Sādhanā Vol. 26, Part 6, December 2001, pp. 549–590. © Printed in India

A set of pathological tests to validate new finite elements

K MALLIKARJUNA RAO and U SHRINIVASA*

Department of Mechanical Engineering, Indian Institute of Science,
Bangalore 560 012, India
e-mail: udipi@mecheng.iisc.ernet.in

MS received 3 August 2001; revised 6 November 2001

Abstract. The finite element method entails several approximations. Hence it is essential to subject all new finite elements to an adequate set of pathological tests in order to assess their performance. Many such tests have been proposed by researchers from time to time. We present an adequate set of tests, which every new finite element should pass. A thorough account of the patch test is also included in view of its significance in the validation of new elements.

Keywords. New finite elements; pathological tests; patch test.

1. Introduction

The finite element method is an approximation technique and thus entails errors. Hence researchers have designed several pathological tests to validate any new finite element. The tests should be able to display most of the parameters which affect finite element accuracy. A representative set of tests should include patch tests, beam, plate and shell problems. We propose a problem set to help developers of finite element programs to ascertain the accuracy of particular finite elements in various applications. This problem set cannot however be used as a bench mark for cost comparison since the problems are too small for this purpose. Inaccuracies of the elements are brought in by the presence of spurious mechanisms/rank deficiencies, locking (excessive stiffness for particular loadings and or irregular shapes), elementary defects like violation of rigid body property and invariance to node numbering etc. Parameters which affect accuracy are loading, element geometry, problem geometry, material properties etc. The member should be subjected to significant loadings and boundary conditions, for each type of deformation like extension, bending, in-plane shear, out-of-plane shear and twist etc. Care should be taken to test non-standard element shapes involving aspect ratio, skew, taper, and warp etc. Problem geometries like curvature and double curvature, which need many elements for representation, should also be considered. Slenderness ratio and the kind of support boundary conditions have to be considered. Poisson's ratio has strong effect on element accuracy when testing with incompressible materials as it approaches 0.5. Plasticity and material anisotropy also affect finite element accuracy. We provide details of the proposed pathological tests in table 1, and a list of pathological tests in table 2. with relevant figure and table numbers indicated therein.

*For correspondence

Table 1. Details of the proposed pathological tests.

Author(s)	Test proposed
Razzaque (1986)	Discuss the patch test as a tool for convergence, i.e continuity, capability to represent rigid body modes, capability to represent constant strain condition.
Melosh (1963)	Continuity, rigid body invariance property.
Irons & Barlow (1964)	Constant strain condition.
Taylor *et al* (1976)	Patch test as a tool for assessment of robustness of algorithms.
MacNeal & Harder (1985)	Element failure modes defined. Throwaway tests proposed. A beam with different element shapes subjected to extension should result in more than 98% accuracy. If not this element is to be thrown away.
Stricklin (1977), Bäcklund (1978), Gifford (1979), Lee & Bathe (1993)	Locking of distorted elements discovered
Pian & Sumihara (1984), Cheung & Chen (1988, 1992)	Proposed distortion sensitivity tests.
Robinson (1986)	Proposed aspect ratio sensitivity tests.
MacNeal (1951) & Levy (1953)	Alternative methods to solve plate problems where theoretical solutions are not available.
White & Abel (1989)	Plate patch tests and low energy deformation mode tests.
Belytschko *et al* (1985), White & Abel (1989)	Tests for checking membrane and shear locking.
Choi & Lim (1995)	General curved beam tests
Sorin & Bordan (1999)	Warping torsion test for beams
Claudio & Maria (1998)	Beams of varying cross section
Bigdeli & Kelly (1997)	Convergence tests, ill-conditioning tests
John *et al* (1999)	Convergence and stress concentration
Watson (1995)	Convergence and cracks in plane-strain
Sze (1992)	Frame invariance tests

2. Patch test

This test verifies whether elastic solid material behaviour is reproduced by a particular element by using an arbitrary 'patch' of elements and applies boundary displacements consistent with constant straining. We hope to establish here the current status of the test since its inception. Refer to figure 1 and table 3 for an overview of the patch test. To conduct the test, choose a finite element mesh consisting of an assemblage of arbitrary shaped elements, with at least one internal node (that is surrounded by elements). Assign displacements/rotations to boundary nodes corresponding to an arbitrary constant stress field. Internal nodes are free. Check for the correctness of displacements of internal nodes and stresses everywhere in the field. If exact values are reproduced, the patch test is said to have been passed.

2.1 Patch test for equilibrium and displacement models

Veubeke (1974) demonstrates that the patch test is contained in the variational formulation of the finite element methods at the assembling level which requires connecting loads to have no virtual work at the interface. Discretization of the zero virtual work condition provides

Table 2. List of pathological tests.

Figure(s) no.	Type of test	Type of loading	Results in +
2	Single element test	Tension, bending, shear	4
-	Eigenvalue test	Tension	
3	Constant strain patch tests	Tension	5
3	Higher order patch tests	Bending	5
4	3-D patch tests	Tension, bending	5
5	Slender beam tests	Tension, bending, in-plane shear, Out-of-plane shear	6
6	Curved beam tests	In-plane shear, Out-of-plane shear	7
7a	Pinched ring test	Point load	-
7b	Spring test	Axial load	-
8 (a), (b)	Warping torsion test	Twist	-
9	Twisted beam tests	In-plane shear, Out-of-plane shear	7
10	Cheung & Chen tests	Bending, shear	7
11	Tapered beam test	Tension, bending, end shear	11
-	Cantilever beam of varying cross-section	Uniform load	9a
-	SS beam of varying cross-section	Uniform load	9b
12	Aspect ratio, sensitivity tests	Bending	12
13	Distortion, sensitivity test	Bending, shear	13
14	Boussinesq problem	Point load	14 and 15
15	Cantilever beam tests*	End shear	16
16	Thick cylinder problem*	Radial pressure	17
17	Membrane problem	End shear	18
18	Cantilever plate test	Tip moment	19
		End shear	19
19	Plate patch test	Bending, Shear, Twisting	20, 21, 22
20	Rectangular SS plate Convergence test	Point load, UDL	23
20	Rectangular clamped plate	Point load convergence test	23
20	Rectangular plate locking test	Point load	23
21	Skew cantilever plate	UDL	24
22	Circular plate Convergence test	Point load, UDL	25
23	Axially loaded plate	Point load	26
24	Corner-supported plate	UDL	
25	Scordelis–Lo roof	Self weight	
26	Pinched cylinder	Point load	
27	Spherical shell	Point load	
28	Torsion bending	Moment	
29a	Convergence	UDL	Figure 29b
30	Convergence	Tension	27
31	Convergence	Tension	28, 29, 30
32a	Ill-conditioning	-	Figure 32b
33a	Stress polution	-	Figure 33b
34a, b	Frame invariance	-	-

* Tests with nearly incompressible materials. UDL: Uniformly distributed load. SS: Simply supported.

+ Results are in table number as given in column, unless specified otherwise

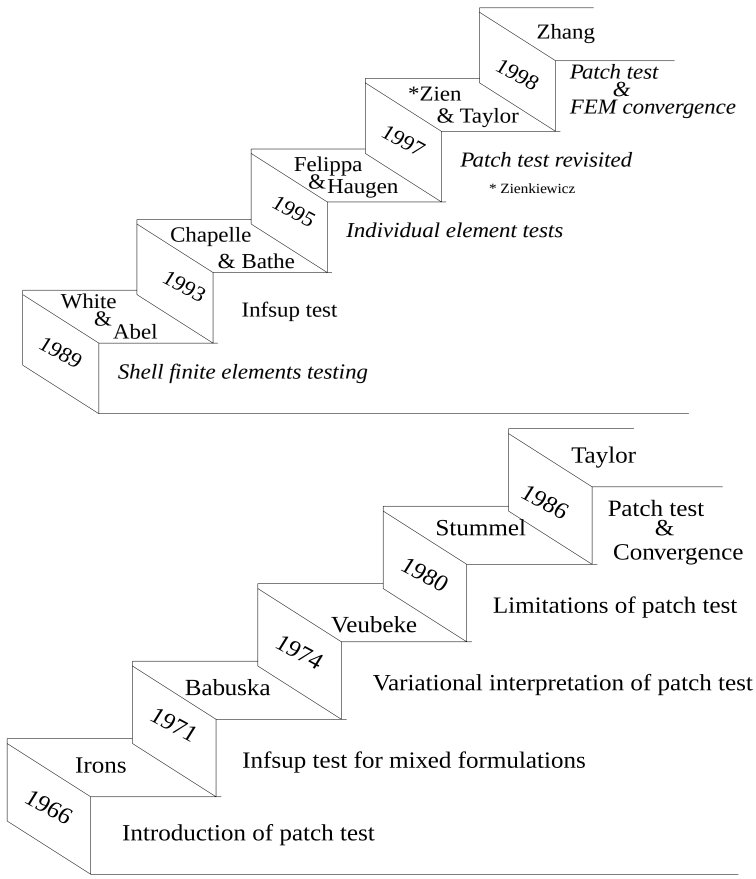


Figure 1. Patch test literature review.

opportunity for the systematic construction of 'non-conforming' elements that pass the patch test. Veubeke (1974) uses a functional that generates all the equations of linear elasticity theory in the form of variational derivatives and natural boundary conditions for a very general three-field principle. With the help of stress functionals, he verifies the patch test and shows that it is identical to the construction of hybrid models. He establishes that the higher rate of convergence is associated with higher degree of polynomial approximation of the field variable. Only the Kirchhoff plate and triangular elements are discussed. Also, extension of this method to three-dimensional cases proves difficult due to the complexity of the stress tensor.

Taylor *et al* (1976) develop a non-conforming element for stress analysis. Properties of some non-conforming elements for plane and three-dimensional analysis are presented. Starting from the minimum potential energy principle for the Q6 non-conforming element, they add two non-conforming nodes to the dependent variable, to arrive at QM6, the new element. It passes the patch tests only for rectangular shapes. Q6 passes the patch test for parallelogram shapes. Several numerical examples are considered and good results obtained.

Oliveira (1977) considers the patch test an advance in convergence analysis, as it provides a general criterion for non-conforming cases. He establishes that passing the patch test together with completeness guarantees convergence. He provides a mathematical basis for the patch

Table 3. Some significant contributions to patch test.

Author(s)	Remarks
Irons (1966)	Introduced patch test. It verifies whether an arbitrary patch of assembled elements reproduce exactly the behaviour of an elastic solid material when subjected to boundary displacements consistent with constant straining
Babuska (1971, 1973)	Ellipticity, and inf-sup condition
Irons & Razzaque (1972)	Experiences with the patch test
Brezzi (1974)	Ellipticity, and inf-sup condition
Veubeke (1974)	Variational interpretation of patch test
Sander & Beckers (1977)	The influence of choice of connectors in FEM
Oliveira (1977)	The patch test and the general convergence criteria
Stummel (1980)	Limitations of patch test: examples of non-conforming finite elements that passed the patch test of Irons (1966) and Strang & Fix (1973) but did not converge
Hayes (1981)	Stability test for under integrated and selectively integrated elements is proposed
Irons & Loikkanen (1983)	Convergence criterion is proved for all eligible FE formulations. Patch test is proved to be universal when it is combined with adequate test of stability
MacNeal & Harder (1985)	Standard set of problems to test finite element accuracy
Belytschko *et al* (1985)	Scordelis–Lo roof, hemispherical shell without holes, pinched cylinder
Taylor *et al* (1986)	Patch test and convergence
Razzaque (1986)	Patch tests
Zienkiewicz *et al* (1986)	Patch test for mixed formulations
Graf *et al* (1986)	Three dimensional thick shell elements using a hybrid/mixed formulation
Noor & Babuska (1987)	Techniques for assessing the reliability of finite element. Techniques for *a posteriori* error estimation and the reliability of the estimators
Verma & Melosh (1987)	Redefined the convergence requirements for finite element models. Model admissibility and preferentiality are introduced. New tests for assessing above requirements. Eigendata test, sub-divisibility test for membrane finite elements
Belytschko & Lasry (1988)	A fractal patch test
White & Abel (1989)	Testing of shell finite element accuracy and robustness
Babuska & Scapollo (1989)	Bench mark computation and performance evaluation of rhombic plate bending problem
White & Abel (1989)	A suite of tests to determine the accuracy and robustness of shell finite elements for linear elastic and geometrically nonlinear problems.
Smith (1990)	Benchmark tests for geometrically non-linear two dimensional beams.
Chapelle & Bathe (1993)	Inf-sup test
Felippa & Haugen (1995)	Patch test and evolved versions of the patch test. Individual Element Test (IET) decomposes element stiffness equations into basic and higher order parts. Original multi-element patch test is translated into IET. Finite element templates are developed. A template is a parametrised algebraic form that yields a continuum of convergent finite elements, of fixed type i.e., an element selected for a specific application and with a given dof configuration
Sze (1996)	Admissible matrix formulation for efficient construction of multifield finite element models which employ patch test to identify the constraints on stiffness/flexibility matrices
Zhang & Chen (1997)	The patch test conditions for some multivariable finite element formulations
Zienkiewicz & Taylor (1997)	Patch test is proved to meet convergence, validation and error estimation–“Patch test revisited”

test. An expression for the upper bound of error is established. This expression is used to show that error is bounded as the elements become smaller and smaller which is necessary for passing the patch test. He holds that passing of a higher order patch test or satisfying a higher order compatibility condition may not necessarily contribute to decreasing the error. He considers continuous structural models followed by hybrid discrete models generated by the potential energy method. He then develops an approximation theorem that leads to a general expression of the error.

Sander & Beckers (1977) show that non-conforming models that pass the patch test are equivalent to hybrid models. Patch test allows a rational way of defining connection modes between elements. A change in definition of the connectors is sought when there is no convergence. He holds, on the experience of his numerical experiments, that the patch test is too conservative. Some of the elements which do not pass the variational patch test, yield convergence in practice.

Taylor *et al* (1986) discuss various forms of the patch test and show that passing this test is necessary for convergence. The test is also applied to verify stability, asymptotic convergence and robustness. It is described as a guide to develop certain incompatible elements. They consider the patch test as applied to a finite element solution of a set of differential equations, and have defined convergence, consistency, and stability as follows. Convergence means that the approximate solution should tend to exact solution when the size of the element approaches zero. Consistency means that as the size of the element tends to zero, the approximate equation should represent the exact differential equation. Stability means that the solution of the discrete (approximate) equation is unique and no spurious mechanisms pollute the solution. Then they start with the Taylor expansion of the displacement within an element and establish the theoretical condition to be satisfied for passing the patch test. They describe, with several numerical examples, the cases of higher order patch tests, plate bending, incompatible elements and the weak patch test. By weak patch test they mean that the element passes the test only when its shape is a parallelogram that holds at mesh refinement to infinitely small size. They classify the patch tests into three forms.

In test A displacements are prescribed at all nodes and the FEM equation is verified. In test B displacements are prescribed only on the boundary nodes and the displacement at the internal node is verified. Test C verifies the boundary 'load' condition where the natural tractions are specified. It also verifies the stability condition. Though the element passes all the patch tests, sometimes convergence is very slow. For such cases, the higher order patch test is recommended. The order can be increased until the patch test is passed in a weak sense. With the use of higher order patch tests exact solution may be easily computed everywhere in the model. Thus accurate rates of convergence may be established.

