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Direct spatial resonance in the laminar boundary layer due to a rotating-disk

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Abstract. Numerical treatment of the linear stability equations is undertaken to investigate the occurrence of direct spatial resonance events in the boundary layer flow due to a rotating-disk. A spectral solution of the eigenvalue problem indicates that algebraic growth of the perturbations shows up, prior to the amplification of exponentially growing instability waves. This phenomenon takes place while the flow is still in the laminar state and it also tends to persist further even if the non-parallelism is taken into account. As a result, there exists the high possibility of this instability mechanism giving rise to nonlinearity and transition, long before the unboundedly growing time-amplified waves.

Keywords. Rotating-disk flow; linear stability; direct spatial resonance.

1. Introduction

Many types of instability mechanisms may be operational during the physical process of transition from laminar flow to turbulence in fully three-dimensional boundary-layer flows. In this paper, we primarily deal with the linear stability, in particular, direct spatial resonance of eigenmodes of the three-dimensional boundary-layer flow due to a rotating-disk. The significance of studying this flow is that the stability properties are similar to those due to the flow over a swept-back wing in the sense that both flows are subject to inviscid crossflow vortex instability induced by the inflexional character of the mean velocity profile.

The flow over a rotating-disk has been investigated experimentally, theoretically and numerically by a number of researchers. These studies have revealed the highly complicated nature of instabilities present and both absolute as well as convective instabilities have been found.

The early works on the stability of rotating-disk flow enlightened many of the aspects relating to the convective nature of the flow. For example, the experimental works of

Gregory *et al* (1955) (hereafter called as GSW), Wilkinson & Malik (1983), Kohama (1984) and Kohama *et al* (1987) have highlighted that primary and secondary instabilities in the form of stationary and/or non-stationary crossflow vortices govern the motion of a rotating-disk flow. Later, numerical and theoretical aspects of linear inviscid and viscous disturbance modes have been examined by a number of researchers including GSW, Hall (1986), Malik (1986), Bassom & Gajjar (1988), Balakumar & Malik (1990), Balachandar *et al* (1992) and Turkyilmazoglu & Gajjar (2000b). All these works investigated the convective instability characteristics rather than any other instability mechanism that might be involved. The wave packet calculations of Mack (1985) using the steepest-descent time integration technique confirmed that in the range of Reynolds number, Re = 250–500, the rotating-disk flow is convectively unstable.

Singularities arising in the dispersion relationship enable the nature of the instability to be determined. Variation of the Reynolds number can cause such points to occur and thus alter the behaviour of a flow from a convectively unstable state to an absolutely unstable regime. These points form whenever modes associated with waves propagating in opposite or same directions coalesce. If the coalescing branches originate from the waves propagating in opposite directions, the singularity which causes *resonance* is said to be of pinch type. For linearly unstable systems, such a direct resonance point separates an absolutely unstable region from a convectively unstable region. For the concept of absolute instability, the reader can see the reviews by Briggs (1964), Bers (1975) and Huerre & Monkewitz (1990). Examples of flows demonstrating this phenomenon are near-wake flows (Betchov & Criminale 1966; Monkewitz 1988) and mixing layer flows with backflow (Huerre & Monkewitz 1985). On the other hand, if the two coalescing modes originate from waves propagating in the same direction, then the corresponding singularity is of double-pole type. When these coalescing modes are nearly neutral, the damping rates are very small and thus a resulting short-term algebraic growth for small times or short distances may carry the whole system into a nonlinear stage long before the exponentially growing mode does. This is the case addressed by, amongst others, Benney & Gustavsson (1981), Koch (1986) and Shanthini (1989) for plane-Poiseiulle flow and Blasius boundary-layer flow.

