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Advances in nonlinear vibration analysis of structures. Part-I. Beams

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Abstract. The development of nonlinear vibration formulations for beams in the literature can be seen to have gone through distinct phases – earlier continuum solutions, development of appropriate forms, extra-variational simplifications, debate and discussions, variationally correct formulations and finally applications. A review of work in each of these phases is very necessary in order to have a complete understanding of the process of evolution of this field. This paper attempts to achieve precisely this objective.

Keywords. Nonlinear vibrations; beams.

1. Introduction

The study of large amplitude vibration of simply supported beams can be traced to the work of Kreiger (1950), wherein the governing partial differential equations were reduced to ordinary differential equations and the solution was obtained in terms of elliptic functions using a one-term approximation. Similarly, Burgreen (1951) gave the solution for the large amplitude vibration problems of hinged beams based on the classical continuum approach. Srinivasan employed the Ritz–Galerkin technique to solve the governing nonlinear differential equation of dynamic equilibrium for free (Srinivasan 1965) and forced (Srinivasan 1966) vibration of simply supported beams and plates. Evensen (1968) extended the study for various boundary conditions using the perturbation method.

Ray & Bert (1969) carried out experimental studies to verify the analytical solutions for the nonlinear vibrations of simply supported beam and compared the solution schemes such as the Assumed Space Mode (ASM), Assumed Time Mode (ATM) and Ritz–Galerkin methods and concluded that the latter two are better than the former. Pandalai & Sathyamoorthy (1973) developed modal equations for the nonlinear vibrations of beams, plates, rings and shells using Lagrange's equation and highlighted the difference in the nature of the modal equations for beams and plates *vis-á-vis* rings and shells.

Lou & Sikarskie (1975) employed *form-function approximations* to study the nonlinear forced vibrations of buckled beam. Rehfield (1975) used an approximate method for nonlinear vibration problems with material nonlinear effects for various boundary conditions.

In the context of finite element solutions, Mallett & Marcal (1968) proposed a scheme for writing the strain energy in a systematic way, known as 'appropriate form', containing three symmetric matrices – linear matrix K, nonlinear matrices N_1 and N_2, and also expressed equilibrium and linear incremental equations using these matrices. Rajasekaran & Murray (1973) in their seminal paper presented an exact procedure/expression for deriving these matrices (proposed by Mallett & Marcal 1968) for various elements in an elegant manner. Chen & Huang (1986) extended the appropriate forms to the potential energy due to external loads also, as the work done by the external loads depend on the nodal degrees of freedom in the case of large deformation problems.

Reddy (1979) presented a review of the formulations of the 1970s related to structural vibrations. Sathyamoorthy (1982) compiled the work on classical methods for the analysis of beams with material, geometric and other types of nonlinearities and also on finite element analysis of nonlinear beams under static and dynamic loads. Another survey of work on shear deformation theories, finite elements and buckling (Kapania & Raciti 1989a), along with those on free, forced, linear, nonlinear vibrations and wave propagation etc., with reference to laminated structures (Kapania & Raciti 1989b) was reported.

Mei (1972, 1973) presented the finite element formulations for large amplitude vibrations of beams and plates. In all his work, the axial deformation was neglected and the average axial force was assumed to be a constant over the element length. Rao *et al* (1976) also studied the nonlinear vibrations of beams and plates and of beams with shear deformation and rotatory inertia. In these formulations, the axial deformation was not considered and the nonlinear strain-displacement relationship was linearized. Later, Raju *et al* (1976) studied the large amplitude vibration problem of beams and plates using Rayleigh–Ritz method by incorporating the inplane deformation as well as inertia, which were absent in the earlier studies, and also by retaining the equivalent linearization function.

Prathap (1977) contested the linearization procedure, inadequate interpretation of w and also the conclusion made by Raju *et al* (1976) that '*the inclusion of inplane terms is insignificant*'. Prathap & Varadan (1978) studied the nonlinear vibrations of simply supported beams using the actual nonlinear equilibrium equations, exact nonlinear expression for curvature and the nonlinearity arising out of axial force. Also, they proposed criteria for definition of the degree of nonlinearity.