Razzaque (1986) emphasizes the necessity of convergence tests and of the patch test in particular. The contribution of the patch test as a test of convergence and in the formulation and improvement of elements is discussed. The finite element method approximates the continuum mechanics problem possessing infinite degrees of freedom, the performance being dependent on the assumed displacement/stress functions. The element characteristics should satisfy some necessary conditions to ensure convergence of the computed results to true solution. The completeness condition requires continuity of displacements, rigid body mode representation and constant strain representation. Convergence is assured if completeness is satisfied. However performance in coarse mesh is not predictable.

Reduced integration and derivative smoothing introduce incompatibility. Irons & Razzaque (1972) suggest an alternative test for incompatible elements since constant stress patch test does not guarantee convergence.

The patch need not be of coarse size. Convergence criteria are only concerned with convergence to the true solution as the elements size tends to zero. Irons & Loikkanen (1983) present proof of the sufficiency of the patch test for convergence. The patch test is described as instrumental in the improvement and development of a number of new elements, like iso-p elements with reduced integration, triangular bending elements with derivative smoothing. It is used as a debugging tool to check the implementation. The test is inexpensive to carry out, and is far more practical than any other convergence test.

2.2 Patch test for mixed formulations

Several problems in solid and fluid mechanics cannot be solved with a single field variable efficiently and thus we have to use more than one variable. The fundamental difficulties in such problems are due to the constraints to which the variables are subjected. The key to whether a formulation is actually valuable lies in its convergence properties. The ellipticity condition and the inf-sup condition of Brezzi (1974) and Babuska (1971, 1973) govern the stability conditions for mixed formulations. This condition is nothing but an extension of the patch test to mixed formulations. Zienkiewicz *et al* (1986) give a simple form to algebraic conditions of Babuska–Brezzi and a conceptual application of patch tests to these conditions. They point out the instability of several formulations for incompressible problems and mixed displacement strain formulations of elasticity.

They start with the standard formulation, apply the non-singularity of the mixed model matrices and arrive at the Babuska–Brezzi condition for the example cases taken up. They also discuss the deviatoric stress, displacement and pressure approximation for incompressible elasticity. Thus they present a methodology to derive the necessary and sufficient conditions for convergence of mixed elements.

Chapelle & Bathe (1993) argue that the existing checks to test mixed formulations for stability by solvability tests and counting rules are deficient in predicting and are misleading as well. Hence they propose a numerical test which guarantees the fulfillment of the inf-sup condition. The inf-sup condition is obtained by extending the patch test formulations. They derive the inf-sup condition for incompressible elasticity and claim that the same holds and can be extended to other problems including all three-dimensional problems. The mathematical analysis starts with the potential energy expression which is minimized with respect to the displacement, subject to the constraint that bulk modulus goes to infinity in the limit. They however do not consider the ellipticity condition and feel it is not an immediate requirement. They illustrate several two-dimensional numerical examples which are in perfect agreement with exact results. This test can be applied even to macro elements. It is especially useful in situations like plate and shell formulations where mathematical results are still sparse. It is also useful in finite difference control volume schemes. This test is an empirical one like the patch test currently being used. The inf-sup condition can be a difficult criterion to apply because analytical expression needs to be derived whereas the proposed numerical test could be carried out with little effort.

Iosilevich *et al* (1997) evaluate the inf-sup condition for Reissner–Mindlin plate bending elements. They follow the idea of the numerical inf-sup test proposed by Chapelle & Bathe (1993) and apply the test to the MITC (Mixed Interpolation of Tensorial Components) and Chapelle & Bathe (1993) elements, and to the displacement-based elements. The MITC elements pass the test while the displacement-based elements fail. They choose the MITC family of elements due to the advantages offered, namely their applicability to moderately thick to very thin plates and the fact that only the displacement functions but not their derivatives need to satisfy inter-element continuity conditions. They consider the variational formulation with Dirichlet boundary conditions (clamped plates). Displacement based finite elements, MITC4 (4 stands for element nodes) elements, MITC9 and MITC*n* are treated in their work. The nine node displacement based element passes the complete test for uniform meshes but fails the test for a sequence of distorted meshes. The element therefore is not suitable for use in complex geometries and in general for nonlinear analysis. The MITC*n* family passes all tests including distorted meshes. The inf-sup test is passed only in case of uniform meshes and should be studied further with distorted meshes. Different kinds of distortions may cause different effects.

Zhang & Chen (1997) establish the constant stress patch test conditions (PTC) for analysing and ensuring convergence or robustness of some multivariable finite element displacement formulations (MFE) like incompatible displacement element and hybrid mixed formulations. MFE models include assumed stress, strain and incompatible displacement variables. The proposed PTC are applied to develop new general MFE formulations which pass PTC and a family of MFE for stress analysis. The MFE used in practice are incompatible displacement model, hybrid stress element model, assumed stress hybrid/mixed element model, assumed stress/strain generalised hybrid element model, and assumed strain/stress quasi conforming element model. If the MFE is rank-sufficient or does not have any spurious zero energy modes (ZEM), then the constant stress patch test (PT) can be used to assess its convergence and consistency. Zhang & Chen (1997) develop their theory based on minimum potential energy functional, the Hellinger–Reissner stationary principle and the Hu–Washizu stationary functional. Then they derive the general constant stress PT by including constant strain/stress condition, constant strain/stress preserving condition, strain/stress uncoupling condition and numerical accuracy ensuring condition. They apply numerical tests to incompatible displacement element Q6, generalised hybrid/mixed elements GH/ME, GQ4/QCS4, GQ6, GQM6, GNQ6 and generalised hybrid stress elements GHSE etc. for verifying whether PT is satisfied. They claim this work may be used to develop new elements with some effort.

Belytschko & Lasry (1988) present a fractal patch test to check element consistency. They argue that the conventional patch is too restrictive and some elements which are convergent are disqualified. They suggest a 'fractal patch test'. In this test the patch size is maintained constant and the distorted mesh is refined. No actual fractal dimension is involved though the term fractal is used. The patches are formed by a geometrical pattern repeated at different scales. It is similar to Mandelbrot's fractal curves. They refute the weak patch test recommended by Taylor *et al* (1986) and provide mathematical background. Their procedure of mesh refinement is described below. The parent patch is a square, divided into four quadrilaterals by a centre node. The position of the centre node governs distortion. This patch is used to further generate the centre node for each quadrilateral using Lagrange shape functions and bilinear mapping. They undertake the cases of two-dimensional isoparametric continuum element, four-node quadrilateral plate bending elements based on Mindlin–Reissner theory and an element based on a special operator. They bring out his viewpoint that elements which seem convergent but fail the conventional patch test and weak patch test, pass the fractal patch test. They claim that it serves as a quick assessment of the performance of the element and is a check of the weak patch test. It is not however an alternative to mathematical convergence. It provides valuable guidance when the mesh distortion influences convergence and the element fails the conventional patch test.

Stummel (1980) discusses a one-dimensional boundary value problem (Dirichlet boundary value problem) that passes the patch test of Irons (1966) but does not converge to the given boundary value problem. Hence success in the patch test is not sufficient for convergence nor it is a necessary condition. The same may be extended to two dimensions. Irons's (1966) patch test idea amounts to linear approximation of solution in a fine mesh, provided the matrix of approximating equations, K, is positive definite. It is proved by Stummel (1980) that it does not prevent the patches from a disproportionate response. He establishes a generalised patch test which yields the necessary and sufficient conditions for convergence of non-conforming elements. He also brings out the strange properties of non-conforming elements that do not pass the patch test or an equivalent convergence condition.

Irons & Loikkanen (1983) define a 'legalizable finite element formulation' as one for which a conforming displacement version is possible using the same nodal variables. They prove, by strain energy minimization consideration together with stability conditions, that the patch test is universal. They account for Stummel's (1980) counter example as a technical misunderstanding due to inadequate documentation.

Verma & Melosh (1987) redefine convergence requirements for finite element models by introducing the concepts of element model admissibility and preferentiality. Admissibility means that the displacement field includes all terms essential to its representation as Taylor's series expansion. Preferentiality means that constant strains are preferred over all other possible behavioural states as the number of nodal variables approaches infinity. It is a *sine qua non* for convergence. Admissibility alone is not enough to guarantee exact solution in the limit. Irons's (1966) patch test only ensures admissibility and therefore is insufficient. This is demonstrated by considering a patch of membrane elements subjected to constant strain in the *x–y* plane. They describe an eigendata test for checking the element model's admissibility. Using this test, admissibility can be checked at the element level directly without having to construct a patch. There exists a zero residual vector corresponding to each rigid body mode. If there are more zero eigenvalue modes than the rigid body modes, then the model is said to be rank deficient. The lowest eigenvalue of the stiffness matrix provides an estimate of the maximum relative discretization error. The sub-divisibility test determines when constant strain state is preferred over all other modes of behaviour in the limit of mesh refinement. This test together with the eigendata test provides the basis for convergence. It also provides information on the direction of convergence and convergence rate.

Felippa & Haugen (1995) start an unconventional approach in finite element research. Their aim is to construct high performance elements and element level error estimators. They discuss individual element test (IET) and bring out its relation to the patch test and the single element test. IET avoids the need to construct several patches. They develop finite element templates based on patch test and parametrized variational principles. A finite element template is a parametrized algebraic form which generates a continuum of convergent finite elements of fixed type, i.e. an element selected for a specific application and with a given degree of freedom configuration. However, for multidimensional elements, universal templates can become too complex and may lack physical transparency. With the new concept of templates, some mysteries like convergence, mixability, accuracy, locking, spurious modes and distortion sensitivity can be cleared. It also leads to the fundamental decomposition of element stiffness equations into higher order and basic parts. They describe the IET by taking up the case study of BCIZ plate-bending triangular element, which is nonconforming. However IET is stronger than really necessary. An element may fail IET but can pass the multi-element test.

Zienkiewicz & Taylor (1997) summarize clearly the patch test and its implications on a sound conceptual basis. In the early 1970's inf-sup condition is used to verify the convergence of finite element approximations. It is not easy to apply it practically. Patch test provides a checking process to verify approximation theory and also highlights programming errors occurring in implementation. It proves very useful for engineering practitioners. The stability is ensured by the absence of mechanisms and this is not guaranteed for mixed formulations. Later, since the late 1980's, patch test is extended to mixed formulations. They begin with the mathematical problem of linear ordinary/partial differential equations and establish the mathematical basis of patch test. They present several examples, namely displacement model and mixed model for linear elasticity, displacement model for beam on elastic foundation etc. They also describe the procedure to devise new elements based on the patch test with the examples of thick plates and triangles (with quadratic displacements, bubbles and linear shear). Other uses of the patch test for stress gradient recovery and as a measure of effectiveness of error estimators are discussed.

2.3 Summary of patch test

- Irons & Razzaque (1972) indicate that extravariational techniques like reduced integration and derivative smoothing introduce incompatibility.
- Several publications on this test exist and mathematical respectability is added by Strang & Fix (1973). The validity of the test is questioned by many, while these criticisms are also refuted by many others (Zienkiewicz & Taylor 1991).
- According to Oliveira (1977) the patch test is one of the most significant advances in convergence analysis just because it provides a general criterion for the non-conforming case. Passing the patch test together with completeness is indeed sufficient for convergence.
- Sander & Beckers (1977) show that nonconforming models that pass a patch test are equivalent to hybrid models. They hold that the patch test is too conservative. Some of the elements which do not pass the variational patch test yield convergence in practice, according to them.
- The introduction of the patch test by Irons (1966) is one of the greatest contributions to the 'science' and 'practice' of the finite element method (Taylor *et al* 1986). This test verifies whether the elastic solid material behaviour is reproduced by an arbitrary 'patch' of elements when boundary displacements consistent with constant straining are applied.
- Razzaque (1986) observes that though the patch test ensures convergence, performance in coarse mesh is not predictable.
- Verma & Melosh (1987) feel that Irons' patch test is not sufficient for convergence.
- According to Zienkiewicz & Taylor (1991) patch test is a necessary and sufficient condition to test convergence to assess the rate of convergence and to check robustness of an algorithm. Though the element passes patch test, convergence may be very slow unless a very large number of elements are used.
- Felippa & Haugen (1995) define finite element templates based on patch test and parametrized variational principles to avoid patch test. But this becomes too complex and lacks transperency for multidimensional elements.
- This test becomes a widely used procedure to check new finite elements and their coding. It provides a sound and systematic basis for the development of new elements.
- We strongly feel that the patch test should be invariably satisfied by all elements, and preferably also the higher order patch tests.

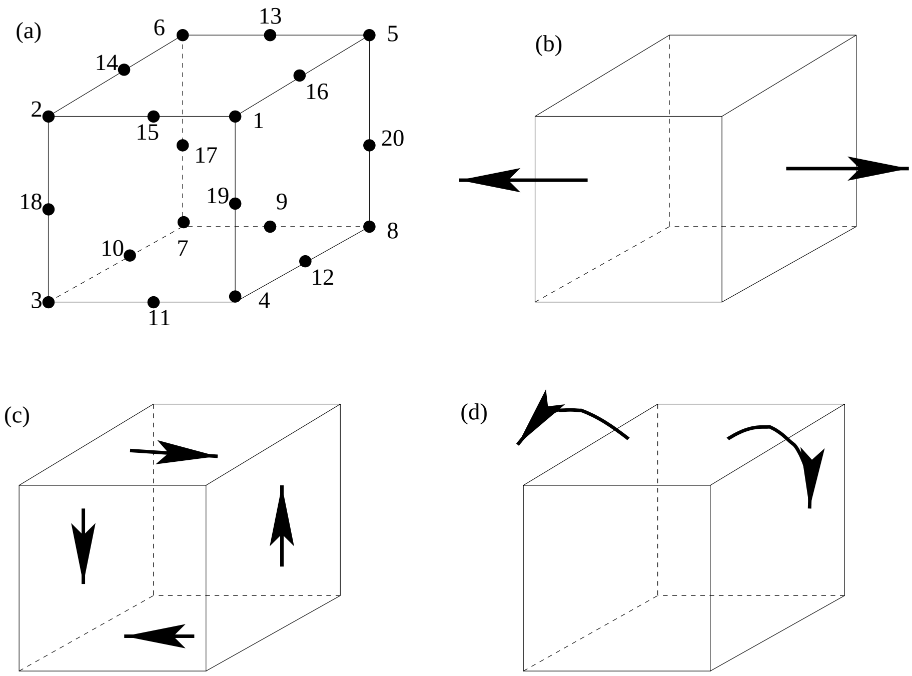


Figure 2. Single element tests for the bi-unit cube ($E = 1$; $\nu = 0.3$). **(a)** 20-node brick with nodal connectivity; **(b)** tension test; **(c)** shear test; **(d)** bending test.

3. Individual tests

3.1 Eigenvalue test

Eigenvalues of the stiffness matrix have to be determined. There should be as many zero values as the number of rigid body modes.

3.2 Single element tests

A bi-unit cube is subjected to tension, bending and shear as shown in figure 2. Theoretical values are given in table 4. Here we have shown the geometry and the nodal connectivity of the proposed element PN6X1 in figure 2a. Its performance is studied in § 4.

Table 4. Single element tests.