The points pertaining to the first class mentioned above have been examined by Cole (1995), Lingwood (1995, 1996) and Turkyilmazoglu *et al* (2000) for the rotating-disk boundary-layer flow and by Turkyilmazoglu & Gajjar (1999) and Turkyilmazoglu *et al* (1999) for the wedge trailing-edge and swept-Hiemenz flows. Making use of the Briggs–Bers criterion and assuming that the flow is parallel, they were able to show that the fluid motion becomes absolutely unstable. Lingwood (1995) concluded that the absolute instability mechanism found in the rotating-disk flow might cause the disturbances to grow exponentially and temporally at a fixed radius, leading to an unbounded linear response that would promote nonlinearity followed by transition. Incidentally, the critical Reynolds number for the flow to undergo an absolutely unstable stage has been found in her study to be very close to the experimentally observed critical Reynolds number for the flow to be transitional. Her experimental study (Lingwood 1996) of absolute instabilities of the rotating-disk boundary-layer flow has confirmed her theoretical results. More recently, the effect of wall compliance on boundary-layer instability over a rotating-disk has been studied by Cooper & Carpenter (1997a). They have shown that wall compliance has a substantial stabilizing effect on the upper branch crossflow instability of GSW. The effect on the viscous lower branch is found to be strongly destabilizing. Cooper & Carpenter (1997b) have also shown that the presence of wall compliance suppresses one of the coalescing eigenmodes postponing the absolute instability to at least a higher Reynolds number. Beyond a critical level of wall compliance the results suggested that complete suppression of the absolute instability is possible, removing a major route to transition in the rotating-disk boundary-layer flow.

Our main motivation is to investigate direct spatial resonance in three-dimensional rotating-disk flow. The question then is whether such a resonance occurs and, if so, which families of eigensolutions are subject to resonance? Benney & Gustavsson (1981) have carried out such an investigation for the stability of parallel shear flows arising from the resonant forcing of the vertical vorticity by the vertical velocity which corresponds to a direct spatial resonance between an Orr–Sommerfeld and a Squire mode. Koch (1986) has also suggested the possibility of direct spatial resonance between two Orr–Sommerfeld modes in his numerical study of the stability of the Blasius boundary layer. Another question would be provided that a direct spatial resonance is present in the specific flow under study, at which Reynolds number does such a resonance appear first? Indeed our analysis demonstrates that the family 1 and 2 branches, both of which originate in the same wave number plane form a direct spatial resonance at a Reynolds number of about 445, at which the flow is still laminar. There is then the possibility that nonlinearity may be first triggered by this instability mechanism.

In line with the literature, the equations of the rotating-disk boundary layer flow are linearized with small disturbances, and either a parallel flow approximation (as by Malik 1986) or the asymptotic approach introduced in this paper is usually undertaken. However, under such assumptions the role of the non-parallel terms is omitted and the results may be inconclusive. This may refer to the inevitable fact that the absolute instability found by Cole (1995), Lingwood (1995) and Turkyilmazoglu (1998) may be misleadingly underestimated, and the absolute instability may occur at a higher Reynolds number than the one found in those studies. Unlike this situation, the direct resonance of two coalescing modes originating from the same wavenumber plane is found to be at Reynolds number 445 in this paper. This implies that even if the non-parallel terms are included in the instability investigations, there may still be the high possibility that the same resonance event persists in the laminar region, causing an algebraic growth of the disturbance amplitudes and so resulting in nonlinearity and transition.

The linear disturbance equations are derived in a rational manner using approximations which are self-consistent. When non-parallel terms are ignored with an additional assumption of r (the radial distance from the centre of the disk) being set to unity, the traditional sixth-order stability equations are obtained. These equations, which have been treated before by Malik (1986) and Balakumar & Malik (1990), are solved here using a different technique based on the spectral Chebyshev collocation with a staggered grid structure. A Runge–Kutta scheme was also employed for checking the self-consistency of our results. The use of spatio-temporal linear stability analysis shows that three families of eigenfunctions are possible in the rotating-disk flow. Monitoring the development of the eigenvalues corresponding to these families shows branch interchanging between distinct waves at certain Reynolds numbers.

This paper is organised as follows. First, the governing equations and the mean flow are given in § 2, followed by the derivation of the linear viscous and inviscid stability equations in § 3. Second, the numerical technique to discretize and solve the stability equations is described in § 4. Third, direct spatial resonance results and discussion are presented in § 5. Finally, conclusions are drawn in § 6.