Later, the effect of neglect of axial displacement of a beam and of the premise to consider the load-displacement relationship as a linear one while deriving the equation of conservation of energy (Szilard 1978) has been demonstrated by Prathap (1980) using an 'one-term approximation' for a simply supported beam. The infeasibility of formulating a Ritz-type finite element, without incorporating longitudinal degree of freedom was then established by Prathap & Bhashyam (1980).

Sarma & Varadan (1982) brought out the errors due to procedures such as equivalent linearization, substitution of inplane boundary conditions at element level rather than at system level and the use of different connotations for w, adopted by many earlier formulations (Mei 1972, 1973; Rao 1976).

The debate on core issues of formulations such as the neglect of longitudinal displacement, equivalent linearization approximation, interpretation of radian frequency *w* and the computation of inplane force etc. with reference to the work by Bhashyam & Prathap (1980) and Sarma & Varadan (1982, 1983), can be seen in the publications of Mei (1984) and Raju & Rao (1984).

Later, Mei (1986) criticized the new definition of criteria for the degree of nonlinearity proposed by Prathap & Varadan (1978) and also questioned the validity of frequency solutions presented by Bhashyam & Prathap (1980) and Sarma & Varadan (1982, 1983).

As a sequel to the points raised by others (Mei 1984; Raju & Rao 1984; Mei 1986), Sarma *et al* (1988) presented a Rayleigh–Ritz solution incorporating the inplane displacement and inertia, and captured the error made in an earlier formulation (Raju *et al* 1976) due to the equivalent linearization approximation. Also, they re-examined the Galerkin (Bhashyam and Prathap 1980), Lagrange-type (Sarma & Varadan 1983), Ritz (Sarma & Varadan 1984) finite element formulations and presented two mixed finite element formulations to critically analyse various assumptions employed earlier for the simply supported beam vibration problem.

Dumir & Bhaskar (1988) traced the errors in the nonlinear finite element formulations of beam and plate vibrations to the presence of a linearizing function in the strain energy evaluation and ascertained the magnitude of error involved due to this function.

Singh *et al* (1990a) derived frequency ratios from the equation of motion, energy balance equation and perturbation method along with the Ritz–Galerkin solution obtained from the following four possible combinations – with/without axial displacement and with/without linearization approximations. They observed that a formulation without axial displacement but with linearization and simple harmonics assumption would yield the same nonlinear frequencies as those of methods (such as Ritz–Galerkin, perturbation etc.) with axial displacement but without linearization and harmonic oscillation assumptions.

Subsequently, Singh *et al* (1990b) reported a formulation for the nonlinear free vibration of beams, wherein the dynamic finite element matrix equations were reduced to a scalar equation (using the converged mode shape), which was then solved using direct numerical integration and concluded that the axial displacements cannot be neglected in any nonlinear vibration analysis.

Bhashyam & Prathap (1980) proposed a Galerkin finite element method for the large amplitude vibration of beams while Sarma & Varadan (1983) published a Lagrange-type finite element formulation for the nonlinear vibration of immovably supported beams. Ritz type finite element formulations for the nonlinear vibration studies of classical beam (Sarma & Varadan 1984) and Timoshenko beam (Sarma & Varadan 1985) have also been reported.

A hybrid approach using both the finite element and perturbation procedure has been proposed for the vibration studies of nonlinear structures by Padovan (1980).

Reddy & Singh (1981) published a total potential energy based traditional element and a Reissner-type variational functional based mixed finite element, with u, w and M_x as dependent variables for the study of nonlinear vibration of beams and shallow arches, considering both the transverse shear and rotatory inertia effects.

Heyliger & Reddy (1988) presented a higher order theory with C^1 element formulation for the static and linear/nonlinear vibration studies of rectangular beams in which they explored the effects of inplane inertia and slenderness ratio on the nonlinear frequency of beams with various boundary conditions.

Kapania & Raciti (1989c) proposed a two-noded Timoshenko beam element with 10 degrees of freedom per node to study the nonlinear vibrations of symmetrically and unsymmetrically laminated composite beams by employing the perturbation method.

Singh *et al* (1991) later studied the large-amplitude vibration problem of unsymmetrically laminated beams based on classical, first-order and higher-order formulations by using the numerical integration technique introduced earlier (Singh *et al* 1990b).

