given by (70) and therefore from the definitions of B_1 and $C_1(z)$, we have

$$B_1 = -\sigma_i \text{dg}(1, 0), \quad (96)$$

$$C_1(z) = -\sigma_i \text{dg}(-a^2, -xg\alpha a^2 p/\beta v). \quad (97)$$

It is easily seen that conditions of theorem are satisfied with $l = \sigma_i$ and hence it follows that

$$\sigma_i = 0, \quad (98)$$

which establishes result (c). This result is due to Pellew & Southwell (1940).

Finally for the thermohaline instability problem of the Veronis type, with $p_i \neq 0$, the matrices B and $C(z)$ given by (71) and therefore from the definitions of B_1 and $C_1(z)$, we have

$$B_1 = -p_i \text{dg}(1, 0), \quad (99)$$

$$C_1(z) = -p_i \text{dg}\left(\frac{R_s \pi^2 \alpha^2}{|p|^2} - \frac{\pi^2 \alpha^2}{\sigma}, -R\right). \quad (100)$$

It is easily seen that conditions of theorem 1 are satisfied with $l = p_i$ and hence it follows that

$$|p|^2 < R_s \sigma, \quad (101)$$

which establishes result (d). This result is due to Banerjee (1972) and is in agreement with the calculations of Veronis (1965).

3. Some more mathematical models and an extended theory

We describe below the analysis of some more mathematical models representing systems of equations and boundary conditions of higher order on the lines shown earlier and bring out the inadequacy of the present unified theory to tackle such cases which correspond to more general physical situations. These are as below.

3.1 Thermohaline instability model of Veronis type

Consider the double eigenvalue problem for p described by the following nondimensional system of ordinary linear and homogenous differential equations with linear and homogeneous boundary conditions, namely

$$(D^2 - \pi^2 \alpha^2)(D^2 - \pi^2 \alpha^2 - \frac{p}{\sigma})\psi = -R\pi\alpha T + R_s\pi\alpha S, \quad (102)$$

$$(D^2 - \pi^2 \alpha^2 - p)T = \pi\alpha\psi \quad (103)$$

$$[\tau(D^2 - \pi^2 \alpha^2) - p]S = \pi\alpha\psi, \quad (104)$$

$$\psi = 0 = T = S, \text{ at } z = 0 \text{ and } z = 1, \quad (105)$$

$$\text{and either } D\psi = 0, \text{ at } z = 0 \text{ and } z = 1, \quad (106)$$

$$\text{or } D^2\psi = 0, \text{ at } z = 0 \text{ and } z = 1, \quad (107)$$

$$\text{or } D\psi = 0, \text{ at } z = 0 \text{ and } D^2\psi = 0 \text{ at } z = 1, \quad (108)$$

$$\text{or } D^2\psi = 0, \text{ at } z = 0 \text{ and } D\psi = 0 \text{ at } z = 1, \quad (109)$$

where τ is a positive constant, $S(z) = S_r(z) + i S_i(z)$ is the dependent variable which is, in general, a complex valued function of z in $[0, 1]$ such that $S_r(z)$ and $S_i(z)$ are real valued functions of z in $[0, 1]$ and the various other symbols used in the above equations have the same mathematical meanings as in (30)–(36).

We state the following result.

Result (e). If (p, ψ, T, S) with $p_r \geq 0$ and $p_i \neq 0$ is a solution of the double eigenvalue problem described by (102)–(109), then

$$|p|^2 < \sigma R_s. \quad (110)$$

Demonstration. Since p_i is nonzero, we can write (104) in the form

$$S = \frac{\tau}{p}(D^2 - \pi^2 \alpha^2)S - \frac{\pi\alpha\psi}{p}, \quad (111)$$

and combine (102) and (111) to derive

$$\begin{aligned} (D^2 - \pi^2 \alpha^2)(D^2 - \pi^2 \alpha^2 - \frac{p}{\sigma})\psi \\ = -R\pi\alpha T + R_s\pi\alpha \left[\frac{\tau}{p}(D^2 - \pi^2 \alpha^2)S - \frac{\pi\alpha\psi}{p} \right]. \end{aligned} \quad (112)$$

We multiply (102) throughout by ψ^* , substitute for ψ^* appropriately from (103) and (104) on the right hand side of the resulting equations, and integrate the equation thus obtained over the range of z to get

$$\begin{aligned}
& \int_0^1 \psi^* (D^2 - \pi^2 \alpha^2) (D^2 - \pi^2 \alpha^2 - \frac{p}{\sigma}) \psi \, dz \\
&= -R \int_0^1 T (D^2 - \pi^2 \alpha^2 - p^*) T^* \, dz \\
&+ \frac{\tau R_s}{p} \int_0^1 (D^2 - \pi^2 \alpha^2) S [\tau (D^2 - \pi^2 \alpha^2) - p^*] S^* \, dz \\
&- \pi^2 \alpha^2 \frac{R_s}{p} \int_0^1 |\psi|^2 \, dz. \tag{113}
\end{aligned}$$

Integration of each term of (113) by parts an appropriate number of times with the help of the boundary conditions (105)–(109) yields

$$\begin{aligned}
& \int_0^1 (|D^2 \psi|^2 + 2\pi^2 \alpha^2 |D \psi|^2 + \pi^4 \alpha^4 |\psi|^2) \, dz + \frac{p}{\sigma} \int_0^1 (|D \psi|^2 + \pi^2 \alpha^2 |\psi|^2) \, dz \\
&= R \int_0^1 (|DT|^2 + \pi^2 \alpha^2 |T|^2 + p^* |T|^2) \, dz + \frac{\tau^2 R_s}{p} \int_0^1 |(D^2 - \pi^2 \alpha^2) S|^2 \, dz \\
&+ \frac{\tau R_s p^*}{p} \int_0^1 (|DS|^2 + \pi^2 \alpha^2 |S|^2) \, dz - \frac{\pi^2 \alpha^2 R_s}{p} \int_0^1 |\psi|^2 \, dz. \tag{114}
\end{aligned}$$

The imaginary part of (114) gives (since p_i is nonzero)

$$\begin{aligned}
& \frac{1}{\sigma} \int_0^1 |D \psi|^2 \, dz + \pi^2 \alpha^2 \left(\frac{1}{\sigma} - \frac{R_s}{|p|^2} \right) \int_0^1 |\psi|^2 \, dz \\
&= -R \int_0^1 |T|^2 \, dz - \frac{2\tau p_r R_s}{|p|^2} \int_0^1 (|DS|^2 + \pi^2 \alpha^2 |S|^2) \, dz. \tag{115}
\end{aligned}$$

Since $p_r \geq 0$, $R > 0$, $R_s > 0$, $\sigma > 0$ and $\tau > 0$, it follows from (115) that

$$|p|^2 < \sigma R_s, \tag{116}$$

which establishes the result.

It may be noted that (102)–(104) and (105)–(109) respectively represent the governing equations and boundary conditions of the instability problem of Veronis's thermohaline model

of a Boussinesq liquid, wherein either both the boundaries are rigid or dynamically free or any one of the boundaries is rigid while the other one is dynamically free. The physical meanings of the various symbols used in these equations are the same as in (30)–(37) except that τ is the Lewis number and $S(z)$ is the concentration.

3.2 Thermohaline instability model of Stern type

Consider the double eigenvalue problem for p described by the following nondimensional system of ordinary linear and homogenous differential equations with linear and homogeneous boundary conditions, namely

$$(D^2 - \pi^2 \alpha^2)(D^2 - \pi^2 \alpha^2 - \frac{P}{\sigma})\psi = R\pi\alpha T - R_s\pi\alpha S, \quad (117)$$

$$(D^2 - \pi^2 \alpha^2 - p)T = \pi\alpha\psi, \quad (118)$$

$$[\tau(D^2 - \pi^2 \alpha^2) - p]S = \pi\alpha\psi, \quad (119)$$

$$\psi = 0 = T = S, \text{ at } z = 0 \text{ and } z = 1, \quad (120)$$

$$\text{and either } D\psi = 0, \text{ at } z = 0 \text{ and } z = 1, \quad (121)$$

$$\text{or } D^2\psi = 0, \text{ at } z = 0 \text{ and } z = 1, \quad (122)$$

$$\text{or } D\psi = 0, \text{ at } z = 0 \text{ and } D^2\psi = 0, \text{ at } z = 1, \quad (123)$$

$$\text{or } D^2\psi = 0, \text{ at } z = 0 \text{ and } D\psi = 0, \text{ at } z = 1, \quad (124)$$

which can be obtained from (102)–(109) by replacing R and R_s by $-R < 0$ and $-R_s < 0$ respectively so that the various symbols used in (117)–(124) have the same mathematical and physical meanings as in (102)–(109). Thus Veronis's thermohaline instability model and Stern's thermohaline instability model are gravitationally opposite to each other.