2. Basic equations

2.1 *Governing equations of the flow*

We consider the three-dimensional boundary-layer flow of an incompressible fluid on an infinite disk which rotates about its axis with a constant angular velocity Ω. The Navier–Stokes equations are non-dimensionalized with respect to a length scale $L = r_e^*$, velocity scale $U_c = L\Omega$, time scale L/U_c and pressure scale ρU_c^2, where ρ is the fluid density. This leads to a global Reynolds number $\mathcal{R}e = U_c L/\nu = \mathrm{Re}^2$, where Re is the Reynolds number based on the displacement thickness $\delta = (\nu/\Omega)^{1/2}$. It can be seen that our non-dimensionalization is somewhat different from that employed by Malik (1986). Thus relative to non-dimensional cylindrical polar coordinates (r, θ, z) which rotate with the disk, the full time-dependent, unsteady Navier–Stokes equations governing the viscous fluid flow are the usual momentum and the continuity equations, and these are given as follows:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - 2v - r = -\frac{\partial p}{\partial r} + \frac{1}{\mathrm{Re}^2}\left[\nabla^2 u - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2}\right],\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + 2u = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{\mathrm{Re}^2}\left[\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right],\\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}^2}[\nabla^2 w],\\
&\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0.
\end{aligned}
\tag{1}
$$

In this analysis the fluid is assumed to lie in the $z \geq 0$ semi-infinite space. In the above equations the viscous, streamline curvature effects as well as the effects stemming from the Coriolis forces are all present. It is now well-known that these terms have a strongly stabilizing impact in the linear stability theory, as pointed out by Malik & Poll (1985) and Wilkinson & Malik (1985).

2.2 *Mean flow*

Dimensionless mean flow velocities and pressure are given by Von Kármán's exact self-similar solution of the Navier–Stokes equations for steady laminar flow. The boundary-layer coordinate Z, which is of order $O(1)$ is defined as $Z = z\mathrm{Re}$, and the self-similar equations take the form

$$
(u_B, v_B, w_B, p_B) = \left(rF[Z], rG[Z], \frac{1}{\mathrm{Re}}H[Z], \frac{1}{\mathrm{Re}^2}P[Z]\right), \tag{2}
$$

where the functions F, G, H and P satisfy the following ordinary differential equations

$$
\begin{aligned}
&F^2 - (G+1)^2 + F'H - F'' = 0,\\
&2F(G+1) + G'H - G'' = 0,\\
&P' + H'H - H'' = 0,\\
&2F + H' = 0.
\end{aligned}
\tag{3}
$$

Here, primes denote derivatives with respect to Z and the appropriate boundary conditions are given as

$$\begin{aligned} &F = G = H = 0 \text{ at } Z = 0, \\ &F = 0,\ G = -1,\ H = h_\infty \text{ as } Z \to \infty. \end{aligned} \tag{4}$$

The unknown h_∞ is a constant vertical velocity of the rotating fluid in the far-field above the disk, and its value has to be found numerically in the course of the solution of (3) and (4).

3. Linear stability equations

3.1 *Viscous disturbance equations*

We are interested here in perturbation solutions of Von Kármán's self-similarity velocity profiles, (2). The instantaneous non-dimensionalized velocity components imposed on the basic steady flow are u, v, w and the pressure component p and they can be expressed as

$$\begin{aligned} u[r,\theta,z,t] &= u_B + u'[r,\theta,z,t], \\ v[r,\theta,z,t] &= v_B + v'[r,\theta,z,t], \\ w[r,\theta,z,t] &= w_B + w'[r,\theta,z,t], \\ p[r,\theta,z,t] &= p_B + p'[r,\theta,z,t]. \end{aligned}$$