Tests		Theory
Tension	Displacement (u)	2.0
	Stress (σ_{xx})	1.0
Bending	Displacement (u)	2.0
	Stress (σ_{xx})	1.0
Shear	Displacement (v)	5.2
	Stress (σ_{xy})	1.0

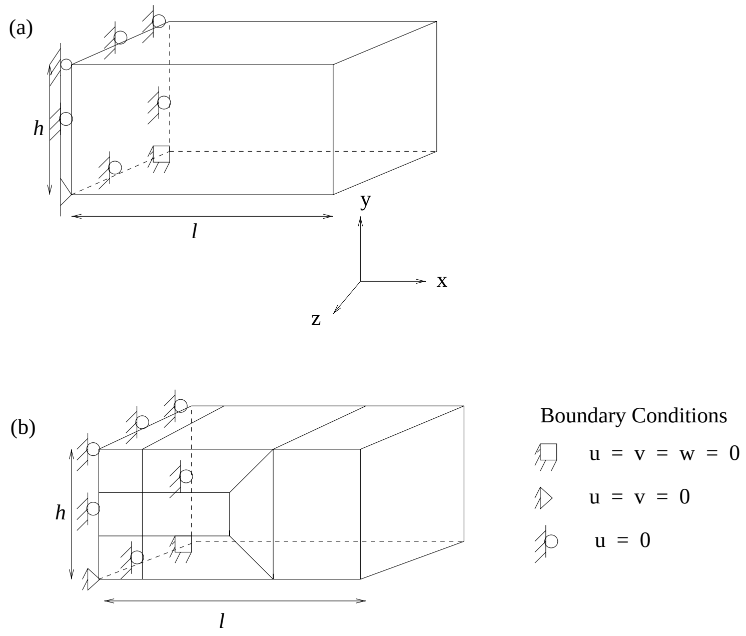


Figure 3. Constant strain and higher order patch tests: **(a)** regular mesh, and **(b)** irregular mesh. ($h = 3$, $l = 6$, $E = 1$, tension = 3, $M = 12$, $\nu = 1/4$.)

3.3 Patch tests

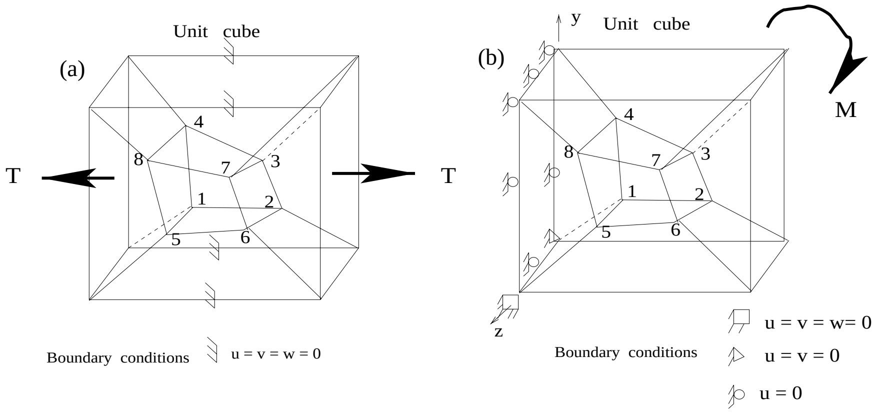
This test consists of constant and higher-order patch tests with regular, distorted and warped elements.

3.3a Constant strain patch tests: The configuration and loading given by Wilson & Ibrahimbegovic (1990) are considered. Choose regular and distorted patches given in figures 3a and b respectively. Theoretical values of the constant strain patch test are given in table 5

3.3b Higher-order patch tests: Configurations are shown in figures 3a and b with the bending load indicated. The computed stresses and displacements should be exactly equal to the expected values. This is to be expected as we have developed the element to represent linear stress field correctly. Table 5 gives the theoretical results.

Table 5. Patch tests.

	Constraint strain		Higher order strain (bending)	
Element type	Displacement	Stress	Displacement	Stress
Regular	6.0	1	72	6
Distorted	6.0	1.0	72	6
3-D warped	1.0	1.0	12	12



Location	X	Y	Z
1	0.249	0.342	0.192
2	0.826	0.288	0.288
3	0.850	0.649	0.263
4	0.273	0.750	0.230
5	0.320	0.186	0.643
6	0.677	0.305	0.683
7	0.788	0.693	0.644
8	0.165	0.745	0.702

Figure 4. Patch test with three-dimensional warped elements: **(a)** tension, **(b)** bending ($E = 1$, tension $= 4$, $M = 4\nu - 0.25$).

3.3c Patch tests with three-dimensional warped elements: Perform a truly 3-D patch test as proposed by MacNeal & Harder (1985). The details of the patch of elements are shown in figure 4. Theoretical results under uniform tension and constant bending moment are shown in table 5.

3.4 Beam tests

Consider straight, curved and twisted beams with the element shapes proposed by MacNeal & Harder (1985). Also consider deep beams in the seven configurations recommended by Cheung & Chen (1988), taper beams and beams of varying cross sections. An additional test, a beam with warped surfaces, not included here, is given by Chandra *et al* (2001).

3.4a Straight slender beam: Consider straight beams made up of six rectangular, trapezoidal and parallelopiped prism elements as shown in figure 5. Perform slender beam tests for in-plane shear, out-of-plane shear, unit tensile and constant bending moment loads. The element should predict displacements accurately for all loading and element shapes considered. Stresses should be accurate for tension and constant bending moment loading, while for endshear loading they should be close to the expected values. Error in predicting the stresses could be attributed to the presence of quadratic stress fields due to endshear load. MacNeal (1987) observes trapezoidal locking in his 4-node quadrilateral element in straight beams with trapezoidal mesh under in-plane loading. Sze & Ghali (1993) expect similar phenomenon with 8-node solid elements. Theoretical results are given in table 6.

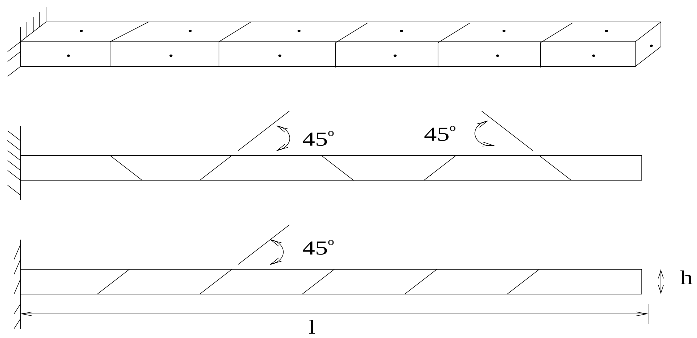


Figure 5. Slender beam tests: **(a)** Regular, **(b)** trapezoidal, and **(c)** parallelopiped meshes (tension $= 1$, $M = 100$, shear $= 1l = 60$, $h = 0.2$, depth $= 0.1$, $E = 1 \times 10^7$, $\nu = 0.3$).

3.4b Curved beam: Consider slender curved beam and apply unit in-plane and out-of-plane shear loads as recommended by MacNeal & Harder (1985). Figure 6 shows the details of the curved beam and table 7 contains the theoretical values.

3.4b (i) Tests for curved beam elements – The following are tests for curved beam elements.

- *Pinched ring test* – Choi & Lim (1995) use the pinched ring test to evaluate the performance of general curved beam elements. The details are shown in figure 7a.
- For *warping torsion test* for thin walled beams (TWB), Sorin & Bogdan (1999), suggest a twisting torsion test for L-shaped beams, with I-sections. The cross-section is oriented such that the central segment and the centre of the I are contained in the XOY plane. The out-of-plane displacement of the cross-section is constrained to be zero at points A and B. The beam is loaded with a torque $M = 10$ Nm acting along the OX axis at point C. Geometry and loading are given in figures 8a and b. Thereotical results are provided in table 8.

Table 6. Slender beam tests.

Elem. shape	Load type	Tip displacements (theory)	Stresses at root (theory)
Regular	I. shear	0.1081	9000.0
	O. shear	0.4321	18000.0
	Tension	3.0×10^{-5}	50.0
	Bending	0.2777	1500.0
Trapezoidal	I. shear	0.1081	9000.0
	O. shear	0.4321	18000.0
	Tension	3.0×10^{-5}	50.0
	Bending	0.2777	1500.0
Parallelopiped	I. shear	0.1081	9000.0
	O. shear	0.4321	18000.0
	Tension	3.0×10^{-5}	50.0
	Bending	0.2777	1500.0

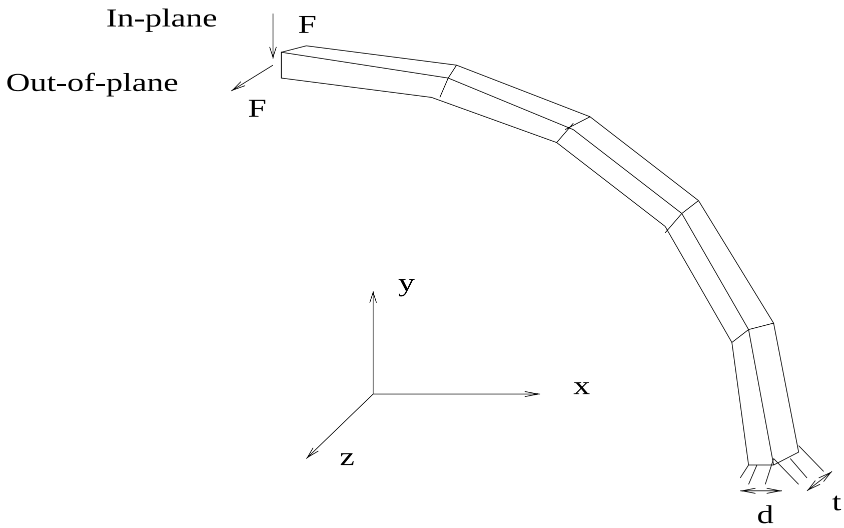


Figure 6. Curved beam tests ($d = 0.2$, $E = 1 \times 10^7$, $t = 0.1$, $F = 1$, $\nu = 0.25$, inner rad = 4.12, outer rad = 4.32, $E = 29 \times 10^6$, $\nu = 0.22$, width = 1.1, depth = 0.32, 0.0032, $F = 1$)

- *Spring test* – Choi & Lim (1995) suggest the following test. A spring becomes a typical example of a general curved beam, including out-of-plane bending, shear and torsional deformations. Figure 7b shows the geometry of a helical spring and its material properties. It shows one turn of a spring with a pitch angle of 10°. The deflection of the spring can be obtained from Castigliano's second theorem to be 3.13.

3.4b (ii) Tests for beams of varying cross section – Claudio & Maria (1998), propose three finite elements for the static analysis of Euler–Bernouli beams with varying rectangular cross-sections. They propose the following test cases for validating their elements.

- *Cantilever* – A cantilever beam with span $L = 10$ m, is subjected to a uniformly distributed load $q = 1\,\text{tm}^{-1}$. In table 9a the vertical displacement at the free end is listed, for various cross-section variation laws, and for various finite element discretizations. The first row refers to a beam with unit depth and linearly varying width between 2 m at the clamped end and 0.25 m at the free end (case A).

  The second and third rows refer to a beam with unit width and varying depth between 2 m at the clamped end and 0.25 m at the free end. In the second row the depth is assumed to vary according to the linear law (case B), whereas in the third row the variation law is supposed to be quadratic (case C). In all cases Young's modulus is equal to $E = 300,000\,\text{tm}^{-2}$.

Table 7. Curved beam and twisted beam test results.

	Theoretical deflection	
Loading	Curved beam	Twisted beam
In-plane shear	0.08734	0.005424
Out-of-plane shear	0.5022	0.001754

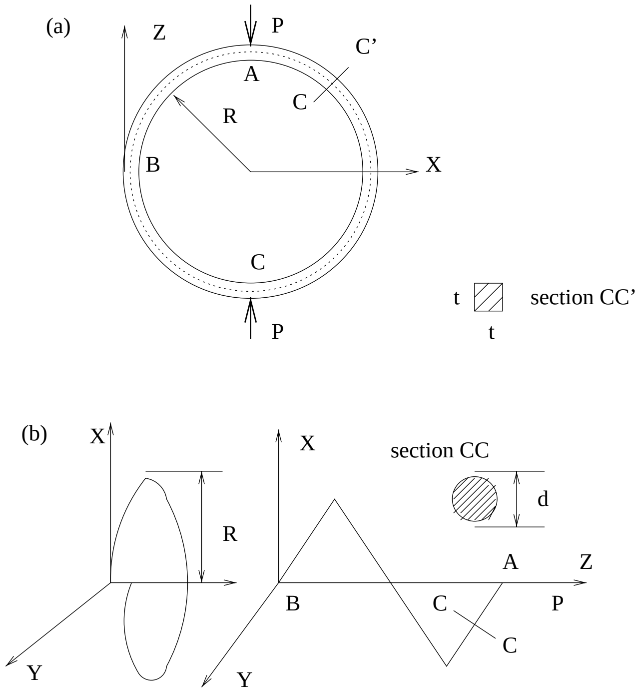


Figure 7. Pinched ring test ($P = 45.45$ kg, $R = 12.58$ cm, $t = 0.2388$ cm, $b = 2.54$ cm, $E = 0.7398 \times 10^6$ kgf/cm^2, $\nu = 0.3125$, $\beta^2 = 0.85$, $W_A = 3.258$). Helical spring test ($P = 45.45$ kg, $R = 25.4$ cm, $\alpha = 10^\circ$, $\nu = 0.3$, $\beta^2 = 0.886$, $d = 2.54$ cm, $E = 0.71 \times 10^6$ kgf/cm^2, deflection $= 7.95$ cm).

In the second column the vertical displacement is reported, obtained using a single finite element with varying cross-section, the third and fourth columns contain the same displacement obtained using two and five finite elements of the same kind.

- *Simply supported beam* – In table 9b midspan vertical displacement is reported for a simply supported beam with span $L = 10$ m. The first row refers to a beam with unit depth and linearly varying width between 0.5 m at the supported ends and 2 m at the midspan (case A).

  The second and third rows refer to a beam with unit width and varying depth between 0.5 m at the ends and 2 m at the midspan. In the second row the depth is assumed to vary according to the linear law (case B), whereas in the third row the variation law is supposed to be quadratic (case C). In all cases Young's modulus is equal to $E = 300,000\,\text{tm}^{-2}$. The columns descriptions are the same as in the first example.

3.4c Twisted beam: MacNeal & Harder (1985) propose this test to verify element performance for pre-twisted geometry. The twist between the two faces of each element along the length is 7.5°. See figure 9 for geometry and table 7 for the theoretical results.

3.4d Cheung and Chen tests: The configurations of the tests suggested by Cheung & Chen (1992) are shown in figure 10. The tests involve the effects of aspect ratio, geometric distortion and higher-order stress fields. Figure 10 shows the seven configurations with boundary conditions and loading. See table 10 for the theoretical results.

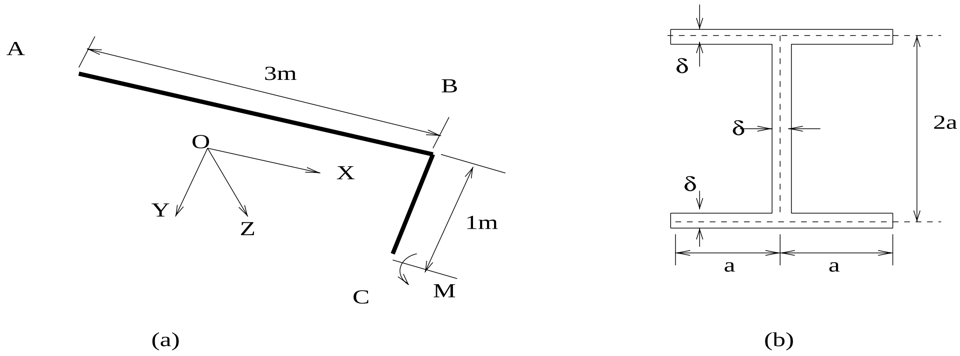


Figure 8. L-shaped thin walled beam with I-section ($\delta = 0.01\,\text{m}$, $a = 0.05\,\text{m}$, $M = 10\,\text{Nm}$).