We state and prove the following results.

Result (f). If (p, ψ, T, S) with $p_r \geq 0$ and $p_i \neq 0$ is a solution of the double eigenvalue problem described by (117)–(124), then

$$|p|^2 < \sigma R. \quad (125)$$

Demonstration. Since p_i is nonzero, we can write (118) in the form

$$T = \frac{1}{p} (D^2 - \pi^2 \alpha^2) T - \frac{\pi\alpha\psi}{p}, \quad (126)$$

and combine (117) and (126) to derive

$$\begin{aligned} & (D^2 - \pi^2 \alpha^2) \left(D^2 - \pi^2 \alpha^2 - \frac{P}{\sigma} \right) \psi \\ &= \frac{R\pi\alpha}{p} (D^2 - \pi^2 \alpha^2) T - \frac{R\pi^2 \alpha^2}{p} \psi - R_s\pi\alpha S. \end{aligned} \quad (127)$$

We multiply (127) by ψ^* , substitute for ψ^* appropriately from (118) and (119) in the right hand side of the resulting equation, and integrate the equation thus obtained over the range of z to get,

$$\begin{aligned}
 & \int_0^1 \psi^* (D^2 - \pi^2 \alpha^2) (D^2 - \pi^2 \alpha^2 - \frac{p}{\sigma}) \psi \, dz \\
 &= \frac{R}{p} \int_0^1 (D^2 - \pi^2 \alpha^2) T (D^2 - \pi^2 \alpha^2 - p^*) T^* \, dz - \frac{R \pi^2 \alpha^2}{p} \int_0^1 |\psi|^2 \, dz \\
 & \quad - R_s \int_0^1 S [\tau (D^2 - \pi^2 \alpha^2) - p^*] S^* \, dz. \tag{128}
 \end{aligned}$$

Integration of each term of (128) by parts an appropriate number of times with the help of the boundary conditions (120)–(124) yields

$$\begin{aligned}
 & \int_0^1 (|D^2 \psi|^2 + 2\pi^2 \alpha^2 |D\psi|^2 + \pi^4 \alpha^4 |\psi|^2) \, dz + \frac{p}{\sigma} \int_0^1 (|D\psi|^2 + \pi^2 \alpha^2 |\psi|^2) \, dz \\
 &= \frac{R}{p} \int_0^1 |(D^2 - \pi^2 \alpha^2) T|^2 \, dz + \frac{R p^*}{p} \int_0^1 (|DT|^2 + \pi^2 \alpha^2 |T|^2) \, dz \\
 & \quad - \frac{R \pi^2 \alpha^2}{p} \int_0^1 |\psi|^2 \, dz + R_s \int_0^1 [\tau (|DS|^2 + \pi^2 \alpha^2 |S|^2) + p^* |S|^2] \, dz. \tag{129}
 \end{aligned}$$

The imaginary part of (129) gives (since p_i is non-zero)

$$\begin{aligned}
 & \frac{1}{\sigma} \int_0^1 |D\psi|^2 \, dz + \pi^2 \alpha^2 \left(\frac{1}{\sigma} - \frac{R_s}{|p|^2} \right) \int_0^1 |\psi|^2 \, dz \\
 &= -\frac{R}{|p|^2} \int_0^1 |(D^2 - \pi^2 \alpha^2) T|^2 \, dz - \frac{2p_r R}{|p|^2} \int_0^1 (|DT|^2 + \pi^2 \alpha^2 |T|^2) \, dz \\
 & \quad - R_s \int_0^1 |S|^2 \, dz. \tag{130}
 \end{aligned}$$

Since $p_r \geq 0$, $R > 0$, $R_s > 0$ and $\sigma > 0$, it follows from (130) that

$$|p|^2 < \sigma R, \tag{131}$$

which establishes the result.

It may be noted that (117)–(119) and (120)–(124) respectively represent the governing equations and boundary conditions of the instability problem of Stern's thermohaline model of a Boussinesq liquid.

If we apply theorem 1 to the system of equations and boundary conditions (written in the present unified framework under consideration) that correspond to the thermohaline instability problems of the Veronis and the Stern types and attempt to derive in a unified manner the results regarding the complex growth rate of an arbitrary oscillatory perturbation that is neutral or unstable otherwise as given above, we do not succeed. The reason for this failure is that the representation of the systems of equations and boundary conditions in the framework of the present generalized biharmonic equation with associated boundary conditions which govern the two instability problems give rise to such matrices A , B , C that while the Hermitian character of A is preserved (which was taken to be necessary by Banerjee *et al* (1978) for the unified framework) the conditions of theorem 1 are not satisfied. Several such results exist concerning problems in hydrodynamic stability for which theorem 1 is not applicable in its present form for the reasons just cited and which therefore cannot be brought under the present scheme of unification. The need for the establishment of an extended fundamental theorem in a framework which retains the essential character of the original one and at the same time can also account for such additional cases is a pressing one and Gupta *et al* (1985) have made significant contribution to the theory by giving a twist to the idea of 'representation' and successfully handling the resulting consequences of such a twist. Their work has extended the scope of the original theory by a considerable extent without sacrificing its elegant mathematical structure and in the following we describe their version of the fundamental theorem in all its essential points.

We prove the following lemma:

Lemma 2. If a solution χ satisfying (66) and (67) exists then we have

$$\int_V (\nabla^2 \chi)^\dagger A_1 (\nabla^2 \chi) dv - \int_V (\text{grad} \chi)^\dagger B_1 (\text{grad} \chi) dv + \int_V \chi^\dagger C_1(x) dv = 0, \quad (132)$$

where $A_1 = (A - A^\dagger)/2i$, A is an $n \times n$ matrix with complex constant entries and various other symbols are as in lemma 1.

Proof. Multiplication of (66) to the left by $\chi^\dagger$ gives

$$\chi^\dagger A \nabla^4 \chi + \chi^\dagger B \nabla^2 \chi + \chi^\dagger C(x) \chi = 0. \quad (133)$$

Now

$$\chi^\dagger A \nabla^4 \chi = \nabla^2 (\chi^\dagger A \nabla^2 \chi) - (\nabla^2 \chi)^\dagger A (\nabla^2 \chi) - 2(\text{grad} \chi^\dagger) \cdot A (\text{grad} (\nabla^2 \chi)), \quad (134)$$

so that (133) can be written in an alternative form as

$$\begin{aligned} \nabla^2 (\chi^\dagger A \nabla^2 \chi) - (\nabla^2 \chi)^\dagger A (\nabla^2 \chi) - 2(\text{grad} \chi)^\dagger \cdot A (\text{grad} (\nabla^2 \chi)) \\ + \chi^\dagger B \nabla^2 \chi + \chi^\dagger C(x) \chi = 0. \end{aligned} \quad (135)$$

Integration of (135) over the domain V and use of Gauss's theorem in R^m and boundary conditions (67) as in lemma 1 leads to

$$\int_V (\nabla^2 \chi)^\dagger A (\nabla^2 \chi) dv - \int_V (\text{grad} \chi)^\dagger B (\text{grad} \chi) dv + \int_V \chi^\dagger C(x) \chi dv = 0. \quad (136)$$

Imaginary part of (136) gives

$$\int_V (\nabla^2 \chi)^\dagger A_1 \nabla^2 \chi dv - \int_V (\text{grad} \chi)^\dagger B_1 (\text{grad} \chi) dv + \int_V \chi^\dagger C_1(x) \chi dv = 0, \quad (137)$$

and this completes the proof of the lemma. It may be noted that the derivation of (137) is essentially on the same lines as given in the derivation of (78) and further that the two equations are identical when A is a Hermitian matrix. Using the above lemma we prove the following theorem.