The form of the base flow (u_B, v_B, w_B, p_B) is given as in the previous section, and infinitesimally small disturbances u', v', w' and p' are superimposed on the steady flow obtained from the solution of (3)–(4). The disturbance components of the above system are determined by solving the form of the Navier–Stokes equations that result from substituting these quantities into (1), and subtracting the mean flow equations, satisfying (3). Having linearized the equations for small perturbations, we find that the linearized Navier–Stokes operator has coefficients independent of θ and hence the disturbances can be decomposed into a normal mode form proportional to $\exp^{[i\,\mathrm{Re}(\beta\theta-\bar{\omega}t)]}$. Such an approximation leads the disturbances to be wave-like, separable in θ and t. Consequently, the perturbations may be assumed to be of the form

$$(u', v', w', p') = (\tilde{u}[r,Z], \tilde{v}[r,Z], \tilde{w}[r,Z], \tilde{p}[r,Z])\exp^{[i\,\mathrm{Re}(\beta\theta-\bar{\omega}t)]} + \text{c.c.},$$

where β and $\bar{\omega}$ respectively are the wave number in the azimuthal direction and the scaled frequency of the wave propagating in the disturbance wave direction and c.c. denotes the complex conjugate. Note that disturbances are not of the form assumed by Malik (1986), for example $u' = f[Z]\ \exp^{[i(\alpha r+\beta \mathrm{Re}\theta-\bar{\omega}t)]} + c.c.$ Spalart (1990) criticizes this method of decomposing the disturbances, suggesting that the natural structure of the mean flow profile supports disturbances in the form $rf[Z]\exp^{[i(\alpha r+\beta \mathrm{Re}\theta-\bar{\omega}t)]} + c.c.$ In spite of his criticism, he concludes that the above assumption accurately predicts the disturbance wave field.

The separation in θ and t simplifies the linear system of equations. However no such simplification arises as far as the r-dependence is concerned (except in the limit as $\mathrm{Re} \to \infty$) and the full linearized partial differential system has to be solved subject to suitable initial conditions to determine the stability of the flow. Consider next the limit

$\mathrm{Re} \to \infty$ and introduce the scale $X = \mathrm{Re}\,r$ which is the appropriate scale on which the disturbances develop. After allowing for the multiple-scale replacement of $(\partial/\partial r)$ by

$$\mathrm{Re}\frac{\partial}{\partial X} + \frac{\partial}{\partial r},$$

and keeping only terms of up to $O(1/\mathrm{Re})$ the following linear system is obtained

$$\begin{aligned}
&-i\bar{\omega}\tilde{u} + rF\frac{\partial \tilde{u}}{\partial X} + i\beta G\tilde{u} + r\frac{dF}{dZ}\tilde{w} + \frac{\partial \tilde{p}}{\partial X} \\
&\qquad = -\frac{1}{\mathrm{Re}}\left[H\frac{\partial \tilde{u}}{\partial Z} + F\tilde{u} - 2(G+1)\tilde{v} - \nabla_2^2\tilde{u}\right] - \frac{1}{\mathrm{Re}}\left[rF\frac{\partial \tilde{u}}{\partial r} + \frac{\partial \tilde{p}}{\partial r}\right],\\
&-i\bar{\omega}\tilde{v} + rF\frac{\partial \tilde{v}}{\partial X} + i\beta G\tilde{v} + r\frac{\mathrm{d}G}{\mathrm{d}Z}\tilde{w} + i\frac{\beta}{r}\tilde{p} \\
&\qquad = -\frac{1}{\mathrm{Re}}\left[H\frac{\partial \tilde{v}}{\partial Z} + F\tilde{v} + 2(G+1)\tilde{u} - \nabla_2^2\tilde{v}\right] - \frac{1}{\mathrm{Re}}\left[rF\frac{\partial \tilde{v}}{\partial r}\right],\\
&-i\bar{\omega}\tilde{w} + rF\frac{\partial \tilde{w}}{\partial X} + i\beta G\tilde{w} + \frac{\partial \tilde{p}}{\partial Z} + \frac{1}{\mathrm{Re}}\left[H\frac{\partial \tilde{w}}{\partial Z} + \frac{dH}{dZ}\tilde{w} - \nabla_2^2\tilde{w}\right] = -\frac{1}{\mathrm{Re}}\left[rF\frac{\partial \tilde{w}}{\partial r}\right],\\
&\qquad \frac{\partial \tilde{u}}{\partial X} + \frac{1}{\mathrm{Re}\,r}\tilde{u} + \frac{i\beta}{r}\tilde{v} + \frac{\partial \tilde{w}}{\partial Z} = -\frac{1}{\mathrm{Re}}\frac{\partial \tilde{u}}{\partial r}.
\end{aligned} \tag{5}$$