Table 8. Results of warping torsion test for thin walled beams.

	Analytical	Semiloof elements	Macro elements
w(m)	0.291×10^{2}	0.289×10^{2}	0.289×10^{-2}
σ (N/m^2)	0.3×10^{6}	0.299×10^{6}	0.299×10^{6}
σ_w (N/m^2)	0.198×10^{7}	0.109×10^{7}	0.109×10^{7}

Table 9. Results for cantilever and simply supported beams.

(a) Cantilever

Case	$n = 1$	$n = 2$	$n = 5$
A	3.15715	3.15715	3.15715
B	1.54308	1.54308	1.54308
C	2.41424	2.41424	2.41424

(b) Simply supported

Case	$n = 2$	$n = 4$
A	3.73507	3.73507
B	2.86824	2.86824
C	2.11590	2.11590

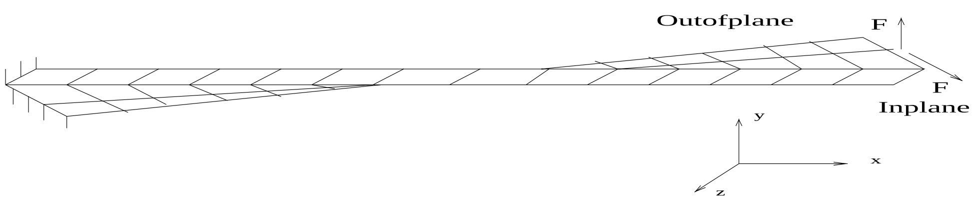


Figure 9. Twisted beam tests ($L = 12$, mesh: 12×2).

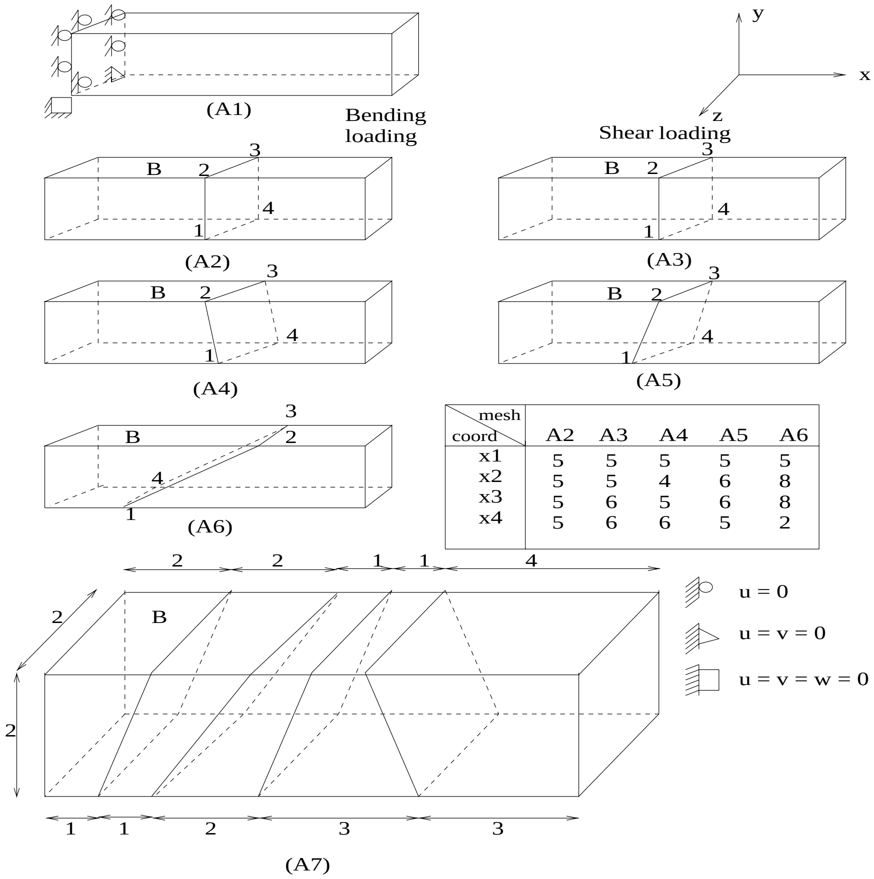


Figure 10. Cheung and Chen tests ($E = 1500$, $\nu = 0.25$, $M = 8000$, shear $= 600$).

3.4e Tapered beam: Many of the engineering structures include tapered beams. Consider a beam with a taper ratio 2:1 and length 20. Apply tension, pure bending and endshear. We observe the convergence of displacements and stresses for mesh size 6×1. Theoretical values are shown in figure 11 and table 11.

Table 10. Cheung & Chen tests for uniform bending and shear.

Mesh	Uniform bending		Uniform shear	
	Displacements (v) (theory)	Stresses at B (σ_{xx}) (theory)	Displacements (v) (theory)	Stresses at B (σ_{xx}) (theory)
A1	100	−3000	102.6	−2250.0
A2	100	−3000	102.6	−3375.0
A3	100	−3000	102.6	−3262.5
A4	100	−3000	102.6	−3375.0
A5	100	−3000	102.6	−3150.0
A6	100	−3000	102.6	−32700.0
A7	100	−3000	102.6	−4050.0

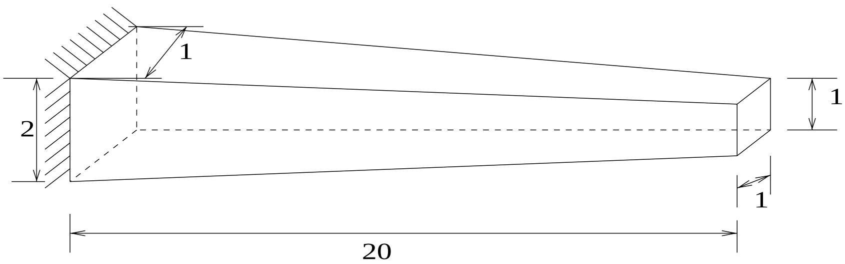


Figure 11. Tapered beam. Loading: Unit tension, bending and shear. Taper ratio $= 2:1$, $E = 1000$, $\nu = 0.3$).

3.5 Sensitivity tests

The aspect ratio sensitivity test and the variable distortion sensitivity tests are proposed by Robinson (1986) and Cheung & Chen (1992) respectively.

3.5a Aspect ratio sensitivity test: This test is to assess the sensitivity of an element for locking when the aspect ratio $(l/2b)$ is high. Take aspect ratios (length to depth) as high as 16 and apply the bending moment. See figure 12 for the test configuration and table 12 for the results.

3.5b Variable distortion sensitivity test: This test is recommended by Cheung & Chen (1992) to assess the sensitivity of an element when the distortion parameter e varies from 0 to 4. Refer to figure 13 for the configuration used. Theoretical values of tip displacements and the stresses at the fixed end are shown in table 13.

3.6 Boussinesq problem

Model one quadrant of the semi-infinite body shown in figure 14 (Bachrach 1987). Constrain the outer curved and bottom surfaces at $y = -45$. All the nodes in xy and yz planes are constrained in z and x directions respectively. Apply a point load of $P/4$ units. See tables 14 and 15 for the results.

3.7 Tests for near-incompressible materials

Cantilever beam and thick cylinder problems are useful to assess the element performance at the nearly incompressible limit where the Poisson ratio is close to 0.5.

Table 11. Tapered beam results (tip displacement).

Loading	Displacement	Stresses σ_{xx}
Tensile displacement (u)	0.0139	0.999
Bending displacement (v)	0.5950	5.920
Shear displacement (v)	6.5725	29.17

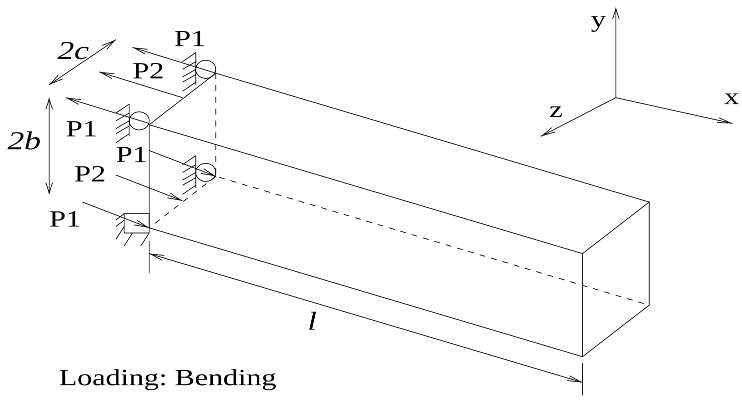


Figure 12. A pect ratio sensitivity tests ($b = c = 0.06$ m, $E = 206 \times 10^9$ N/m^2, $\nu = 0.25$, $M = 1656$ N).

Table 12. Aspect ratio sensitivity tests – tip deflection (v)§ stress.

Aspect ratio	Theory ($\times 10^{-6}$)	Theory
1	3.333	5.75
2	13.33	5.75
4	53.33	5.75
8	213.3	5.75
16	853.3	5.75

§ Source: Chandra & Prathap (1989)

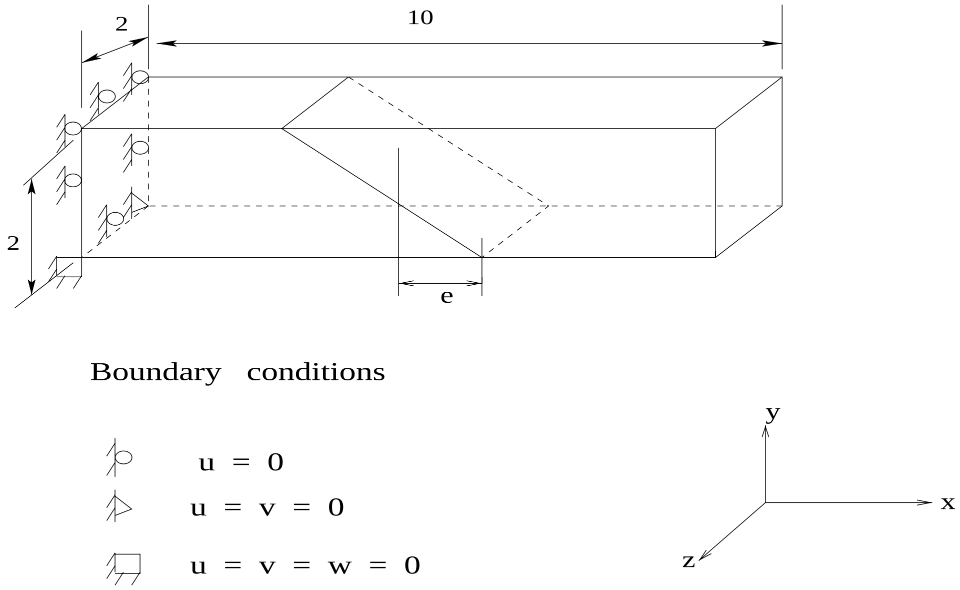


Figure 13. Distortion sensitivity tests ($M = 400$, $F = 600$).

Table 13. Variable distortion sensitivity tests.

Loading	Tip displacement(v)	Stresses at fixed end σ_{xx}
Bending	100	3000
Shear	103.75	3000

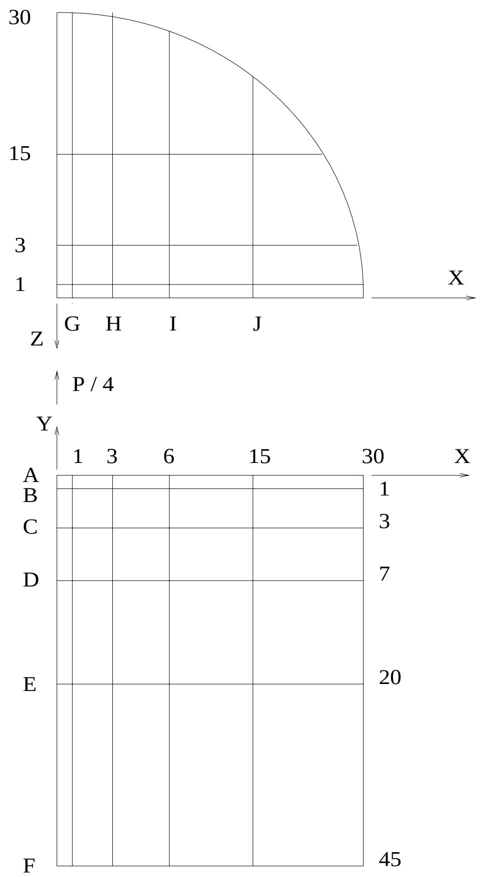


Figure 14. Boussinesq problem ($P = 1 \times 10^4$, $E = 1 \times 10^7$, $\nu = 0.3$).

Table 14. Boussinesq problem (v displacements).

Elements	Displacement along y-axis ($\times 10^{-5}$)				Displacement along x-axis ($\times 10^{-5}$)		
	A	B	C	D	G	H	I
Theory	∞	49.6	16.06	7.09	29.0	9.66	4.83

Table 15. Boussinesq problem (stress σ_{yy} averaged between two nodes along the y-axis).

Elements	AB	BC	CD	DE
Theory	19100	1194	191	26

3.7a Straight cantilever beam with different Poisson's ratios: Consider a straight beam with regular and trapezoidal elements subjected to endshear loading as shown in figure 15. Table 16 gives theoretical results.

3.7b Thick cylinder: MacNeal & Harder (1985) propose this test to verify the effect of near incompressibility. Figure 16 shows the quarter cylinder with boundary conditions and loading. Calculate the displacements for $\nu = 0.25$, 0.49, 0.499 and 0.4999 (table 17).

3.8 Membrane problem

Membrane locking is usually examined by the twisted beam test by reducing the thickness 100 times and checking for deterioration in displacement. We also find the following tests in the literature provided by Bergan & Felippa (1985). Consider a membrane with skewed elements and apply unit distributed endshear load as shown in figure 17. Table 18 gives the expected results.

3.9 Single element cantilever plate tests

Bretl & Cook (1979) propose the tests shown in figure 18 involving beams and plates in bending. Table 19 provides the expected results.

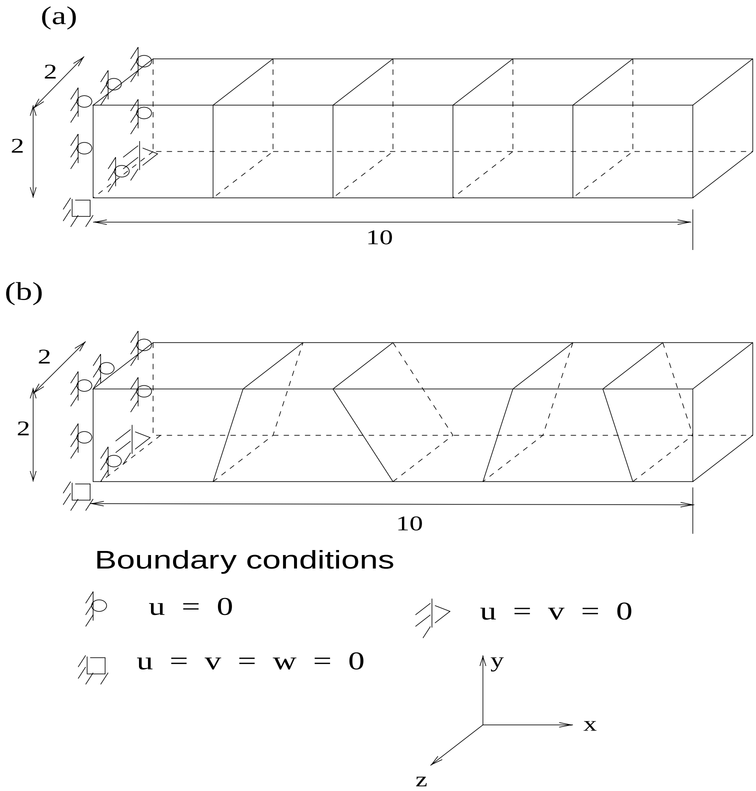


Figure 15. Cantilever beams with different Poisson's ratios: **(a)** Regular, and **(b)** trapezoidal meshes (Vertical endshear, $F = 300$).