Theorem 2. (An extended fundamental theorem): Under the hypothesis of lemma 2 if $A_1 = l A_2 \cdot B_1 = -l B_2$, $C_1(x) = -l C_2(x)$, then

$$\text{Sup} \{ \text{eigenvalues of } C_2(x) \} > 0, \quad (138)$$

where l is a nonzero real number, A_2 is a non-negative definite Hermitian matrix, B_2 is a positive definite Hermitian matrix, $C_2(x)$ is a Hermitian matrix and 'Sup' is taken over all the eigenvalues of $C_2(x)$ over all x in V .

Proof. Since A_2 is non-negative definite and B_2 is positive definite, we have

$$\int_V (\nabla^2 \chi)^\dagger A_2 (\nabla^2 \chi) dv + \int_V (\text{grad} \chi)^\dagger B_2 (\text{grad} \chi) dv > 0. \quad (139)$$

Lemma 2 then gives

$$\int_V \chi^\dagger C_2(x) \chi dv > 0. \quad (140)$$

Now proceeding exactly as in the proof of theorem 1, we have

$$\text{Sup} \{ \text{eigenvalues of } C_2(x) \} > 0. \quad (141)$$

This completes the proof of the theorem.

COROLLARY 1

Under the hypothesis of theorem 2 if

$$C_2(x) = C_3(x) - C_4(x),$$

then,

$$\text{Sup} \{ \text{eigenvalues of } C_3(x) \} > \text{Inf} \{ \text{eigenvalues of } C_4(x) \}, \quad (142)$$

where 'Sup' and 'Inf' are taken in the usual sense.

The proof readily follows from theorem 2.

COROLLARY 2

If equation (66) is replaced by

$$[A\nabla^4 + B\nabla^2 + C(x) + i \sum_{j=1}^n \gamma_j \frac{\partial}{\partial x_j}] \chi(x) = 0, \quad (143)$$

γ_j 's being real constants, then the conclusions of lemma 2 and theorem 2 are still valid.

The proof follows exactly as in the case of corollary 2 of theorem 1.

COROLLARY 3

Under the hypothesis of theorem 2 if $B_2 = B_3 - B_4$, $C_2(x) = C_5(x) - C_6(x)$ and there exists a positive definite Hermitian matrix $H(x)$ such that

$$\int [(\text{grad} \chi)^\dagger B_1 (\text{grad} \chi) dv + \chi^\dagger C_5(x) \chi] dv < \int \chi^\dagger H(x) \chi dv, \quad (144)$$

then

$$\text{Sup}\{\text{eigenvalues of } H(x)\} > \text{Inf}\{\text{eigenvalues of } C_6(x)\}, \quad (145)$$

where $B_3, B_4, C_5(x), C_6(x)$ are Hermitian matrices with B_3, B_4 , and $C_5(x)$ being positive definite.

The proof readily follows from theorem 2.

In the following we apply theorem 2 to the systems of equations and boundary conditions that correspond to the thermohaline instability problem of the Veronis and the Stern types (to which the fundamental theorem does not apply as pointed out earlier) and derive thereupon in a unified manner the already established results that are associated with these systems, namely the results (e) and (f).

For the thermohaline instability problem of the Veronis type we represent (102)–(104) in the following convenient forms, namely

$$\begin{aligned} (D^2 - \pi^2 \alpha^2) \left(D^2 - \pi^2 \alpha^2 - \frac{p}{\sigma} \right) \psi + R\pi\alpha T \\ - R_s\pi\alpha \left[\frac{\tau}{p} (D^2 - \pi^2 \alpha^2) S - \frac{\pi\alpha}{p} \psi \right] = 0, \end{aligned} \quad (146)$$

$$- R \left[(D^2 - \pi^2 \alpha^2 - p) T - \pi\alpha \psi \right] = 0, \quad (147)$$

$$\frac{\tau^2 R_s}{p^*} (D^2 - \pi^2 \alpha^2) \left[\left(D^2 - \pi^2 \alpha^2 - \frac{p}{t} \right) S - \frac{\pi\alpha}{\tau} \psi \right] = 0, \quad (148)$$

which is permissible since p_i is non-zero in the context. Equation (66) reduces to the above equations with

$$A = \text{dg}(1, 0, \tau^2 R_s / p^*), \quad (149)$$

$$B = \begin{bmatrix} -\left(2\pi^2\alpha^2 + \frac{p}{\sigma}\right) & 0 & -\tau R_s \frac{\pi\alpha}{p} \\ 0 & -R & 0 \\ -\tau R_s \pi \frac{\alpha}{p^*} & 0 & \frac{-\tau^2 R_s}{p^*} \left(2\pi^2\alpha^2 + \frac{p}{\tau}\right) \end{bmatrix}_{3 \times 3}, \quad (150)$$

$$C = \begin{bmatrix} \pi^4\alpha^4 + \pi^2\alpha^2 \frac{p}{\sigma} + R_s \pi^2 \frac{\alpha^2}{p} & R\pi\alpha & \tau R_s \pi^3 \frac{\alpha^3}{p} \\ R\pi\alpha & R(\pi^2\alpha^2 + p) & 0 \\ \tau R_s \pi^3 \frac{\alpha^3}{p^*} & 0 & R_s \frac{\tau^2}{p^*} \left(\pi^4\alpha^4 + p\pi^2 \frac{\alpha^2}{\tau}\right) \end{bmatrix}_{3 \times 3}, \quad (151)$$

$$\chi = \begin{bmatrix} \psi(z) \\ T(z) \\ S(z) \end{bmatrix}_{3 \times 1}. \quad (152)$$

Further, boundary conditions on χ include all the boundary conditions on ψ , T and S and two more boundary conditions on S to complete the mathematical problem, since the order of the differential system in the present representation exceeds that of the original one by two, though the two representation are completely equivalent.

Also

$$A_1 = p_i \operatorname{dg} (0, 0, \tau^2 R_s / |p|^2), \quad (153)$$

$$B_1 = -p_i \operatorname{dg} (1/\sigma, 0, 2\tau R_s (\tau^2 \pi^2 \alpha^2 + p_r) / |p|^2), \quad (154)$$

$$C_1 = p_i \operatorname{dg} \left(\pi^2 \alpha^2 \left(\frac{R_s}{|p|^2} - \frac{1}{\sigma} \right), -R, -\left(\tau R_s \pi^2 \frac{\alpha^2}{|p|^2} \right) (\tau \pi^2 \alpha^2 + 2p_r) \right) \quad (155)$$

It is easily seen that conditions of the extended fundamental theorem are satisfied with $l = p_i$ and hence it follows that

$$|p|^2 < R_s \sigma, \quad (156)$$

which establishes result (e) (Banerjee *et al* 1981).

For the thermohaline instability problem of the Stern type we represent (117)–(119) in the following convenient forms, namely

$$\begin{aligned} & (D^2 - \pi^2 \alpha^2) \left(D^2 - \pi^2 \alpha^2 - \frac{p}{\sigma} \right) \psi \\ & - R\pi\alpha \left[\frac{1}{p} (D^2 - \pi^2 \alpha^2) T - \frac{\pi\alpha}{p} \psi \right] + R_s \pi \alpha S = 0, \end{aligned} \quad (157)$$

$$\frac{R}{p^*} (D^2 - \pi^2 \alpha^2) [(D^2 - \pi^2 \alpha^2 - p) T - \pi \alpha \psi] = 0, \quad (158)$$

$$-R_s [\{\tau (D^2 - \pi^2 \alpha^2) - p\} S - \pi \alpha \psi] = 0, \quad (159)$$

which are permissible since p_i is nonzero in the context.

Equation (66) reduces to the above equations with

$$A = \text{dg}(1, R/p^*, 0), \quad (160)$$

$$B = \begin{bmatrix} -(2\pi^2 a^2 + p/\sigma) & -R\pi\alpha/p & 0 \\ -R\pi\alpha/p^* & -(R/p^*)(2\pi^2 \alpha^2 + p) & 0 \\ 0 & 0 & -\tau R_s \end{bmatrix}_{3 \times 3}, \quad (161)$$

$$C = \begin{bmatrix} \pi^2 \alpha^2 \left(\pi^2 a^2 + \frac{p}{\sigma} + \frac{R}{p} \right) & R\pi^3 \frac{\alpha^3}{p} & R_s \pi \alpha \\ R\pi^3 \frac{\alpha^3}{p^*} & \left(R\pi^2 \frac{a^2}{p^*} \right) (\pi^2 \alpha^2 + p) & 0 \\ R_s \pi \alpha & 0 & R_s (\tau \pi^2 \alpha^2 + p) \end{bmatrix}_{3 \times 3}, \quad (162)$$

$$\chi = \begin{bmatrix} \psi(z) \\ T(z) \\ S(z) \end{bmatrix}_{3 \times 1} \quad (163)$$

Further, boundary conditions on χ include all the boundary conditions on ψ , T and S and two more boundary conditions on T to complete the mathematical problem, since the order of the differential system in the present representation exceeds that of the original one by two, though the two representations are completely equivalent.