The operator ∇_2^2 is defined by

$$\nabla_2^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{\beta^2}{r^2}.$$

The terms on the right hand side of (5) reflect the non-parallelism of the basic flow and appear at the same order as the other $O(1/\mathrm{Re})$ terms which are retained in the familiar 'parallel flow approximation'. In the formal limit $\mathrm{Re} \to \infty$ and with $(\partial/\partial X)$ replaced by $i\alpha$ we obtain Rayleigh's equation. It is only in this limit that the full normal mode decomposition can be justified. If we neglect the terms on the right hand side of (5) and replace $(\partial/\partial X)$ by $i\alpha$ *together with* $r = 1$, the sixth-order system studied by several people (including Mack 1984, Malik 1986, Balakumar & Malik 1990 and Lingwood 1995, amongst others) is retrieved. Given that the neglect/inclusion of terms of $O(1/\mathrm{Re})$ leads to significant differences in the results especially near the critical Reynolds number, the neglect of the non-parallel terms cannot in general be justified, see for instance, Saric & Nayfeh (1975, 1977), Smith (1979), Fasel & Konzelmann (1990), Malik & Li (1992), Malik *et al* (1994), and Turkyilmazoglu & Gajjar (1999). The comment made in several papers including that by Balakumar & Malik (1990) that the non-parallel terms are only important when the (scaled) wave number is small is thus misleading since the same may also be true for $O(1)$ wave numbers. Setting $r = 1$ as highlighted above is tantamount to considering the stability at the local station r_e^*.

The reduced system of equations stemming from these approximations can be written in the following form ($\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{p}$ are replaced by f, g, h and p respectively),

$$\begin{aligned}
&f'' - Hf' - [i\mathrm{Re}(\alpha F + \beta G - \bar{\omega}) + \lambda^2 + F]f + 2(G+1)g - \mathrm{Re}F'h - i\alpha\mathrm{Re}p = 0,\\
&g'' - Hg' - [i\mathrm{Re}(\alpha F + \beta G - \bar{\omega}) + \lambda^2 + F]g - 2(G+1)f - \mathrm{Re}G'h - i\beta\mathrm{Re}p = 0,\\
&h'' - Hh' - [i\mathrm{Re}(\alpha F + \beta G - \bar{\omega}) + \lambda^2 + H']h - \mathrm{Re}p' = 0,\\
&\bar{\alpha}f + i\beta g + h' = 0,
\end{aligned} \tag{6}$$

where $\lambda^2 = \alpha^2 + \beta^2$, $\bar{\alpha} = i\alpha + (1/\mathrm{Re})$ and $\omega = \bar{\omega}\mathrm{Re}$. The linear equation system above is identical to the one used by previous investigators, such as Malik (1986).

The boundary conditions for this set of equations are $f = g = h = 0$ at the solid wall ($Z = 0$). Considering the decaying property of disturbances the boundary conditions to be imposed far away from the disk surface are derived from the asymptotic form of (6). The specific form of the far-field boundary conditions can be found in the work of Turkyilmazoglu (1998).

3.2 *Inviscid Rayleigh equation*

It is possible to obtain the familiar Orr–Sommerfeld equation for the normal velocity component from the set of equations (6), by ignoring the streamline curvature and Coriolis effects. Further neglect of all the terms of order of Re^{-1} in (6) leads to the well-known Rayleigh equation

$$[(\alpha F + \beta G - \omega)(D^2 - \lambda^2) - (\alpha D^2 F + \beta D^2 G)]h = 0, \tag{7}$$

where $\lambda^2 = \alpha^2 + \beta^2$, $\omega = \bar{\omega}$ and $D = (\partial/\partial Z)$.