Table 16. Cantilever beam for different Poisson's ratios.

Mesh	Poisson's ratio	Theory ($\times 10^{-3}$)
Regular	0.3	7.7340
	0.4	7.7520
	0.49	7.7682
	0.499	7.7698
	0.4999	7.7699
	0.49999	7.7699
	0.499999	7.7699
	0.4999999	7.7699
Trapezoidal	0.3	7.7340
	0.4	7.7520
	0.49	7.7682
	0.499	7.7698
	0.4999	7.7699
	0.49999	7.7699
	0.499999	7.7699
	0.4999999	7.7699

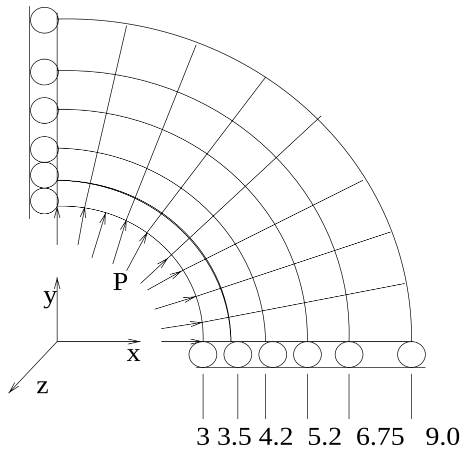


Figure 16. Thick cylinder (thickness $= 1$, $E = 1 \times 10^3$, mesh: 9×5, $\nu = 0.03, 0.49, 0.499, 0.4999$, $P = 1$)

Table 17. Thick cylinder-radial displacements.

Poisson's ratio	0.25	0.49	0.499	0.4999
Theory ($\times 10^{-3}$)	4.5825	5.0399	5.0602	5.0623

Table 18. Cook's membrane problem.

	Deflection at C	Stress at A	Stress at B
Best results*	23.90	0.236	−0.201

* Bergan & Felippa (1985)

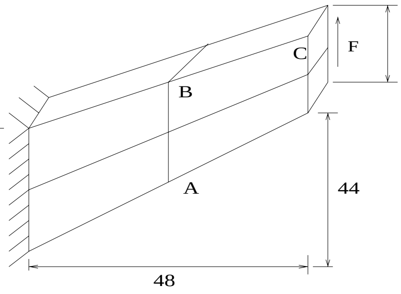


Figure 17. Cook's membrane problem ($E = 1$, thickness $= 1$, $\nu = 0.333$, $F = 1$, mesh: 2×2).

3.10 Plate patch tests

Zienkiewicz *et al* (1993) observe the following: "*In plate problems the importance of the patch test in both design and testing of the elements is paramount and this test is never to be omitted*". If the element passes the plate patch tests, convergence is assured in plate and shell problems. We present the tests proposed by White & Abel (1997). Figure 19 shows the three types of meshes which are used, the boundary conditions and the loading. Apply constant bending moment, out-of-plane shear and twisting moment loads.

3.10a Constant bending moment patch test: Figure 19d shows boundary conditions and loading. Calculate vertical deflection and bending stress σ_{xx} at the tip of the plate. Table 20 gives the theoretical results.

3.10b Out-of-plane shear load patch test: Figure 19e shows boundary conditions and end-shear loading. Calculate vertical deflection of the tip, and bending stress σ_{xx} at the root of the plate. Table 21 provides theoretical results.

3.10c Constant twisting moment patch test: Figure 19f shows the boundary conditions and figure 19g the twisting moment loads. Table 22 gives the results.

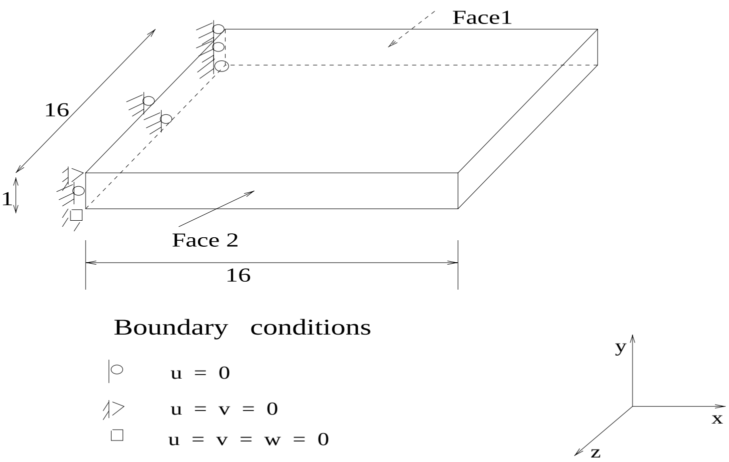


Figure 18. Single element cantilever plate tests ($E = 1 \times 10^6$, $\nu = 0.25$, vertical endshear, $F = 10/3$, endmoment, $M = 100$).

Table 19. Single element cantilever plate tests.

Tests	Constraints		Theory
Tip moment	z†	Displacement	0.0096
		Stress	37.5
	z‡	Displacement	0.0090
		Stress	37.5
Tip shear	z†	Displacement	0.01024
		Stress	60.00
	z‡	Displacement	0.0096
		Stress	60.00

† All the nodes of faces 1 and 2 are not restrained in z-direction
‡ All the nodes of faces 1 and 2 are restrained in z- direction

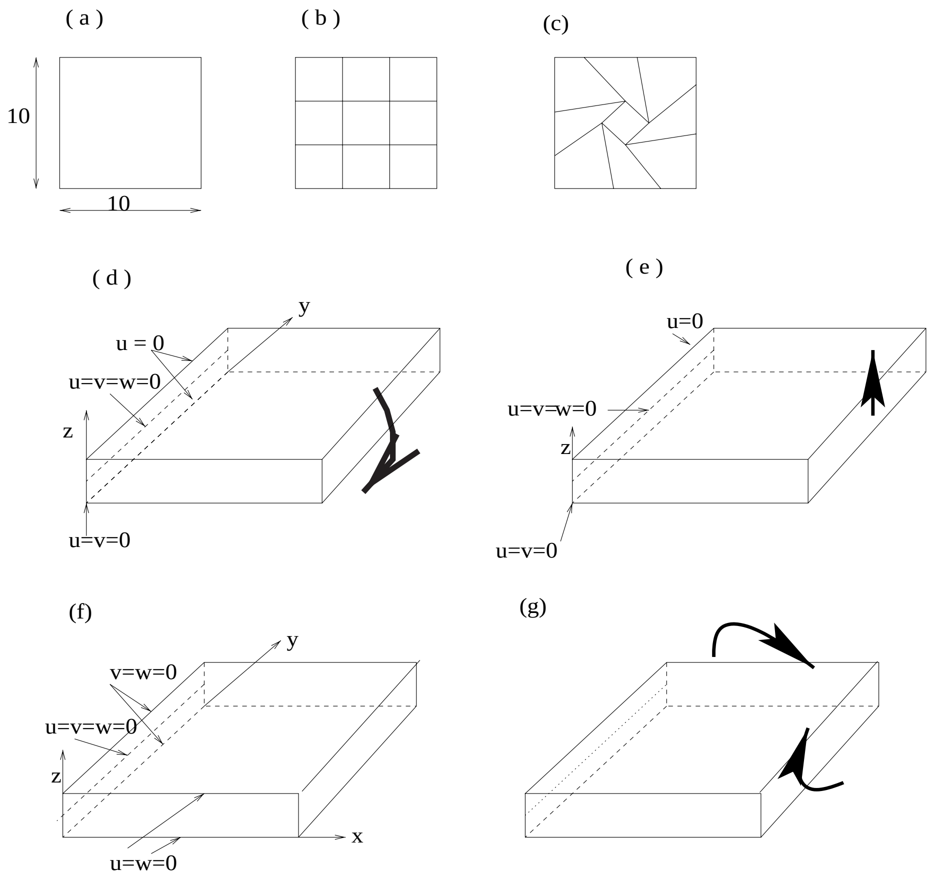


Figure 19. Plate patch tests: **(a)** Regular 1×1, **(b)** regular 3×3, **(c)** irregular 3×3, **(d)** constant bending moment test, **(e)** out-of-plane shear load test, **(f)** twisting test: boundary conditions, **(g)** twisting test: loading ($E = 1 \times 10^4$, $\nu = 0.3$, $t = 1$ (d), $M = 2$ (e), $F = 4/3$ (e), $T = 2$).

Table 20. Constant bending moment patch test for plates.

Tip deflection w (theory)	0.012
Stresses σ_{xx} (theory)	1.2

Table 21. Out-of-plane shear load patch test for plates.

Tip deflection w (theory)	0.16
Stresses σ_{xx} (theory)	24.0

Table 22. Constant twisting moment patch test for plates.

Tip deflection w (theory)	0.0312
Stresses τ_{xy} (theory)	1.2

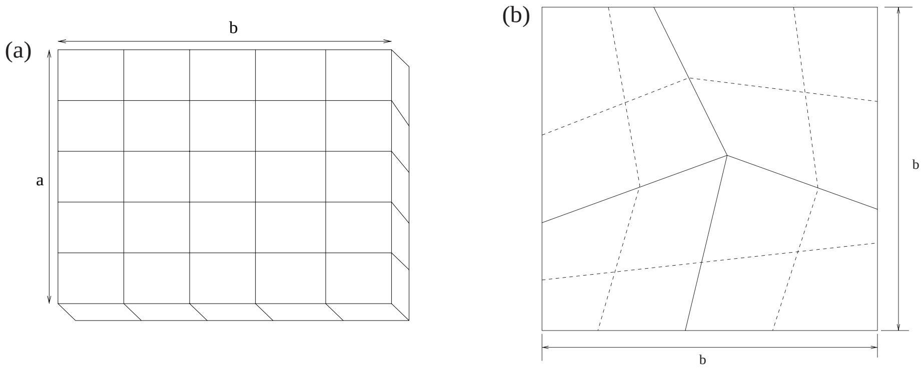


Figure 20. Rectangular plate for convergence and locking tests **(a)** Mesh: $N \times N, a = 2$, $b = 2$ or 10, thickness $= 0.01$, $E = 1.7472 \times 10^7$, $\nu = 0.3$, uniform pressure, $q = 1 \times 10^{-4}$, point load, $P = 4 \times 10^{-4}$ at the plate centre. **(b)** Distorted mesh.

Table 23. Theoretical solutions for rectangular plates.

Boundary supports	Aspect ratio	Displacement at centre	
	b/a	Uniform pressure	Concentrated load
Simple	1.0	4.062	11.60
Simple	5	12.97	16.96
Clamped	1.0	1.26	7.23
Clamped	5.0	2.56	7.23

3.11 Convergence tests for rectangular plates

Study of convergence with rectangular plates has become almost a *de facto* standard test for newly developed elements. Consider thin plates having aspect ratios 1 and 5 (mesh $N \times N$ for full plate) with simply supported and clamped edges, subjected to point and uniformly distributed loads. Figure 20 shows the rectangular plate with material properties. For point load cases, apply the load at the centre node. For the uniformly distributed load calculate the equivalent nodal loads for each element separately. Normalize the predicted deflections with the expected values given by MacNeal & Harder (1985). Expected displacement at the centre for each case is listed in table 23.

3.12 Locking tests for rectangular plates

Consider a square plate with $a = b = 2$ units, as shown in figure 20 (MacNeal & Harder 1985). Use 8×8 (full plate) distorted mesh, with simply supported and built-in edges, subjected to point and uniformly distributed loads. Keep the sides of the plate constant and vary only their thicknesses to perform locking tests for length to thickness ratios varying from 100 to 10000. Many available elements lock due to the presence of excessive shear for high a/t ratios. Expected results for $a/t = 10$ are given in table 23. For other a/t values, results may be extrapolated suitably.

3.13 Convergence tests for a skew cantilever plate

Zienkiewicz *et al* (1993) and also a few others attempt this problem using plate elements.

The skew cantilever plate shown in figure 21 is subjected to uniformly distributed transverse load. Table 24 gives the theoretical results (White & Abel 1989).

3.14 Convergence tests for circular plates

To assess the performance in shear locking use distorted elements, reduce the thickness 100 times and check for any deterioration in displacement. Consider circular plates with simply supported and clamped edges subjected to point and uniformly distributed loads (Timoshenko

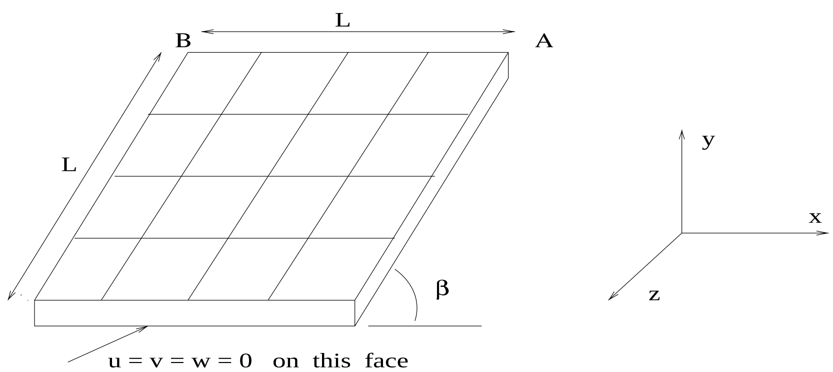


Figure 21. A skew cantilever plate under unit uniform pressure ($q = 1$, $L = 100$, $E = 100$, $t = 4$, $\nu = 0.3$, $\beta = 70^\circ, 50^\circ, 30^\circ$)

& Kreiger 1959). Owing to their symmetry, consider only one quarter of the plate and apply a unit point load at the centre to represent plates with central loads. Consider a quarter of the plate with 3, 12 and 48 elements. Normalize the theoretical deflections at the centre of the plate with Kirchoff solution given in table 25. Figure 22 gives the configuration.

3.15 Tests for low-energy spurious modes

Generally, the stiffness matrix is subjected to the eigenvalue test to detect the spurious energy modes. The following tests are also suggested by other researchers Low-energy deformation modes usually appear in quadrilateral elements while using reduced integration which sometimes leads to singular stiffness matrix. Verhegghe & Powel (1986) and many other authors discuss control of zero-energy modes. According to White & Abel (1989) the validation of an element is not complete without tests to check whether these modes are sufficiently restrained to prevent any degradation of results.

3.15a Axially loaded plate: Consider a flat plate loaded by a concentrated axial load. Restrain one side of the plate as shown in figure 23. If there are zero-energy modes in the element, they lead to erroneous displacements and stresses away from the restrained ends. Table 26 gives the theoretical results.