Also

$$A_1 = p_i \text{dg}(0, R/|p|^2, 0), \quad (164)$$

$$B_1 = -p_i \text{dg} \left(\frac{1}{\sigma}, (2R/|p|^2) (\pi^2 \alpha^2 + p_r), 0 \right), \quad (165)$$

$$C_1 = -p_i \text{dg} (\pi^2 \alpha^2 R/|p|^2 - (1/\sigma)), -R\pi^2 \alpha^2 (\pi^2 \alpha^2 + 2p_r)/|p|^2, -R_s). \quad (166)$$

It is easily seen that conditions of theorem 2 are satisfied with $l = p_i$ and hence it follows that

$$|p|^2 < R\sigma, \quad (167)$$

which establishes result (f) (Banerjee et al 1981).

Next, we apply corollary 3 of theorem 2 to the system of equations and boundary conditions given in the following sections corresponding to the thermohaline instability problems of the Veronis and Stern types for a viscoelastic liquid and derive thereupon in a unified manner the results already established by the usual Pellew and Southwell technique with slight modifications (Gupta *et al* 1985).

Governing equations of the thermohaline instability problem of the Veronis type under Boussinesq approximation, when the viscoelastic liquid is described by Maxwell's constitutive relation, are given by

$$\begin{aligned} (D^2 - \pi^2 \alpha^2) \left[D^2 - \pi^2 \alpha^2 - \frac{p}{\sigma} (1 + \Gamma p) \right] \psi \\ = -R\pi\alpha (1 + \Gamma p) T + R_s\pi\alpha (1 + \Gamma p) S, \end{aligned} \quad (168)$$

$$(D^2 - \pi^2 \alpha^2 - p) T = \pi\alpha\psi, \quad (169)$$

$$[\tau (D^2 - \pi^2 \alpha^2) - p] S = \pi\alpha\psi, \quad (170)$$

with boundary conditions (105)–(109), where Γ is a non-negative real number characterising the elastic nature of the liquid.

We state and prove the following result.

Result (g). If (p, ψ, T, S) with $p_r \geq 0$ and $p_i \neq 0$ is a solution of (168)–(170) together with the boundary conditions (105)–(109), then

$$|p|^2 < \frac{1}{4} \left[R\Gamma\sigma + (R^2\Gamma^2\sigma^2 + 4R_s\sigma)^{1/2} \right]^2. \quad (171)$$

Demonstration. Since p_i is nonzero, we represent (168)–(170) in the following convenient forms, namely

$$\begin{aligned} (D^2 - \pi^2 \alpha^2) \left[D^2 - \pi^2 \alpha^2 - \frac{p}{\sigma} (1 + \Gamma p) \right] \psi + R\pi\alpha (1 + \Gamma p) T \\ - R_s\pi\alpha (1 + \Gamma p) R_s\pi\alpha (1 + \Gamma p) \left[\frac{\tau}{p} (D^2 - \pi^2 \alpha^2) S - \frac{\pi\alpha}{p} \psi \right] = 0, \end{aligned} \quad (172)$$

$$-R(1 + \Gamma p^*) [(D^2 - \pi^2 \alpha^2 - p) T - \pi\alpha\psi] = 0, \quad (173)$$

$$\left[\tau^2 R_s \left(\frac{1 + \Gamma p^*}{p^*} \right) \right] (D^2 - \pi^2 \alpha^2) \left[\left(D^2 - \pi^2 \alpha^2 - \frac{p}{\tau} \right) S - \frac{\pi\alpha}{\tau} \psi \right] = 0. \quad (174)$$

Equation (66) reduces to the above equations with

$$A = \text{dg} (1, 0, (\tau^2 R_s / p^*) (1 + \tau p^*)), \quad (175)$$

$$B = \begin{bmatrix} -\{2\pi^2\alpha^2 + \frac{p}{\sigma}(1 + \Gamma p)\} & 0 & -\tau R_s \frac{\pi\alpha(1 + \Gamma p)}{p} \\ 0 & -R(1 + \Gamma p^*) & 0 \\ -\tau R_s \frac{\pi\alpha}{p^*}(1 + \Gamma p^*) & 0 & -\frac{\tau^2 R_s}{p^*} \\ & & \left\{ \frac{2\pi^2\alpha^2}{p^*}(1 + \Gamma p^*) + \frac{p(1 + \Gamma p^*)}{\tau p^*} \right\} \end{bmatrix}_{3 \times 3}, \quad (176)$$

$$C = \begin{bmatrix} \pi^2\alpha^2 \left\{ \pi^2\alpha^2 + \left(\frac{p}{\sigma} + \frac{R_s}{p} \right) (1 + \Gamma p) \right\} & R\pi\alpha(1 + \Gamma p) & \tau R_s \frac{\pi^3\alpha^3(1 + \Gamma p)}{p} \\ R\pi\alpha(1 + \Gamma p^*) & R(\pi^2\alpha^2 + p)(1 + \Gamma p^*) & 0 \\ \tau R_s \frac{\pi^3\alpha^3}{p^*}(1 + \Gamma p^*) & 0 & R_s \frac{\tau^2\pi^2\alpha^2}{p^*}(1 + \Gamma p^*) \\ & & \left(\pi^2\alpha^2 + \frac{p}{\tau} \right) \end{bmatrix}_{3 \times 3}, \quad (177)$$

$$\chi = \begin{bmatrix} \psi(z) \\ T(z) \\ S(z) \end{bmatrix}_{3 \times 1}. \quad (178)$$

Further, boundary conditions on χ include all the boundary conditions on ψ , T and S and two further boundary conditions on S to complete the mathematical problem, since the order of the differential system in the present representation exceeds that of the original one by two, although the two representations are completely equivalent.

$$A_1 = p_i \text{dg}(0, 0, \tau^2 R_s / |p|^2), \quad (179)$$

$$B_1 = -p_i(B_3 - B_4), \quad (180)$$

$$C_1 = -p_i(C_5 - C_6), \quad (181)$$

where

$$B_3 = \text{dg} \left(\left(\frac{(1 + 2\Gamma p_r)}{\sigma}, 0, \tau^2 R_s \left\{ \frac{2\pi^2\alpha^2}{|p|^2} + \frac{(\Gamma + 2p_r/|p|^2)}{\tau} \right\} \right) \right), \quad (182)$$

$$B_4 = \text{dg}(0, R\Gamma, 0), \quad (183)$$

$$C_5 = \text{dg}(0, R\pi^2\alpha^2\Gamma, 0), \quad (184)$$

$$C_6 = \text{dg}\left(\pi^2\alpha^2\left(\frac{1}{\sigma} - \frac{R_s}{|p|^2} + \frac{2\Gamma p_r}{\sigma}\right), R, \right. \\ \left. \tau^2\pi^2\alpha^2 R_s \left\{ \frac{\pi^2\alpha^2}{|p|^2} + \frac{(2p_r/|p|^2 + \Gamma)}{\tau} \right\} \right). \quad (185)$$

Further, with

$$H = \text{dg}\left(\frac{R\pi^2\alpha^2\Gamma}{|p|}, 0, 0\right), \quad (186)$$

it follows upon using (169) and Schwartz's inequality that

$$\int_0^1 [(\text{grad}\chi)^\dagger B_4 (\text{grad}\chi) + \chi^\dagger C_5 \chi] dz < \int_0^1 \chi^\dagger H \chi dz. \quad (187)$$

It is now easily seen that conditions of corollary 3 of theorem 2 are satisfied with $l = p_i$ and hence it follows that

$$\frac{1}{\sigma} - \frac{R_s}{|p|^2} - \frac{R\Gamma}{|p|} < 0. \quad (188)$$

It follows from inequality (188) that

$$|p| < \frac{1}{2} \left[R\Gamma\sigma + (R^2\Gamma^2\sigma^2 + 4R_s\sigma)^{1/2} \right] \quad (189)$$

which establishes the result.