Various types of solutions to the nonlinear equation of motion such as Galerkin, harmonic balance method and simple harmonic oscillations based method were proposed and analysed by Pillai & Rao (1992) while solutions by the method of multiple scales and ultra-spherical polynomial approximation method have been suggested by Srirangarajan (1994). Shi & Mei (1996) proposed a finite element time domain modal formulation for the large amplitude free vibration analysis of beams and plates.

2. Distinct phases of development

The complete development of nonlinear vibration theory of beams is presented topic-wise in tables 1 and 2. While table 1 presents the earlier literature with continuum solution, the important phase of development of appropriate forms with first and second-degree nonlinear matrices is given in table 2a. Various reviews covering the 1970s and 80s are tabulated in table 2b.

Modelling the nonlinear vibration problems using finite elements, albeit with a couple of extra-variational simplifications (table 2c), stirred the proverbial hornet's nest. A series of papers on these issues, starting from 1977 is presented in table 2d.

Variationally correct formulations appeared in the literature from the 1980s (table 2e) and the applications of this area to problems such as composites etc. are presented in table 2f.

Table 1. Earlier continuum solutions.

Type of solution	Author(s)	Year
Elliptic functions one-term approximation	Kreiger	1950
Classical continuum approach	Burgreen	1951
Ritz–Galerkin technique	Srinivasan	1965, 1966
Perturbation method	Evensen	1968
Experimental, ASM, ATM and Ritz–Galerkin	Ray & Bert	1969
Lagrange's equation	Pandalai & Sathyamoorthy	1973
Form-function approximation	Lou & Sikarskie	1975

Table 2. Distinct phases of development.

Details	Author(s)	Year
(a) Development of appropriate forms		
Appropriate form	Mallett & Marcal	1968
First and second degree nonlinear matrices	Rajasekaran & Murray	1973
Appropriate forms for energy due to external loads	Chen & Huang	1986
(b) Review papers		
Structural vibrations	Reddy	1979
Nonlinear analysis of beams	Sathyamoorthy	1982a, 1982b
Studies on laminated beams	Kapania & Raciti	1989a, 1989b
(c) Extra-variational simplifications (EVS)		
Neglect of inplane displacement	Mei	1972, 1973a, 1973b
Quasi-linearisation	Rao *et al*	1976a, 1976b
Quasi-linearisation with inplane displacement	Raju *et al*	1976
(d) Debate and discussions on EVS		
On quasi-linearisation procedure	Prathap	1977
Effect of neglect of axial displacement	Prathap	1980
Inplane displacement for Ritz type formulation	Prathap & Bhashyam	1980
Errors due to various simplifications	Sarma & Varadan	1982
Discussion on EVS	Mei	1984
	Raju & Rao	1984
Discussion on definition criteria for nonlinearity of Prathap & Varadan (1978)	Mei	1986
Summary of various methods and errors due to EVS	Sarma *et al*	1988
Errors due to quasi-linearisation	Dumir & Bhaskar	1988
Debate on EVS	Singh *et al*	1990
(e) Variationally correct formulations		
Galerkin finite element	Bhashyam & Prathap	1980
Lagrange-type finite element	Sarma & Varadan	1983
Ritz f.e. – classical beams	Sarma & Varadan	1984
Ritz f.e. – Timoshenko beams	Sarma & Varadan	1985
(f) Applications		
Perturbation solution with finite element	Padovan	1980
Higher-order mixed finite element	Reddy & Singh	1981
Higher-order C^1 element	Heyliger & Reddy	1988
Composite beam with 10 d.o.f. per node	Kapania & Raciti	1989c
Nonlinear vibrations of composite beam	Singh *et al*	1991
Comparison of solutions	Pillai & Rao	1992
Polynomial approximation	Srirangarajan	1994
Time-domain model	Shi & Mei	1996

3. Conclusions

It can be concluded that the most significant phase has been that of the development of appropriate forms and of variationally correct formulations and perhaps the crucial phase was that of debate and discussions. This study, it is hoped, will serve the purpose of providing a state-of-the-art overview of the nonlinear vibration theory of beams.

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