3.15b Corner-supported plate : Consider a corner-supported plate (figure 24), which is known to be extremely sensitive to the existence of zero-energy modes. We consider a square plate of variable length and unit thickness, use a 16×16 (full plate) mesh and apply equivalent nodal forces to represent the uniformly distributed load. The expected value of displacement is 2.53×10^{-9} units for $a/t = 10$.

3.16 Shell tests

Belytschko *et al* (1989) discuss the common drawbacks of the 9-node Lagrangian shell elements, which lock in shear and membrane action. Shear-locking occurs in the elements when they are loaded in pure bending and the elements generate spurious transverse shear energy. This can be avoided by using selective reduced integration which however causes difficulties for some boundary conditions, such as corner supporting of plates (the global stiffness matrix becomes singular). On the other hand, membrane-locking occurs when the element is subjected to pure bending. Then again the element exhibits spurious or parasitic membrane stresses. Since the membrane stiffness is much higher than the bending stiffness, the appearance of parasitic membrane energy stiffens the element, and is called membrane-locking. To test whether the newly developed element locks in shear or in membrane action, earlier workers have proposed problems such as the Scordelis–Lo roof, the pinched cylinder and the spherical shell, which are discussed below.

3.16a Scordelis–Lo roof: This is a single-curved shell proposed by MacNeal & Harder (1985). The cylindrical concrete shell roof with diaphragm boundary conditions is shown in figure 25. Calculate and apply equivalent nodal loads to represent the self-weight of the roof. The theoretical midside vertical displacement is 0.3086.

Table 24. Displacements at the tips of skew cantilever plates.

$\beta = 70^\circ$		$\beta = 50^\circ$		$\beta = 30^\circ$	
$w_A.Et^3/qL^4$	$w_B.Et^3/qL^4$	$w_A.Et^3/qL^4$	$w_B.Et^3/qL^4$	$w_A.Et^3/qL^4$	$w_B.Et^3/qL^4$
1.42970	1.04420	1.18180	0.54680	0.85020	0.15620

w_A and w_B are deflections at points A and B respectively

Table 25. Deflection at the centre of the circular plate: theory.

Boundary supports	Loading	Deflection†
Simple	Point	5.050
Simple	Uniform pressure	39.832
Clamped	Point	1.989
Clamped	Uniform pressure	9.787

† Source: Timoshenko & Krieger (1959)

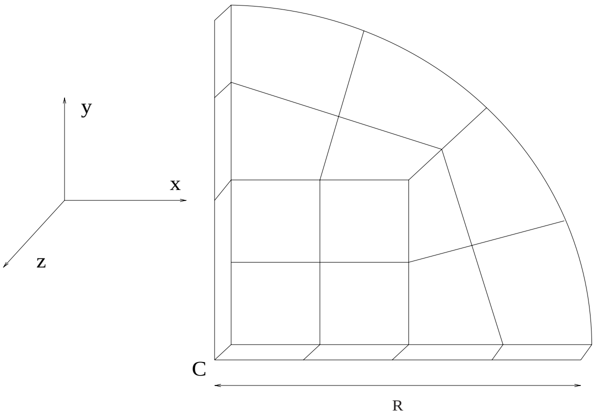


Figure 22. Circular plate (mesh: 12, $E = 10920$, $t = 0.01$, $\nu = 0.3$, $R = 5, 50$, $P = 1$ at C).

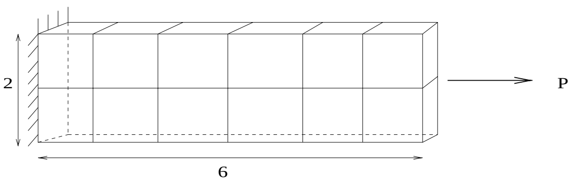


Figure 23. Axially loaded plate ($P = 1$, $t = 0.1$, $E = 1 \times 10^7$, $\nu = 0.2$).

Table 26. Axially loaded plate.

Element	Displacements ($\times 10^6$)	Stresses (σ_{xx})
PN6X1	**3.31**	**4.751**
LAG9 ‡	4.161	†
9-node V-P ‡	4.134	†

† Range of results 4.917 to 5.038

‡ Source: White & Abel (1989)

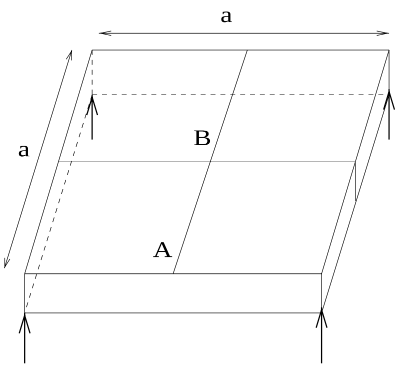


Figure 24. Corner-supported plate ($a = 10$ and 1000, $t = 1$, $E = 1 \times 10^4$, mesh : 16×16, $\nu = 0.3$, uniform load, $q = 1 \times 10^{-8}$).

3.16b Pinched cylinder: Consider a thin cylindrical shell with diaphragm boundary conditions and apply a point load at the centre of the cylindrical surface. In the pinched cylinder, shear-locking is more severe than membrane-locking. Since it is symmetrical, model one-eighth of the cylinder with $N \times N$ mesh as shown in figure 26. The expected deflection under load is 1.82488×10^{-5}.

3.16c Pinched spherical shell: This is a doubly curved shell test proposed by MacNeal & Harder (1985). Model only a quadrant of the hemisphere with $N \times N$ mesh as shown in figure 27. The skewness of the element aggravates both membrane- and shear-locking problems. The expected deflection under load is 0.0924.

3.16d Torsion bending of thin sections: This test is presented in figure 28, and provides an important check of the accuracy and correctness of shell elements for problems which involve facets and junctions of shell surfaces. Here a thin Z-section is considered as given by White & Abel (1989).

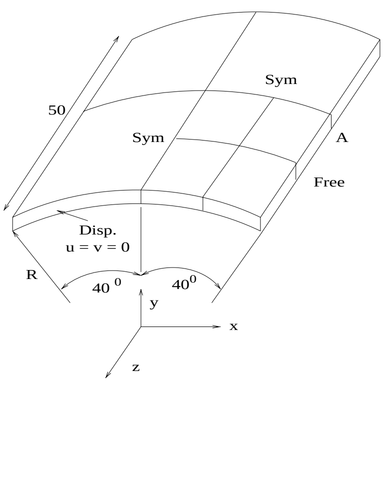


Figure 25. Scordelis–Lo roof (mesh: $N \times N$, $R = 25$, length $= 50$, $t = 0.25$, $E = 4.32 \times 10^8$, self-weight 90/area, $\nu = 0.01$).

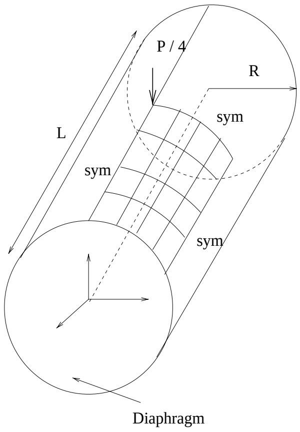


Figure 26. Pinched cylinder with rigid end diaphragms ($R = 300$, $L = 600$, $t = 3$, $\nu = 0.3$, $E = 30 \times 10^6$).

3.17 Convergence tests

Bigdeli & Kelly (1997) propose a test for assessing the convergence characteristics. They choose an L-shaped domain with stress singularity shown in figure 29a. They study h-convergence and p-convergence under plane stress conditions. Results expected are presented in figure 29b. For p-convergence a sequence of meshes consisting of linear, quadratic, cubic and quartic elements are produced, keeping the number of elements fixed at 27. For h-convergence meshes containing 27, 108, and 432 four node bilinear elements are constructed.

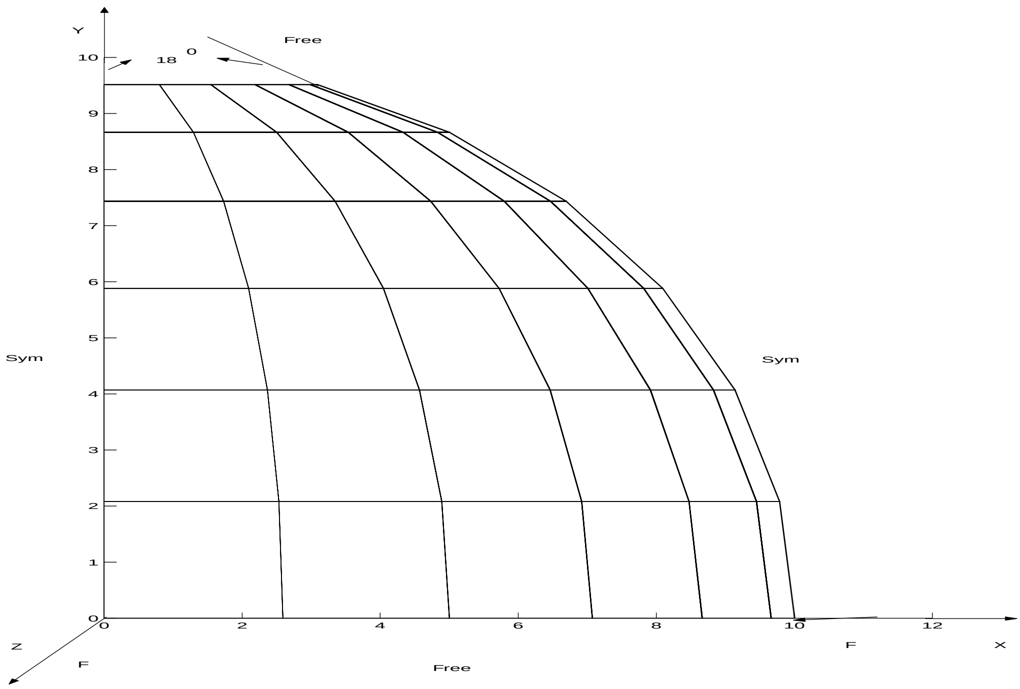


Figure 27. Spherical bending of thin sections ($R = 10$, $F = 2$; $t = 0.04$, 0.02; $\nu = 0.03$; $E = 6.825 \times 10^7$).

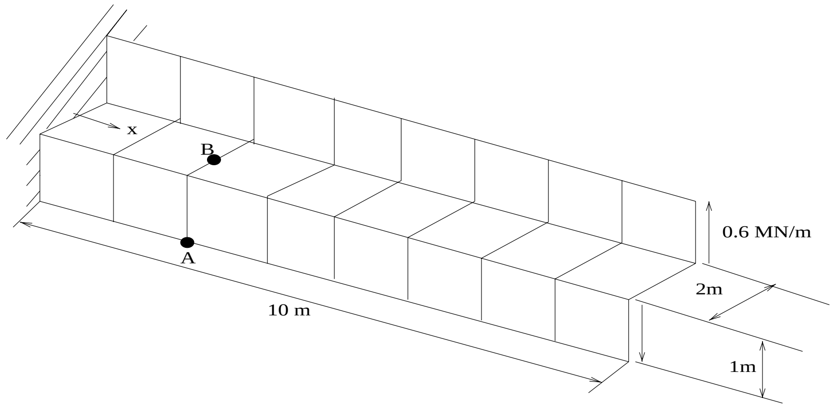


Figure 28. Torsion bending of thin sections ($t = 0.1$, $E = 210$ GPa, $\nu = 0.3$, $\sigma_{xxA} = 108$ MPa, $\sigma_{xxB} = 36$ MPa).

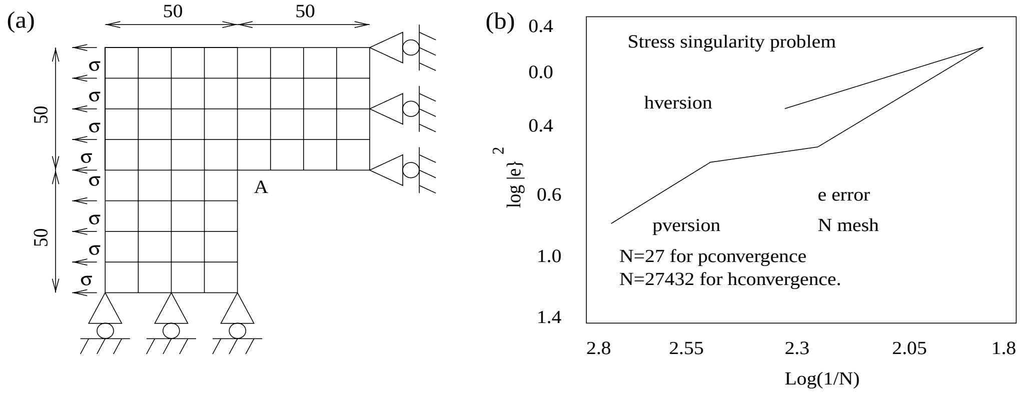


Figure 29. L-shaped domain.

Table 27. Stress concentration factors (analytical).

$1/a$	100	10	8	6	4	3	2	1
$\nu = 0$	3.000	2.878	2.824	2.729	2.545	2.389	2.169	1.889
$\nu = 0.5$	3.000	2.937	2.908	2.855	2.743	2.639	2.476	2.231

Table 28. Stress intensity factors for inclined crack in a plate.

a/w	0.1	0.2	0.3	0.4	0.5	0.6
$k_1/(\pi a)^{1/2}$	0.5046	0.5181	0.5406	0.5719	0.6119	0.6611
$k_2/(\pi a)^{1/2}$	0.5018	0.5072	0.5162	0.5290	0.5458	0.5674

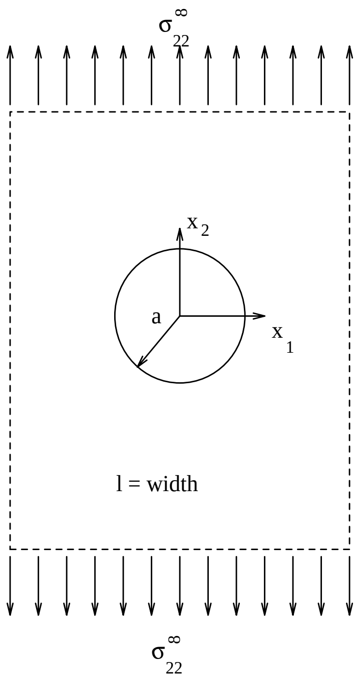


Figure 30. Plate with stress concentration for convergence.

3.17a Stress concentration problem: In order to examine the convergence properties of the various finite elements for a 2-D problem, John *et al* (1999) consider an infinite solid containing a circular cylindrical hole. The solid is subjected to remote uniform tension, as shown in figure 30. Table 27 provides the analytical results in the form of stress concentration factor at the edge of the hole; values are obtained as a function of normalized hole radius a/l for Poisson's ratios $\nu = 0$ and $\nu = 0.5$.

3.17b Tests for cracks in plane strain: Cracks of different geometries are considered here (Watson 1995).