The governing equations of the thermohaline instability problem of the Stern type under Boussinesq approximation, when the viscoelastic liquid is described by Maxwell's constitutive relation, are given by

$$(D^2 - \pi^2\alpha^2) \left[D^2 - \pi^2\alpha^2 - \frac{P}{\sigma} (1 + \Gamma p) \right] \psi \\ = R\pi\alpha (1 + \Gamma p) T - R_s\pi\alpha (1 + \Gamma p) S, \quad (190)$$

$$(D^2 - \pi^2\alpha^2 - p) T = \pi\alpha\psi, \quad (191)$$

$$[\tau (D^2 - \pi^2\alpha^2) - p] S = \pi\alpha\psi, \quad (192)$$

with boundary conditions (120)–(124).

We state and prove the following result.

Result (h). If (p, ψ, T, S) with $p_r \geq 0$ and $p_i \neq 0$ is a solution of (190)–(192) together with the boundary conditions (120)–(124), then

$$|p|^2 < \left[\frac{1}{2} \left[R_s\Gamma\sigma + (R_s^2\Gamma^2\sigma^2 + 4R_s\sigma)^{1/2} \right] \right]^2. \quad (193)$$

Demonstration. Since p_i is nonzero we represent (190)–(192) in the following convenient forms, namely

$$\begin{aligned} & (D^2 - \pi^2 \alpha^2) \left[D^2 - \pi^2 \alpha^2 - \frac{p}{\sigma} (1 + \Gamma p) \right] \psi \\ & - R\pi\alpha (1 + \Gamma p) \left[\frac{1}{p} (D^2 - \pi^2 \alpha^2) T - \frac{\pi\alpha}{p} \psi \right] + R_s \pi\alpha (1 + \Gamma p) S = 0, \end{aligned} \quad (194)$$

$$\left[\frac{R(1 + \Gamma p^*)}{p^*} \right] (D^2 - \pi^2 \alpha^2) [(D^2 - \pi^2 \alpha^2 - p) T - \pi\alpha \psi] = 0, \quad (195)$$

$$- R_s (1 + \Gamma p^*) [\{\tau (D^2 - \pi^2 \alpha^2) - p\} S - \pi\alpha \psi] = 0. \quad (196)$$

Equation (66) reduces to the above equation with

$$A = \text{dg} \left(1, \frac{R(1 + \Gamma p^*)}{p^*} \right), 0, \quad (197)$$

$$B = \begin{pmatrix} -\{2\pi^2 \alpha^2 + \frac{p}{\sigma}(1 + \Gamma p)\} & \frac{-R\pi\alpha(1 + \Gamma p)}{p} & 0 \\ \frac{-R\pi\alpha(1 + \Gamma p^*)}{p^*} & \frac{-R(1 + \Gamma p^*)(2\pi^2 \alpha^2 + p)}{p^*} & 0 \\ 0 & 0 & -\tau R_s(1 + \Gamma p^*) \end{pmatrix}_{3 \times 3}, \quad (198)$$

$$C = \begin{pmatrix} \pi^2 \alpha^2 \left\{ \pi^2 \alpha^2 + \left(\frac{p}{\sigma} + \frac{R}{p} \right) (1 + \Gamma p) \right\} & \frac{R\pi^3 \alpha^3 (1 + \Gamma p)}{p} & R_s \pi\alpha (1 + \Gamma p) \\ R \frac{\pi^3 \alpha^3}{p^*} (1 + \Gamma p^*) & \frac{\{R\pi^2 \alpha^2 (1 + \Gamma p^*) (\pi^2 \alpha^2 + p)\}}{p^*} & 0 \\ R_s \pi\alpha (1 + \Gamma p^*) & 0 & R_s (1 + \Gamma p^*) \\ & & (\tau \pi^2 \alpha^2 + p) \end{pmatrix}_{3 \times 3}, \quad (199)$$

$$\chi = \begin{pmatrix} \psi(z) \\ T(z) \\ S(z) \end{pmatrix}_{3 \times 1}. \quad (200)$$

Further, boundary conditions on χ include all the boundary conditions on ψ , T and S and two further boundary conditions on T to complete the mathematical problem, since the order of

the differential system in the present representation exceeds that of the original one by two, though the two representations are completely equivalent.

$$A_1 = p_i \operatorname{dg}(0, R/|p|^2, 0), \quad (201)$$

$$B_1 = -p_i(B_3 - B_4), \quad (202)$$

$$C_1 = -p_i(C_5 - C_6), \quad (203)$$

where

$$B_3 = \operatorname{dg}\left(\frac{(1 + 2\Gamma p_r)}{\sigma}, R\left(\frac{2\pi^2\alpha^2}{|p|^2} + \frac{2p_r}{|p|^2} + \Gamma\right), 0\right), \quad (204)$$

$$B_4 = \operatorname{dg}(0, 0, \tau R_s \Gamma), \quad (205)$$

$$C_5 = \operatorname{dg}(0, 0, \tau R_s \pi^2 \alpha^2 \Gamma), \quad (206)$$

$$C_6 = \operatorname{dg}\left(\pi^2 \alpha^2 \left(\frac{1}{\sigma} - \frac{R}{|p|^2} + \frac{2\Gamma p_r}{\sigma}\right), \frac{R\pi^2 \alpha^2}{|p|^2} (\pi^2 \alpha^2 + 2p_r + \Gamma |p|^2), R_s\right). \quad (207)$$

Further, with

$$H = \operatorname{dg}\left(\frac{\pi^2 \alpha^2 \Gamma R_s}{|p|}, 0, 0\right), \quad (208)$$

it follows, upon using (192) and Schwartz's inequality that

$$\int_0^1 [(\operatorname{grad} \chi)^\dagger B_4 (\operatorname{grad} \chi) + \chi^\dagger C_5 \chi] dz < \int_0^1 \chi^\dagger H \chi dz. \quad (209)$$

It is now easily seen that conditions of corollary 3 of theorem 2 are satisfied with $l = p_i$ and hence it follows that

$$\frac{1}{\sigma} - \frac{R}{|p|^2} - \frac{R_s \Gamma}{|p|} < 0. \quad (210)$$

It follows from inequality (110) that,

$$|p| < \frac{1}{2} \left[R_s \Gamma \sigma + (R_s^2 \Gamma^2 \sigma^2 + 4R\sigma)^{1/2} \right], \quad (211)$$

which establishes the result.

4. An illustrative example

We now describe another method of unifying results (a)–(h) with eigenvalue p of the problem occurring explicitly.

Consider the system of equations

$$\begin{aligned} p(\nabla^2 - a_1)[\nabla^2 - a_2 - p(1 + Lp)a_3]\psi \\ = p(1 + Lp)a_4\phi - p(1 + Lp)a_5\mu - f(x)\psi, \end{aligned} \quad (212)$$

$$(b_1\nabla^2 - b_2 - b_3p)\phi = -\psi, \quad (213)$$

$$(c_1\nabla^2 - c_2 - c_3p)\mu = -\psi, \quad (214)$$

where $a_i (i = 1, 2, \dots, 5)$, $b_i (i = 1, 2, 3)$, $c_i (i = 1, 2, 3)$ and L are nonnegative constants with a_1, a_3, b_3 and c_3 being nonzero, and $f(x)$ real for each x in the domain V .

The boundary conditions are

$$\psi = 0 = \phi, \text{ and either } (\partial\psi/\partial n) = 0 \text{ or } \nabla^2\psi = 0, \quad (215)$$

on the boundary S of V .

We state and prove the following result.