The homogeneous boundary conditions to be incorporated are given as

$$\begin{aligned} &h = 0 \text{ at } Z = 0, \\ &h \to 0 \text{ as } Z \to \infty. \end{aligned} \tag{8}$$

Equation (7) has a singularity in the flow at the point $Z = Z_c$ satisfying $\alpha F(Z_c) + \beta G(Z_c) = \omega$. This singularity identifies the critical layer and depending upon the non-stationary or stationary character of the flow there might exist two or only a single critical layer. The boundary condition at infinity for the inviscid equation (8) is found analytically from the asymptotic form of (7) so, $h \to e^{-\lambda Z}$ as $Z \to \infty$.

4. Spectral treatment of the stability equations

Several methods are available for solving the linearized stability equations. Malik (1986) solved (6) by eliminating the pressure terms and obtaining a pair of equations for the disturbance vorticity. The method was based on a difference approximation using a two-point fourth-order compact scheme. He finally reduced the equations into a block-tridiagonal matrix form from which he directly computed the eigenvalues and eigenfunctions. Balakumar & Malik (1990) treated the same equations in their primitive form and solved them using the same two-point compact scheme. Lingwood (1995) recently solved the equations by reducing them into first-order ordinary differential equations and using a fourth-order Runge–Kutta integration technique.

In our study we solve these systems of equations by a spectral method. The discretization technique we use here is a spectral collocation method using Chebyshev polynomials as basis functions. A staggered grid is used only in the Z direction. The pressure is defined at the cell centres, at half points, but other components at the cell faces. Since no pressure points fall on the boundary, the critical boundary conditions for the pressure have been avoided. Momentum equations are thus collocated at Gauss–Lobatto points $\cos[k(\pi/N)]$, whereas the continuity equation is imposed at Gauss points $\cos[(k + \frac{1}{2})(\pi/N)]$. The Chebyshev interpolation between these two different grids, from cell centres to cell faces or otherwise is then employed as given by Malik *et al* (1984).

The computational physical plane has been mapped onto spectral space by means of a linear transformation given as

$$\eta = -1 + (2/Z_{\max})Z. \tag{9}$$

Here, $Z_{\max}$ is the far-field boundary of the flow, which throughout the calculations is set to a finite value of 20, about 4 times the boundary-layer thickness.

Standard spectral collocation discretization formulas (see for example Canuto *et al* 1988 and Danabasoglu & Biringen 1989) are used to form matrix differentiation operators and applied to momentum and continuity equations respectively. The mean flow, pre-calculated by a fourth-order Runge–Kutta scheme, is interpolated onto the transformed grid. Finally, transformation matrices are constructed to express the linkage between grid points.

The resulting equations from the discretization ultimately can be assembled in a large generalized matrix-eigenvalue problem in the form

$$LU = 0, \tag{10}$$

where L is the $(4N+3) \times (4N+3)$ full matrix and $U = [f, g, h, p]^{\mathrm{T}}$ is the eigenfunction. The incorporation of boundary conditions only necessitates some modifications to the first and last row of the matrix L. A solution technique for the eigenvalue problem (10) as well as two procedures to compute the branch points making use of the Newton–Raphson method are given by Turkyilmazoglu (1998).

5. Results and discussion

In order to check on the accuracy of the solution technique, we first computed the eigenvalues of the dispersion relation of the mixing-layer problem using the procedure of finding branch points, and reproduced exactly the graphs given by Huerre & Monkewitz (1985).

Prior to displaying the results of direct spatial resonance, for the purpose of checking the accuracy of the methods used for solving the linear stability equations, in the following we first show that the high Reynolds number viscous modes match onto the inviscid Rayleigh modes.