- *Straight crack in an infinite domain* – Consider a crack of length 2 mm in an infinite elastic domain, in which there is uniaxial tension of 1 MPa normal to the crack at infinity. The exact value of the mode I stress intensity factor at the end of crack is $\pi^{0.5}$ and is independent of the elastic constants, which are arbitrarily chosen as $E = 10000$ MPa, $\nu = 0.0$.
- *Inclined crack in a plate* – Figure 31a shows a plate with a buried crack inclined at 45^0 to the axis of loading. Theoretically the stress distribution in the plane of the plate is independent of whether there exists a state of plane strain or of plane stress, and is also independent of the elastic constants. Young's modulus and Poisson's ratio are taken to be 210 000 MPa and 0.3 respectively. The stress intensity factors for symmetric and anti-symmetric modes are listed in table 28 (Murakami 1987).
- *Curved crack in an infinite domain* – Figure 31b shows the geometry and dimensions. Stress intensity factors calculated by Muskhelishvili (1953) and Tada *et al* (1973) are reported in table 29 for comparison. A crack of radius 1 mm is considered in an infinite elastic domain with uniaxial stress 1 MPa at infinity, with Young's modulus and Poisson's ratio arbitrarily taken to be 10000 MPa and 0.0 respectively.

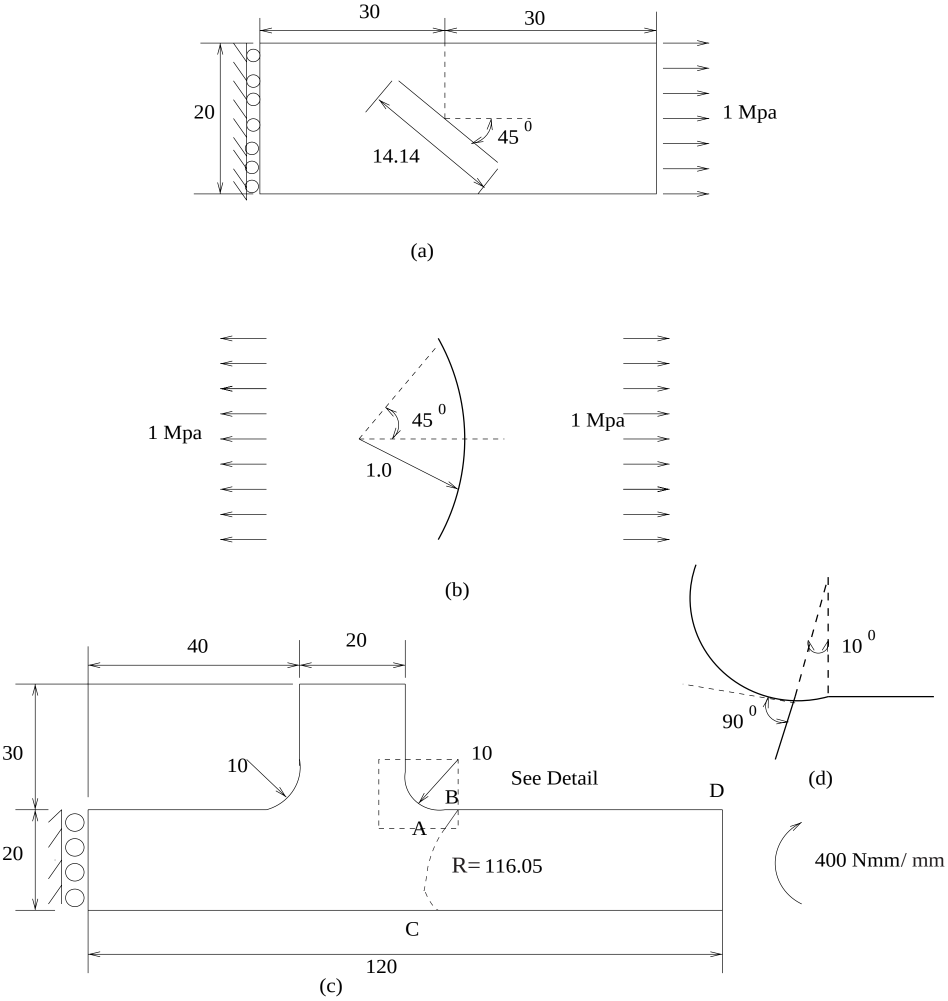


Figure 31. Convergence study of cracks in plane strain.

- *Curved edge crack in tee joint* – Figures 31c and d show the geometry and dimensions of an edge crack in a tee joint. A bending moment of 400 N-mm per millimetre thickness is applied. This moment gives rise to an extreme stress of 6.00 MPa at D. In the absence of a crack the maximum stress at B is 8.4 MPa. The crack is taken to propagate from B along the circular path BC shown in the figure, which meets the edge of the plate at right angles. Young's modulus and Poisson's ratio are taken to be 210 000 MPa and 0.3 respectively. Table 30 shows the best values obtained by Watson (1995).

Table 29. Theoretical stress concentration factors for curved crack.

k_1	0.81067
k_2	0.90616

Table 30. Stress concentration factors for curved edge crack.

a/w	0.1	0.2	0.3	0.4	0.5
$k_1/(\pi a)^{1/2}$	6.863	6.416	6.616	7.277	8.514

3.17c Test for ill-conditioning: Bigdeli & Kelley (1997) propose the condition number test to reduce the risk of losing accuracy of solution for an ill-conditioned system of equations. A small change in coefficients changes the results of the solution set of equations by a large factor in ill-conditioned systems. A numerical measure of ill-conditioning is the condition number.

$$C(k) = \lambda_{\max}/\lambda_{\min},$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of the stiffness matrix. C is the condition number. A large value of C indicates appreciable round off-error. The estimated accuracy loss is given by

$$\text{accurate digits lost} = \log_{10} C(K),$$

p-versions suffer more from growth of condition number by refinement than h-versions.

- *Cantilever* – A cantilever beam with four elements is chosen for comparing different schemes. A series of meshes including C^0 bilinear, quadratic and cubic serendipity elements for the p-version are produced. A series of meshes including 4, 16, 64 elements is constructed for the h-version. The geometry is shown in figure 32a. The condition numbers are plotted against corresponding DOF in figure 32b.

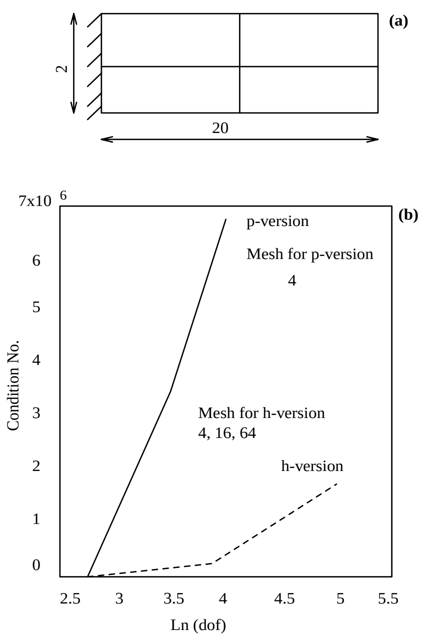


Figure 32. Cantilever beam (condition number analysis). **(a)** Geometry; **(b)** Condition numbers plotted against degrees of freedom.

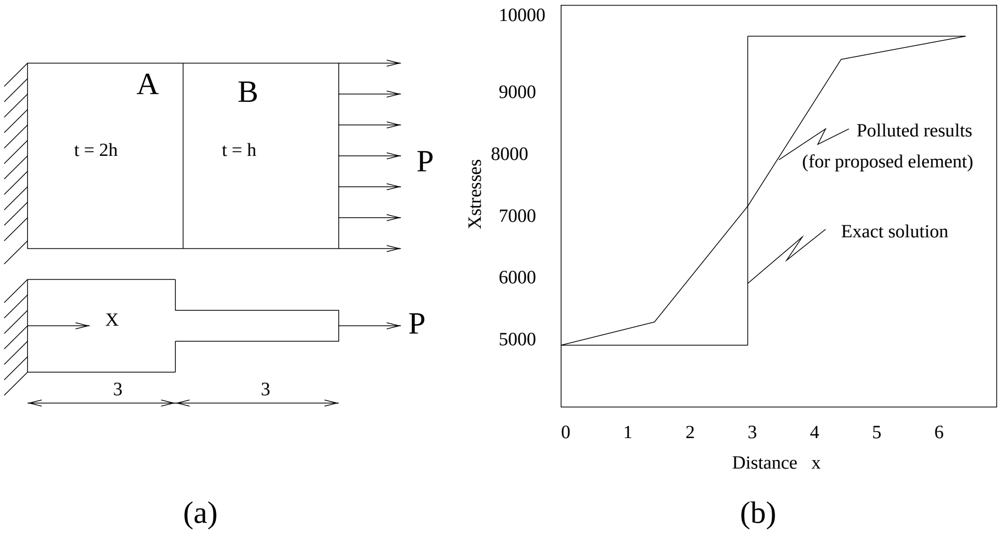


Figure 33. Cantilever beam (stress pollution problem).

- *Stress pollution* – Bigdeli & Kelley (1997) propose a test for stress pollution problems. When different materials or geometric properties are present, it is difficult to include strains as nodal variables. These will result in strain pollution, if calculated based on single derivatives at the nodes as shown in figure 33a. A simple solution based on the use of constraints is chosen. Two sets of degrees of freedom are defined for nodes on the interface between regions with different thickness. Continuity of derivatives across the interface is imposed through Lagrange multipliers. The polluted results are compared with exact results in figure 33b.

3.17d Test for frame invariance: Sze *et al* (1992) propose the following test for frame invariance in two-and three-dimensional problems. The single element structures employed are depicted in figures 34a and b. Local co-ordinate frames $\bar{x}$-$\bar{y}$ and $\bar{x}$-$\bar{y}$-$\bar{z}$ are attached to the bases of the structures as shown. The nodal forces acting are defined with respect to the local

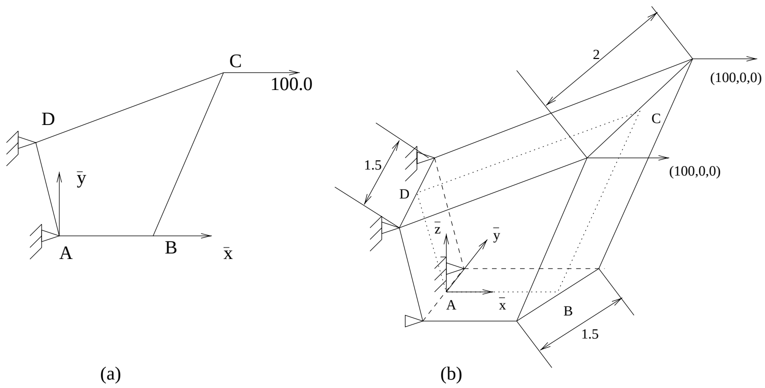


Figure 34. Tests for invariance.

Table 31. List of competing elements.

S.no.	Elements	References
–	**PN6X1**	Proposed element
1	ADJ (softened)	Bretl & Cook (1979)
2	APO	Punch & Atluri (1984)
3	AQ	Allman (1984)
4–6	B8-9P, B8-15P and B8-24P	Weissman (1996)
7	FCB	Chandra & Prathap (1989)
8–9	HEXA(8) & HEX20	MacNeal & Harder (1985)
10–12	HL, HG and PS5β	Pian & Sumihara (1984)
13	LAG9	White & Abel (1989)
14	MAQ	Yunus *et al* (1989)
15	OHB	Bachrach (1987)
16–17	PN30 and PN34	Venkatesh & Shrinivasa (1995, 1996a, 1996b)
18	PN34I	Bhattacharya *et al* (1996)
19	PN340	Bassayya *et al* (2000)
20	QM6	Taylor *et al* (1976)
21	Q_{S11-2}	Cheung & Chen (1988)
22	Q4BL	Zienkiewicz *et al* (1986)
23	RGD_{8-B}	Cheung & Chen (1992)
24–26	RGH4, RGH8 and Q4	Cheung & Chen (1992)
27	SS18β	Sze & Ghali (1993)
28	07β	Sze (1992)
29–31	9-node (3 $\times$ 3, γ and SRI)	Belytschko *et al* (1989)
32	9-node V-P	Verhegghe & Powell (1989)
33	PN5X1	Basayya & Shrinivasa (2000)
34	SOLID95	ANSYS (1996) 20-node solid
35	SHELL93	ANSYS (1996) 9-node shell
36	NKTP4	NISA 20-node solid
37	NKTP20	NISA 8-node shell
38	URI, FI, SS18	Sze *et al* (1977)
39	H20S, H20SS	Sze (1993)
40	ANS5, ANS3DL, ANS3DLr, ANS6Z, ANS3Dq, ANS3DEAS	Hauptman & Schweizerhof (1998)

frames. and are parallel to the $\bar{x}$-direction. To test the invariance of the proposed models, the $\bar{x}$–$\bar{y}$ and the $\bar{x}$–$\bar{z}$ planes of two-and three-dimensional structures respectively are rotated anti-clockwise by angles $\pi/8$, $\pi/4$, $3\pi/8$, and $\pi/2$. The computed displacements of the edges containing point C in the $\bar{x}$ direction are 0.44177 for $P5\beta$ 2D element and 0.53635 for $PT18\beta$. $P5\beta$ is Pian-Sumihara's 5–β plane element. $PT18\beta$ is Pian–Tong's 18–β solid element.

4. Case study

We have developed a new finite element PN6X1, based on Papcovitch–Neuber functions (Mallikarjuna Rao & Shrinivasa 2001). Its geometry and the nodal connectivity are shown in figure 2a. We have subjected this element to most of the tests that are proposed in this article, and have compared its performance with that of other elements listed in table 31. We summarize the performance in figure 35, which shows normalized errors in stresses and displacements. We categorize the test cases according to the induced stress fields as constant,

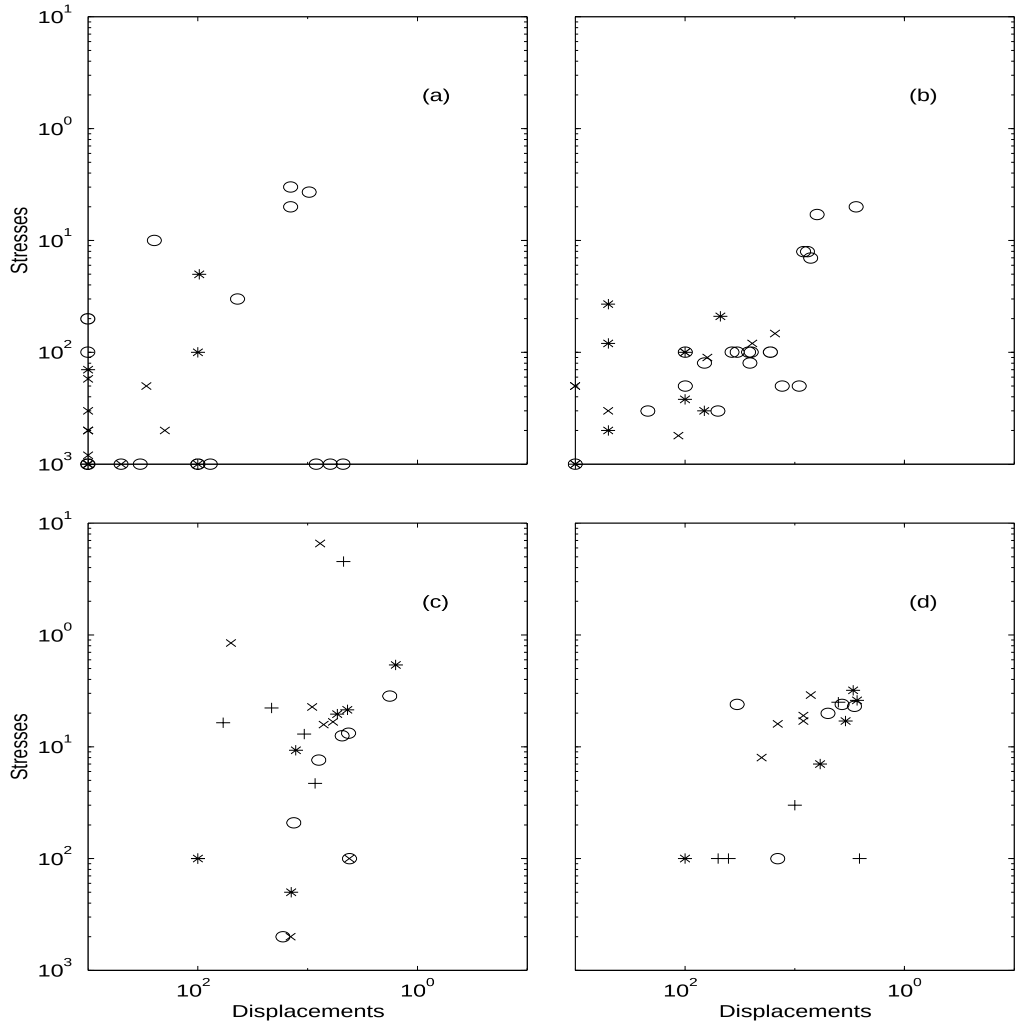


Figure 35. Errors in normalised stresses versus normalised displacements for the pathological tests for some elements: **(a)** PN6X1: ($*$) constant, ($\times$) linear, ($\circ$) quadratic or higher order stress fields; **(b)** PN5X1: ($*$) constant, ($\times$) linear, ($\circ$) quadratic or higher order stress fields; **(c)** QS_{11--2}: ($*$) linear ($\circ$) quadratic; FCB: ($+$) linear, ($\times$) quadratic stress fields; **(d)** RGD_{8--B}: ($*$) linear, ($\circ$) quadratic; APO: ($+$) linear, ($\times$) quadratic stress fields.