Result (i). If (p, ψ, ϕ, μ) with $p_r \geq 0$ and $p_i \neq 0$ is a solution of (212)–(215), then

$$|p| < [\beta + (\beta^2 + 4\alpha\gamma)^{1/2}] / 2\alpha, \quad (216)$$

$$\text{where } \alpha = a_1a_3b_3c_3, \quad (217)$$

$$\beta = La_4c_3, \quad (218)$$

$$\gamma = b_3\{a_5 + c_3(\sup_{x \in V} f(x))\}. \quad (219)$$

Demonstration. Since p_i is nonzero, we represent (212)–(214) in the following convenient forms, namely

$$\begin{aligned} (\nabla^2 - a_1)[\nabla^2 - a_2 - p(1 + Lp)a_3]\psi &= (1 + Lp)a_4\phi \\ &- (1 + Lp)a_5\left[\frac{\psi}{c_3p} + \frac{1}{c_3p}(c_1\nabla^2 - c_2)\mu\right] - \frac{p^*}{|p|^2}f(x), \end{aligned} \quad (220)$$

$$-a_4(1 + Lp^*)[(b_1\nabla^2 - b_2 - b_3p)\phi + \psi] = 0, \quad (221)$$

$$-a_5\frac{(1 + Lp^*)}{c_3p^*}(c_1\nabla^2 - c_2)[(c_1\nabla^2 - c_2 - c_3p)\mu + \psi] = 0. \quad (222)$$

Equation (66) reduces to the above equations with

$$A = \text{dg}\left(1, 0, c_1^2a_5p\frac{(1 + Lp^*)}{c_3|p|^2}\right) \quad (223)$$

$$B = \begin{bmatrix} -\{a_1 + a_2 + a_3 p(1 + Lp)\} & 0 & c_1 a_5 \frac{p^*(1 + Lp)}{c_3 |p|^2} \\ 0 & -b_1 a_4 (1 + Lp^*) & 0 \\ c_1 a_5 \frac{p(1 + Lp^*)}{c_3 |p|^2} & 0 & -a_5 p(1 + Lp^*) \\ & & \frac{(2c_1 c_2 + p c_1 c_3)}{c_3 |p|^2} \end{bmatrix}_{3 \times 3} \quad (224)$$

$$C = \begin{bmatrix} a_1 \{a_2 + a_3 p(1 + Lp)\} + & -a_4 (1 + Lp) & c_2 a_5 \frac{p^*(1 + Lp)}{c_3 |p|^2} \\ a_5 \frac{p^*(1 + Lp)}{c_3 |p|^2} + \frac{f(x)p^*}{|p|^2} & & \\ -a_4 (1 + Lp) & a_4 (b_2 + p b_3) (1 + Lp^*) & 0 \\ -c_2 a_5 \frac{p(1 + Lp^*)}{c_3 |p|^2} & 0 & c_2 a_5 p \frac{(1 + Lp^*)}{c_3 |p|^2} \\ & & (c_2 + p c_3) \end{bmatrix}_{3 \times 3}, \quad (225)$$

$$\chi = \begin{bmatrix} \psi(x) \\ \phi(x) \\ \mu(x) \end{bmatrix}_{3 \times 1}. \quad (226)$$

Further, boundary conditions on χ conform to those of ψ , ϕ and μ with similar remarks as mentioned earlier.

Also

$$A_1 = -p_i \operatorname{dg} \left(0, 0, \frac{c_1^2 a_5}{c_3 |p|^2} \right), \quad (227)$$

$$B_1 = -p_i (B_3 - B_4), \quad (228)$$

$$C_1 = p_i (C_5 - C_6), \quad (229)$$

where

$$B_3 = \operatorname{dg} \left(a_3 (1 + Lp_r), 0, \frac{2c_1 c_2 a_5}{c_3 |p|^2} + \frac{2p_r c_1 a_5}{|p|^2} + L c_1 a_5 \right), \quad (230)$$

$$B_4 = \text{dg}(0, Lb_1a_4, 0), \quad (231)$$

$$C_5 = \text{dg}(0, La_4b_2, 0), \quad (232)$$

$$C_6 = \text{dg}(a_1a_3(1 + 2p_rL) - \frac{a_5}{c_3|p|^2} - \frac{f(x)}{|p|^2}, b_3a_4a_5 \left(\frac{c_2^2}{c_3|p|^2} + Lc_2 + \frac{2p_rc_2}{|p|^2} \right)). \quad (233)$$

We now show that if

$$H = \text{dg} \left(\frac{La_4}{b_3|p|}, 0, 0 \right) \quad (234)$$

then

$$\int_V [(\text{grad}\chi)^\dagger B_4(\text{grad}\chi) + \chi^\dagger C_5\chi] dv < \int_V \chi^\dagger H\chi dv. \quad (235)$$

Multiplication of (213) by its complex conjugate and integration of the resulting equation over the domain V gives

$$\begin{aligned} \int_V |(b_1\nabla^2 - b_2)\phi|^2 dv - b_3p^* \int_V \phi^*(b_1\nabla^2 - b_2)\phi dv \\ - b_3p \int_V \phi(b_1\nabla^2 - b_2)\phi^* dv + b_3^2|p|^2 \int_V |\phi|^2 dv = \int_V |\psi|^2 dv. \end{aligned} \quad (236)$$

Use of Gauss's theorem and boundary conditions of ϕ yields

$$\int_V \phi^*\nabla^2\phi dv = \int_V \phi\nabla^2\phi^* dv = - \int_V |\text{grad}\phi|^2 dv. \quad (237)$$

We use (237) in (236) to get

$$\begin{aligned} \int_V |(b_1\nabla^2 - b_2)\phi|^2 dv + 2b_3p_r \int_V (b_1|\text{grad}\phi|^2 \\ + b_2|\phi|^2) dv + b_3^2|p|^2 \int_V |\phi|^2 dv = \int_V |\psi|^2 dv. \end{aligned} \quad (238)$$

It follows from (238) that

$$\int_V |(b_1\nabla^2 - b_2)\phi|^2 dv < \int_V |\psi|^2 dv, \quad (239)$$

$$\int_V |\phi|^2 dv < \frac{1}{b_3^2|p|^2} \int_V |\psi|^2 dv. \quad (240)$$

Now

$$\begin{aligned}
 \int_V (b_1 |\text{grad} \phi|^2 + b_2 |\phi|^2) dv &= - \int_V \phi^* (b_1 \nabla^2 - b_2) \phi dv \\
 &= \left| - \int_V \phi^* (b_1 \nabla^2 - b_2) \phi dv \right| \\
 &\leq \int_V |\phi| |(b_1 \nabla^2 - b_2) \phi| dv.
 \end{aligned} \tag{241}$$

Use of Schwartz's inequality gives

$$\int_V |\phi| |(b_1 \nabla^2 - b_2) \phi| dv \leq \left(\int_V |\phi|^2 dv \right)^{\frac{1}{2}} \left(\int_V |b_1 \nabla^2 - b_2 \phi|^2 dv \right)^{\frac{1}{2}}, \tag{242}$$

so that

$$\int_V (b_1 |\text{grad} \phi|^2 + b_2 |\phi|^2) dv \leq \left[\int_V |\phi|^2 dv \right]^{\frac{1}{2}} \left[\int_V |(b_1 \nabla^2 - b_2) \phi|^2 dv \right]^{\frac{1}{2}}, \tag{243}$$

We use inequalities (239) and (240) in inequality (243) to get

$$\int_V (b_1 |\text{grad} \phi|^2 + b_2 |\phi|^2) dv < \frac{1}{b_3 |p|} \int_V |\psi|^2 dv. \tag{244}$$

Hence

$$\int_V [(\text{grad} \chi)^\dagger B_4 (\text{grad} \chi) + \chi^\dagger C_5 \chi] dv < \int_V \chi^\dagger H \chi dv. \tag{245}$$

Thus, we see that conditions of corollary 3 of theorem 2 are satisfied with $l = p_i$ and hence it follows that

$$\frac{La_4}{b_3 |p|} > \inf_{x \in V} \left[\frac{-f(x)}{|p|^2} \right] + a_1 a_3 (1 + 2p_r) - \frac{a_5}{c_3 |p|^2},$$

i.e.

$$\alpha |p|^2 - \beta |p| - a_5 b_3 - (\sup_{x \in V} f(x)) b_3 c_3 < 0, \tag{246}$$

where

$$\alpha = a_1 a_3 b_3; \tag{247}$$

$$\beta = La_4 c_3. \tag{248}$$

Inequality (246) clearly implies that

$$|p| < \frac{[\beta + (\beta^2 + 4\alpha\gamma)^{1/2}]}{2\alpha}, \quad (249)$$

where

$$\gamma = b_3\{a_5 + c_3(\sup_{x \in V} f(x))\}. \quad (250)$$

This establishes the result.