The computed absolute instability range from the inviscid Rayleigh equation is shown in figure 1a. In the β interval displayed in the figure, the complex α, complex ω and wave angle $\varepsilon = \tan^{-1}(\beta/\alpha_r)$ parameters which constitute the absolutely unstable regime of the inviscid flow over a rotating-disk boundary layer are identified. The results of Lingwood (1995) are also indicated by the cross marks in figure 1a, which exhibit excellent agreement with the present calculations. As seen from the figure, for long wavelengths near the vicinity of the origin, the flow is neutrally absolutely unstable. In this region the critical layer moves rapidly towards the wall boundary. A rigorous asymptotic approach is used to resolve this region enabling branch points to be calculated analytically, and the results are given elsewhere (Turkyilmazoglu & Gajjar 2000b). Away from this region, as β increases, $(-\omega_r)$ and the wave angle ε linearly increase, and the spatial amplification rate $(-\alpha_i)$ is almost constant, while the real part of the wave number α_r behaves like $O(\beta^{1/2})$. The suffixes r and i denote the real and imaginary part of a variable respectively. The range of absolute instability which is given by the positive sign of ω_i ceases near the azimuthal wave number $\beta = 0.265$. Here, the corresponding eigenvalues are $\omega_r = -0.070$, $\alpha = 0.34 - i0.058$ and $\varepsilon = 38.12°$, respectively. These points are the upper limits of the region of absolute instability.

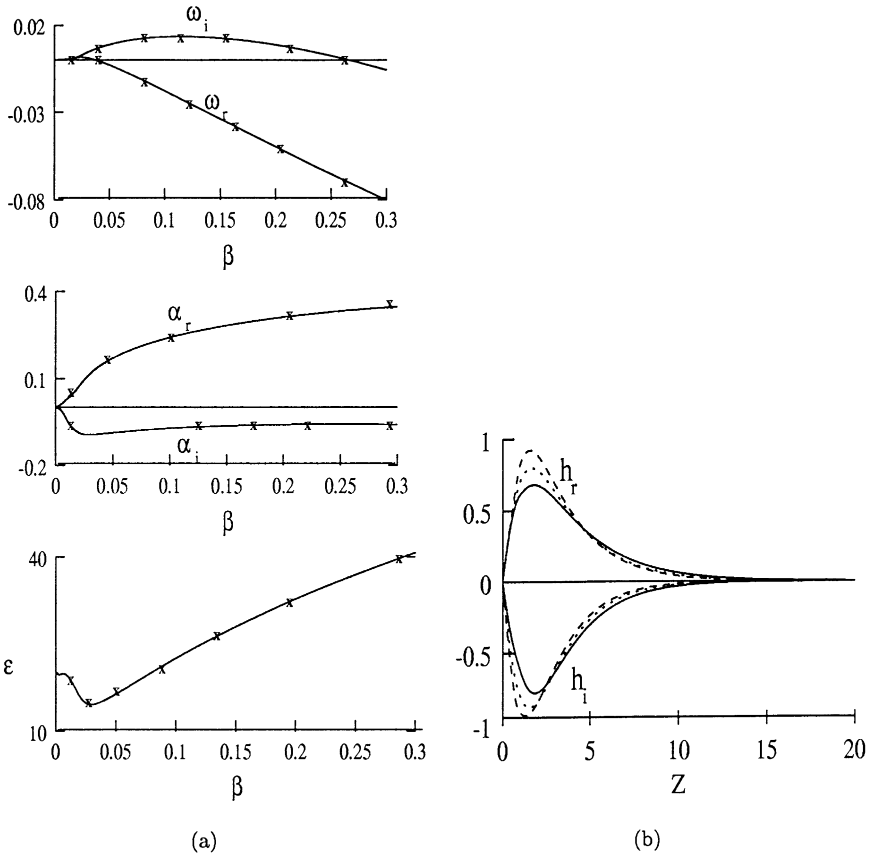

Figure 1. (**a**) Plot showing the locus of the branch points (ω, α) against β. Variation of the wave angle ε is also given. Data is taken from the solution of the inviscid Rayleigh equation (7) and clarifies the range where the rotating-disk flow becomes inviscidly absolutely unstable. Also cross-marked symbols show the results of Lingwood (1995). (**b**) Eigenfunctions of the normal velocity component (h) plotted at selected branch points in the absolutely unstable regime for the inviscid rotating-disk flow. Corresponding branch points are respectively; (− − −) $\beta = 0.31$, $\alpha = (0.334, -0.057)$, $\omega = (-0.0830, -0.0078)$; $(\cdots)$ $\beta = 0.2652$ (upper limit), $\alpha = (0.338, -0.0582)$, $\omega = (-0.0698, 0.0)$; and (—) $\beta = 0.21$, $\alpha = (0.317, -0.0588)$, $\omega = (-0.053, 0.0075)$.