linear, and quadratic or higher. From figure 35 we observe that the errors in constant and linear stress fields are less than 2% irrespective of the geometry of the continuum and shape of the elements. In the case of quadratic stress fields with regular and moderately distorted elements the errors are still less than 0.75%; however, with extreme distortion, the error goes up to 25%. This however can be reduced to acceptable limits on discretization. Here we define error as

$$\text{error} = |\,(1 - \text{ test result } / \text{ theoretical value})\,|.$$

4.1 Pathological tests score

We calculate a score from tests in which we consider the errors in both stresses and displacements. Our Pathological Tests Score (PTS) defined below represents averaged percentage accuracy in both displacements and stresses.

$$\mathrm{PTS} = \left[1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{(\sigma_i^* - 1)^2 + (\delta_i^* - 1)^2}{2}\right)^{1/2}\right] \times 100\%,$$

where n is the number of tests and σ_i^* and δ_i^* are the normalized stresses and displacements from the ith test respectively. For our element, PN6X1, with $n = 69$, we get PTS $= 99.39\%$ which obtains the 'A' grade of MacNeal & Harder (1985).

5. Conclusions

We strongly feel that the set of tests compiled here has to be satisfied by all new finite elements. The possibility is demonstrated with a case study. The tests seem to encompass most of the criteria that need to be satisfied. These pathological tests are different from bench-mark tests which normally accompany many finite element analysis software. For more details on such tests NAFEMS (1988) and its later publications can be referred to.

References

Allman D J 1984 A compatible triangular element including vertex rotation for plane elasticity analysis. *Comput. Struct.* 19: 1–8

ANSYS 1996 Introduction to ANSYS. Release 5.3, Swanson Analysis Systems Inc., Houston, PA

Babuska I 1971 Error bounds for finite element methods. *Number Math.* 16: 322–333

Babuska I 1973 The finite element method with Lagrange multipliers. *Number Math.* 20: 179–192

Babuska I, Scapolla T 1989 Bench mark computation and performance evaluation for a rhombic plate bending problem. *Int. J. Numer. Meth. Engng.* 28: 155–179

Bachrach W E 1987 An efficient formulation of hexahedral elements with high accuracy for bending and incompressibility. *Comput. Struct.* 26: 453–467

Bäcklund J 1978 On isoparametric elements. *Int. J. Numer. Meth. Engng.* 12: 731–732

Bassayya K, Shrinivasa U 2000 A 14-node brick element, PN5X1, exactly representing linear stress fields. *Comput. Struct.* 74: 147–165

Bassayya K, Bhattacharya K, Shrinivasa U 2000 Au eight-node brick element, PN 340, to represent every constant stress field exactly. *Comput. Struct.* 74: 441–460

Belytschko T, Lasry D 1988 A fractal patch test. *IntJ. Numer. Meth. Engng.* 26: 2199–2210

Belytschko T, Stolarski H, Liu W K, Carpenter N, Ong J S J 1985 Stress projection for membrane and shear locking in shell finite elements. *Comput. Meth. Appl. Mech. Engng.* 51: 221–258

Belytschko T, Wong B L, Stolarski H 1989 Assumed strain stabilization procedure for the 9-node Lagrange shell element. *Int. J. Numer. Meth. Engng.* 28: 385–414

Bergan P G, Felippa C A 1985 A triangular membrane element with rotational degrees of freedom. *Comput. Meth. Appl. Mech. Engng.* 50: 25–69

Bhattacharya K, Bassayya K, Shrinivasa U 1996 Development of an eight-node isoparametric brick element using Papcovitch–Neuber functions. *Comput. Struct.* (under review)

Bigdeli B, Kelly D W 1997 C^*- convergence in the finite element method. *Int. J. Numer. Meth. Engng.* 40: 4405–4425

Bretl J L, Cook R D 1979 A new eight-node solid element. *Int. J. Numer. Meth. Engng.* 14: 593–615

Brezzi F 1974 On the existence, uniqueness and approximation of saddle point problems arising from Lagrange multipliers. *Rev. Francaise Autom. Inf. Rech. Oper., Aual. Numer.* 8: 129–151

Chandra S, Prathap G 1989 A field-consistent formulation for the eight-noded solid finite element. *Comput. Struct.* 33: 345–355

Chandra S, Kumar P, Chattopadhya L, James D 2001 Report No. PD-ST-XXX, National Aerospace Laboratories, Bangalore

Chapelle D, Bathe K J 1993 The inf-sup test. *Comput. Struct.* 47: 537–545

Cheung Y K, Chen W 1988 Isoparametric hybrid hexahedral elements for three dimensional stress analysis. *Int. J. Numer. Meth. Engng.* 26: 677–693

Cheung Y K, Chen W 1992 Refined hybrid method for plane isoparametric elements using an orthogonal approach. *Comput. Struct.* 42: 683–694

Choi J K, Lim J K 1995 General curved beam elements based on the assumed strain fields. *Comput. Struct.* 55: 379–386

Claudio F, Maria M 1998 Some finite elements for the static analysis of beams with varying cross-section. *Comput. Struct.* 69: 191–196

Felippa C A, Haugen B J 1995 From the individual element test to finite element templates: evolution of the patch test. *Int. J. Numer. Meth. Engng.* 38: 199–229

Gifford L N 1979 More on distorted isoparametric elements. *Int. J. Numer. Meth. Engng.* 14: 290–291

Graf W, Chang T Y, Saleeb A F 1986 On the numerical performance of three dimensional thick shell elements using a hybrid/mixed formulation. *Finite Elements Anal. Design.* 2: 357–375

Hauptmann R, Schweizerhof K 1998 A systematic development of solid shell element formulations for linear and nonlinear analyses employing only displacement degrees of freedom. *Int. J. Numer. Meth. Engng.* 42: 49–69

Hayes L J 1981 Practical stability test for finite elements with reduced integration. *Int. J. Numer. Meth. Engng.* 17: 1689–1695

Irons B M 1966 Engineering applications of numerical integration in stiffness methods. *AIAA J.* 4: 2035–2037

Irons B M, Barlow J 1964 Comments on ‘Matrices for the direct stiffness method by R J Melosh’. *AIAA J.* 2: 403–404

Irons B M, Loikkanen M 1983 An engineer’s defence of the patch test. *Int. J. Numer. Meth. Engng.* 19: 1391–1401

Irons B M, Razzaque A 1972 *Experience with the patch test for convergence of finite elements in mathematical foundation of the finite element method* (ed.) A K Aziz (New York: Academic Press) pp 557–587

Iosilevich A, Bathe K J, Brezzi F 1997 On evaluating the inf-sup condition for plate bending elements. *Int. J. Numer. Meth. Engg.* 40: 3639–3663

John S Y, Wayne K E, Norman F A 1999 Finite elements for materials with strain gradient effects. *Int. J. Numer. Meth. Engng.* 44: 373–391

Lee N S, Bathe K J 1993 Effect of element distortions on the performance of isoparametric elements. *Int. J. Numer. Meth. Engng.* 36: 3553–3576

Levy S 1953 Structural analysis and influence coefficients for delta wings. *J. Aeronaut. Sci.* 20: 449–454

MacNeal R H 1951 The solutions of elastic plate problems by electrical analogies. *Trans. ASME, J.Appl. Mech.* 18: 59–67

MacNeal R H 1987 A theorem regarding the locking of tapered four-noded membrane elements. *Int. J. Numer. Meth. Engng.* 24: 1793–1799

MacNeal R H, Harder R L 1985 A proposed set of problems to test finite element accuracy. *Finite Elements Anal. Design* 1: 3–20

Mallikarjuna Rao K, Shrinivasa U 2001 A 20-node hexahedral element, PN6X1, exactly representing linear stress fields. *Comput. Struct.* (submitted)

Melosh R J 1963 Structural analysis of solids. *Proc. Am. Soc. Civil Eng.* ST4-89: 205–223

Muskhelishvili N I 1953 *Some basic problems of the mathematical theory of elasticity* (Groningen: Noordhoff)

Murakami Y 1987 *Stress intensity factors handbook* (Oxford: Pergamon) pp 909–911

NAFEMS 1988 Proposed NAFEMS linear bench marks. National Agency for Finite Element Method and Standards, U K

Noor A K, Babuska I 1987 Quality assessment and control of finite element solutions. *Finite Elements Anal. Design* 3: 1–26

Oliveira A E R 1977 The patch test and the serial convergence criteria of the finite element method. *Int. J. Solids Struct.* 13: 159–78

Pian T H H, Sumihara K 1984 Rational approach for assumed stress finite elements. *Int. J. Numer. Meth. Engng.* 20: 1685–1695

Punch E F, Atluri S N 1984 Development and testing of stable, invariant, isoparametric curvilinear 2- and 3-dimensional hybrid stress elements. *Comput. Meth. Appl. Eng.* 47: 331–356

Razzaque A 1986 The patch test for elements. *Int. J. Numer. Meth. Engng.* 22: 63–71

Robinson J 1986 New FEM user project – single element test for aspect ratio sensitivity of solids (Part I). *Finite Element News* 1: 26–32

Sander G, Beckus P 1977 The influence of the choice of connectors in the finite element method. *Int. J. Numer. Meth. Engng.* 11: 1491–1505

Smith M 1990 Bench mark tests for geometrically non-linear two dimensional beams. *Finite Elements Anal. Design* 3: 40–43

Sorin M D, Bordan E I 1999 Study of warping torsion of thin walled beams with open cross-section using macro-elements. *Int. J. Numer. Meth. Engng.* 44: 853–868

Strang G, Fix G J 1973 *An analysis of the finite element method* (Englewood cliffs, NJ: Prentice-Hall).

Stricklin J A, Ho W S, Richardson E Q, Haisler W E 1977 On isoparametric vs linear strain triangular elements. *Int. J. Numer. Meth. Engng.* 11: 1041–1043

Stummel F 1980 The limitations of the patch test. *Int. J. Numer. Meth. Engng.* 15: 177–188

Sze K Y 1992 Efficient formulations of robust hybrid elements using orthogonal stress or strain interpolants and admissible matrix formulation. *Int. J. Numer. Meth. Engng.* 35: 1–20

Sze K Y 1996 Admissible matrix formulation from orthogonal approach to explicit hybrid stabilization. *Finite Elements Anal. Design* 24: 1–30

Sze K Y, Ghali A 1993 Hybrid hexahedral element for solids, plates, shells and beams by selective scaling. *Int. J. Numer. Meth. Engng.* 36: 1519–1540

Sze K Y, Chen W, Cheung Y K 1992 An efficient quadrilateral plane element with drilling degrees of freedom using orthogonal stress modes. *Comput. Struct.* 42: 695–705

Tada H, Paris P C, Irwin G R 1973 *The stress analysis of cracks handbook* (Hellerstown: Del Res. Corp.) p. 4

Taylor R L, Beresford P J, Wilson E L 1976 A non conforming element for stress analysis. *Int. J. Numer. Meth. Engng.* 10: 1211–1219

Taylor R L, Zienkiewicz O C, Simo J C, Chan A H C 1986 The patch test - a condition for asserting term convergence. *Int. J. Numer. Meth. Engng.* 22: 39–62

Timoshenko S P, Krieger S W 1959 *Theory of plates and shells* 2nd edn. (New York: McGraw-Hill) pp 51–143

Venkatesh D N, Shrinivasa U 1995 Generation of eight-node brick element using Papcovitch–Neuber function. *Comput. Struct.* 54: 1077–1084

Venkatesh D N, Shrinivasa U 1996a Hexahedral elements using PN functions – Applications to beams. *Comput. Struct.* 60: 305–313

Venkatesh D N, Shrinivasa U 1996b Plate bending with hexahedral PN elements. *Comput. Struct.* 60: 635–641

Verhegghe B, Powell G H 1986 Control of zero-energy modes in 9-node plane element. *Int. J. Numer. Meth. Engng.* 23: 863–869

Verma A, Melosh R J 1987 Numerical tests for assessing finite element model convergence. *Int. J. Numer. Meth. Engg.* 24: 843–857

Veubeke B F D 1974 Variational principles and the patch test. 1974, *Int. J. Numer. Meth. Engng.* 8: 783–801

Watson J O 1995 Singular boundary elements for the analysis of cracks in plane strain. *Int. J. Numer. Meth. Engng.* 38: 2389–2411

Weissman S L 1996 High accuracy, low-order three-dimensional brick elements. *Int. J. Numer. Meth. Engng.* 39: 2337–2361

White D W, Abel J F 1989 Testing of shell finite element accuracy and robustness. *Finite Elements Anal. Design* 6: 129–151

Wilson E L, Ibrahimbegovic A 1990 Use of incompatible displacement modes for the calculation of element stiffnesses or stress. *Finite Elements Anal. Design* 7: 229–241

Yunus S M, Saigal S, Cook R D 1989 On improved hybrid finite elements with rotational degrees of freedom. *Int. J. Numer. Meth. Engng.* 28: 785–800

Zhang W U, Chen D P 1997 The patch test conditions and some multivariable finite element formulations. *Int. J. Numer. Meth. Engng.* 40: 3015–3032

Zienkiewicz O C, Taylor R L 1991 *The finite element method* (New York: McGraw-Hill)

Zienkiewicz O C, Taylor R L 1997 The finite element patch test revisited. A computer test for convergence, validation and error estimates. *Comput. Methods Appl. Mech. Engg.* 149: 223–254

Zienkiwicz O C, Taylor R L, Nakazawa S 1986 The patch test for mixed formulations. *Int. J. Numer. Meth. Engg.* 23: 1873–1883

Zienkiewicz O C, Xu Z, Zeng L F, Samuelsson F, Wiberg N 1993 Linked interpolation for Reissner-Mindlin plate elements: Part I – A simple quadrilateral. *Int. J. Numer. Meth. Engng.* 36: 3043–3056