5. A semicircle theorem for a random statistical variate and its relevance to theoretical hydrodynamics

While writing down the initial findings of their investigation on a problem concerning moments of a probability distribution, we (M B Banerjee & A Banerjee, B & B) came across a monograph by Shohat & Tamarkin (1963) containing a description of the classical works of Heine and others on the problem of moments, where they have investigated the problem exhaustively and in great generality. The discovery of this monograph did not allow us (B & B) much scope for claiming originality and publishing our results. However, a small portion of our work and that of Banerjee & Shandil (1995) is still worth reporting on account of the originality of the contents and the elementary nature of the methods used, their geometrical flavour and the possibilities of the generalizations inherent in them, and most important of all in the present context, the relationship that they have with the results on bounds of eigenvalues of the Rayleigh and the Taylor–Goldstein equations, as shown through the application of the Pellew and Southwell method, describing respectively the instability problems of inviscid incompressible homogeneous or heterogeneous, parallel shear flows.

Banerjee & Shandil (1995) proved a semicircle theorem stating “If μ and $\sigma (> 0)$ respectively denote the mean and the standard deviation of a continuous random statistical variate X with a range $a \leq x \leq b$ and probability density function $\phi(x)$ then the point (μ, σ) must lie inside the semicircle in the upper half of the $\mu\sigma$ -plane which has the range of X for diameter”, for the first time simply and elegantly. Expressed in terms of mathematical symbols, the semicircle theorem implies that the inequality

$$\left(\mu - \frac{a+b}{2}\right)^2 + \sigma^2 \leq \left(\frac{b-a}{2}\right)^2, \quad (251)$$

must of necessity hold good and since then the semicircle theorem and the associated inequality have come to be known as the Banerjee–Shandil semicircle theorem (Banerjee et al 1999) and the Banerjee–Shandil semicircle inequality (Kapur 1996) respectively.

Kapur & Rani (1995) mathematically proved the general inequality

$$\mu'_r \leq \frac{\mu'_s(b^r - a^r) + b^s a^r - a^s b^r}{b^s - a^s}, \text{ for } r > s \text{ and } a > 0, \quad (252)$$

and making use of the above inequality and the equation

$$\mu'_{2s} - \mu'^2_s = \sigma_s^2, \quad (253)$$

Kapur (1996) showed that a more general inequality than that given by (251) can be derived which states that

$$\left\{ \mu'_s - \frac{b^s + a^s}{2} \right\}^2 + \sigma_s^2 \leq \left(\frac{b^s - a^s}{2} \right)^2 \text{ for } a > 0, \quad (254)$$

where μ'_r , μ'_s and μ_{2s} respectively denote the r th, the s th and $2s$ th moments of a continuous random statistical variate varying over the interval $[a, b]$ with r and s being positive integers. Inequality (254) implies that the point (μ'_s, σ_s) must lie inside the semicircle in the upper half of the $\mu\sigma$ -plane with centre $(b^s + a^s)/2$ and radius $(b^s - a^s)/2$. The more general semicircle theorem and the associated inequality will henceforth, be called the Kapur semicircle theorem and Kapur–Rani inequality. It may be noted from inequality (254) that the case with $s = 1$ corresponds to the Banerjee–Shandil inequality, but only for $a > 0$.

From the Banerjee–Shandil semicircle theorem the following results are obtained.

Result (j). If μ and $\sigma (< 0)$ respectively denote the mean and the standard deviation of a continuous random statistical variate X with range $a \leq x \leq b$ and probability density function $\phi(x)$ then the point (μ, σ') with $0 < \sigma' \leq \sigma$, must lie inside the semicircle in the upper half of the $\mu\sigma$ -plane, with diameter in the range of X , i.e. μ and σ' must satisfy the inequality

$$\left(\mu - \frac{a+b}{2} \right)^2 + \sigma'^2 \leq \left(\frac{b-a}{2} \right)^2, \quad (255)$$

while from the generalized semicircle theorem the following is evident.

Result (k). If M and $S (> 0)$ respectively denote the mean and the standard deviation of a continuous variate Z of the continuous random statistical variate X with range $a \leq x \leq b$ and probability density function $\phi(x)$, then the point (M, S') with $0 < S' \leq S$, must lie inside the semicircle in the upper half of the MS -plane with diameter in the range of Z , i.e. M and S' must satisfy the inequality

$$\left(M - \frac{Z_{\min} + Z_{\max}}{2} \right)^2 + S'^2 \leq \left(\frac{Z_{\max} - Z_{\min}}{2} \right)^2. \quad (256)$$

5.1 Eigenvalue bounds for the Rayleigh equation in hydrodynamic stability

The fundamental equation governing infinitesimal instability in a homogeneous inviscid parallel shear flow in the framework of linear theory is given by the Rayleigh equation

$$(U - c) \left(\frac{d^2}{dy^2} - \alpha^2 \right) \phi - \left(\frac{d^2 U}{dy^2} \right) \phi = 0, \quad (257)$$

where y is the real independent variable such that $y_1 \leq y \leq y_2$, $U(y)$ is the basic velocity field which is twice continuously differentiable, $\phi(y)$ is the y -dependence of the stream function perturbation and stands for the dependent variable, $c = c_r + ic_i$ is the complex wave velocity which is constant, such that c_r and c_i are the real and imaginary parts of c , and α^2 is the square of the wave number which is constant and satisfies the inequality $0 < \alpha^2 < \infty$. The

boundary conditions are that ϕ vanishes on the rigid walls at $y = y_1$ and $y = y_2$ which may recede to $\mp \infty$ in the limiting cases and therefore

$$\phi(y_1) = \phi(y_2) = 0. \quad (258)$$

The requirement of nontrivial solutions of differential equation (257) that satisfy the boundary conditions (258) poses a double eigenvalue problem for the determination of c_r and c_i for prescribed values of $U(y)$ and α^2 and the flow is unstable if such solutions exist for which the imaginary part c_i of c is greater than zero.

We prove the following theorem.

Theorem 3. *If (ϕ, c) with $c_i > 0$ is a solution of the eigenvalue problem described by differential equation (257) and boundary conditions (258) for prescribed values of $U(y)$ and α^2 then*

$$\left(c_r - \frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^2 \leq \left(\frac{U_{\max} - U_{\min}}{2}\right)^2, \quad (259)$$

where $U_{\min}$ is the minimum value $U_{\max}$ is the maximum value of U over the range $y_1 \leq y \leq y_2$.

Proof. Making use of the transformation $F = \phi/(U - c)$, which is valid since $c_i > 0$, (257) and (258) become

$$\frac{d}{dy} \left[(U - c)^2 \frac{dF}{dy} \right] - \alpha^2 (U - c)^2 F = 0, \quad (260)$$

and

$$F(y_1) = 0 = F(y_2). \quad (261)$$

Now, multiplying (260) by F^* (the complex conjugate of F) throughout, integrating the resulting equation over the vertical range of y with the help of the boundary conditions (261) and then separating the real and imaginary parts of the equation thus obtained, we get

$$\int_{y_1}^{y_2} [(U - c_r)^2 - c_i^2] \left[\left| \frac{dF}{dy} \right|^2 + \alpha^2 |F|^2 \right] dy = 0, \quad (262)$$

and

$$\int_{y_1}^{y_2} (U - c_r) \left[\left| \frac{dF}{dy} \right|^2 + \alpha^2 |F|^2 \right] dy = 0, \quad (263)$$

where we have utilized the fact that $c_i > 0$ in deriving (263).

Equations (262) and (263) show that c_r and $c_i (> 0)$ could be regarded as the mean and the standard deviation of a continuous variate $U(y)$ of the continuous random variate y with range $y_1 \leq y \leq y_2$ and probability density function

$$\psi(y) = \left\{ \left| \frac{dF}{dy} \right|^2 + \alpha^2 |F|^2 \right\} / \left\{ \int_{y_1}^{y_2} \left[\left| \frac{dF}{dy} \right|^2 + \alpha^2 |F|^2 \right] dy \right\} \quad (264)$$

Therefore, applying the generalized semicircle theorem as proved above, we derive that

$$\left(c_r - \frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^2 \leq \left(\frac{U_{\max} - U_{\min}}{2}\right)^2, \quad (265)$$

and hence the theorem.

Thus, the complex wave velocity c of an arbitrary unstable perturbation, manifesting in homogeneous inviscid incompressible parallel shear flow, must lie inside the semicircle in the upper half of the $c_r c_i$ -plane with diameter in the range of U . This is one of the most beautiful results in the theory of hydrodynamic stability and is known as Howard's (1961) semicircle theorem and together with the point of inflexion theorem (Rayleigh 1880), the stronger version of the point of inflexion theorem (Fjortoft 1950), the even stronger version of the point of inflexion theorem (Banerjee *et al* 1999), the theorem of Miles (1961) and the theorem of Kochar & Jain (1979) constitutes our basic knowledge of the stability of inviscid nonviscous parallel shear flows. It is a matter of sheer coincidence that we chanced upon this exact replica of Howard's semicircle theorem in the context of mathematical statistics while dabbling with the expression of 'variance' of a random statistical variate. Incidentally, the alternative manner of deriving this important result of Howard's as presented herein not only makes the result evolve out as a natural outcome of the relevant equations but also cuts short to a large measure the adhocism of his mathematical procedure.

5.2 Eigenvalue bounds for Taylor–Goldstein equation in hydrodynamic stability

The fundamental equation governing infinitesimal instability in a heterogeneous inviscid parallel shear flow in the framework of the linear theory is given by the Taylor–Goldstein equation

$$(U - c) \left[\frac{d}{dy} \left(\rho \frac{d\phi}{dy} \right) - \rho \alpha^2 \phi \right] - \frac{d}{dy} \left(\rho \frac{dU}{dy} \right) \phi + \frac{\rho g \beta}{U - c} \phi = 0, \quad (266)$$

where $\rho(y)$ denotes the density field, $\beta(y) = (1/\rho)(d\rho/dy)$ denotes the heterogeneity field and is non-negative everywhere in the flow domain while all the other symbols used in (266), and (267) below are the same as in (257) and (258). As before the boundary conditions are given by

$$\phi(y_1) = \phi(y_2) = 0. \quad (267)$$

We prove the following theorem.

Theorem 4. *If (ϕ, c) with $c_i > 0$ is a solution of the eigenvalue problem described by differential equation (266) and boundary conditions (267) for prescribed values of $U(y)$ and α^2 then*

$$\left(c_r - \frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^2 \leq \left(\frac{U_{\max} - U_{\min}}{2}\right)^2. \quad (268)$$

Proof. Making use of the transformation $F = \phi/(U - c)$, which is valid since $c_i > 0$, (266) and (267) become

$$\frac{d}{dy} \left[\rho(U - c)^2 \frac{dF}{dy} \right] + [\rho g \beta - \rho k^2 (U - c)^2] F = 0, \quad (269)$$

and

$$F(y_1) = F(y_2) = 0. \quad (270)$$

Now, multiplying (269) by F^* throughout, integrating the resulting equation over the vertical range of y with the help of the boundary conditions (270) and then separating the real and imaginary parts of the equation thus obtained, we get

$$\int_{y_1}^{y_2} \rho [(U - c_r)^2 - c_i^2] \left[\left| \frac{dF}{dy} \right|^2 + \alpha^2 |F|^2 \right] dy - \int_{y_1}^{y_2} \rho g \beta |F|^2 dy = 0, \quad (271)$$

and

$$\int_{y_1}^{y_2} (U - c_r) \left[\left| \frac{dF}{dy} \right|^2 + \alpha^2 |F|^2 \right] dy = 0. \quad (272)$$

Equations (271) and (272) clearly show that if we regard $U(y)$ as a continuous variate of the continuous random variate y with range $y_1 \leq y \leq y_2$ and probability function

$$\psi(y) = \left\{ \left| \frac{dF}{dy} \right|^2 + \alpha^2 |F|^2 \right\} / \left\{ \int_{y_1}^{y_2} \left[\left| \frac{dF}{dy} \right|^2 + \alpha^2 |F|^2 \right] dy \right\}, \quad (273)$$

then with

$$c_r = M, \quad (274)$$

and

$$c_i^2 = S'^2 \leq S^2 = \int_{y_1}^{y_2} (U - c_r)^2 \psi(y) dy, \text{ since } \beta(y) \geq 0 \forall y \in [y_1, y_2], \quad (275)$$

all the conditions under which result (k) is valid are satisfied. We therefore have

$$\left(c_r - \frac{U_{\min} + U_{\max}}{2} \right)^2 + c_i^2 \leq \left(\frac{U_{\max} - U_{\min}}{2} \right)^2, \quad (276)$$

and hence the theorem.

Thus, the complex wave velocity c of an arbitrary unstable perturbation, manifesting in inviscid heterogeneous parallel shear flow, must lie inside the semicircle in the upper half of the $c_r c_i$ - plane with diameter in the range U and therefore the upper bound which is valid for the homogeneous case remains valid for the heterogeneous case as well. This result is known in the literature as Howard's semicircle theorem for heterogeneous inviscid parallel shear flow and it is in this context that Howard derived the result first.

We shall show some more applications of the semicircle theorem and its generalized version, in particular in the fields of mechanics of a solid body and the mechanics of a particle, in a subsequent paper.

While the manuscript was under preparation, JRG's youngest son was killed in a road accident. The paper is dedicated herewith to his memory.

References

- Banerjee M B 1971 Thermal instability of nonhomogeneous fluids: a mathematical theory-I. *Indian J. Pure Appl. Math.* 2: 257–269
- Banerjee M B 1972 Thermal instability of nonhomogeneous fluids: a mathematical theory-II. *Indian J. Pure Appl. Math.* 3: 226–240
- Banerjee M B, Kalthia N L 1971 On the rate of growth of disturbances in Rayleigh–Taylor model. *J. Phys. Soc. Jpn.* 30: 1494–1497
- Banerjee M B, Shandil R G 1995 A theorem on mean and standard deviation of a statistical variate. *Ganita* 47: 21–23
- Banerjee M B, Roy R, Gupta J R, Banerjee A 1978 A generalized biharmonic equation and its applications to hydrodynamic stability. *J. Math. Phys. Sci* 12: 19–33
- Banerjee M B, Katoch D C, Dube G S, Banerjee K 1981 Bounds for growth rate of a perturbation in thermohaline convection. *Proc. R. Soc. London A* 287: 301–304
- Banerjee M B, Shandil R G, Shirkot K S, Sharma D 1999 The importance of Tollmien’s counter example. *Studies Appl. Math.* 105: 191–202
- Chandrasekhar S 1961 *Hydrodynamic and hydromagnetic stability* (Oxford: Clarendon)
- Fjortoft R 1950 Application of integral theorems in deriving criteria of stability of laminar flow and for baroclinic circular vortex. *Geofys. Pub.* 17: 1–52
- Gupta J R, Sood S K, Shandil R G, Banerjee M B 1985 A generalized biharmonic equation and its applications to hydrodynamic stability. *J. Math. Anal. Appl.* 110: 31–50
- Howard L N 1961 Note on a paper of John W Miles. *J. Fluid Mech.* 10: 509–512
- Kapur J N, Rani 1995 Testing the consistency of given values of a set of probability distributions. *J. Bihar Math. Soc.* 26: 61–65
- Kapur J N 1996 On inequalities between moments of probability distributions. *Ganita* 47: 37–41
- Kochar G T, Jain R K 1979 Note on Howard’s semi-circle theorem. *J. Fluid Mech.* 91: 489–491
- Miles J W 1961 On the stability of heterogeneous shear flows. *J. Fluid Mech.* 10: 496–508
- Pellew A, Southwell R V 1940 On the maintained convective motion in a fluid heated from below. *Proc. R. Soc. London A* 176: 312–343
- Rayleigh J W S 1880 On the stability, or instability, of certain fluid motions. *Proc. London Math. Soc.* 9: 57–70
- Rayleigh J W S 1916 On the convective currents in a horizontal layer of fluid when the higher temperature is on the underside. *Philos. Mag.* 32: 529–546
- Shohat J A, Tamarkin J D 1963 *The problem of moments. Mathematical Surveys No. 1* (Amer. Math. Soc.)
- Veronis G 1965 On finite amplitude instability in thermohaline convection. *J. Mar. Res.* 23: 1