In figure 1b, the development of the eigenfunction (h) is shown in the absolutely unstable region at some selected azimuthal wave numbers $\beta = 0.21, 0.265$ (upper limit for absolute instability) and 0.31 respectively. Later it will be shown that in the limit of large Reynolds numbers the corresponding viscous eigensolution has also a similar shape.

In the viscous solution procedure, the sixth-order system of equations has been solved and the branch points are found using the Newton–Raphson search procedure. These eigenvalues were first compared with those calculated by Lingwood (1995) and found to agree well.

A neutral absolute instability search yields a critical Reynolds number for the flow to undergo absolute instability at $\mathrm{Re} = 507.40$, and the corresponding eigenvalues are

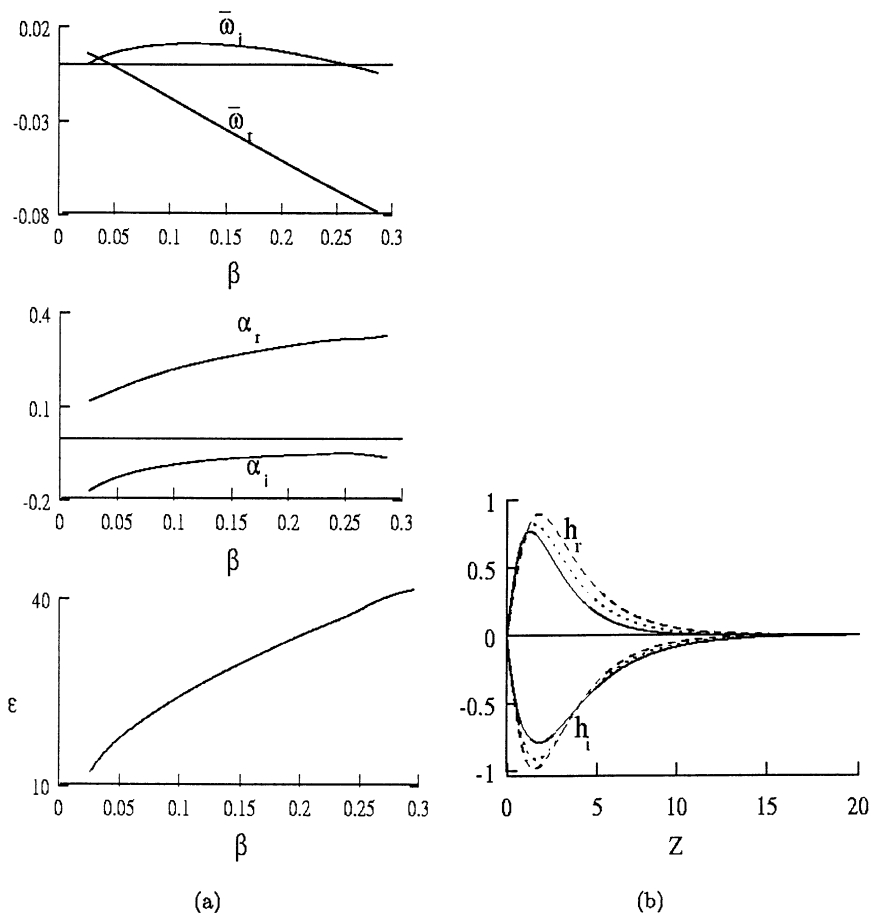

Figure 2. **(a)** Locus of the branch points (ω, α) are given as β varies. Variation of ε is also given. Data are taken from the solution of the full sixth-order viscous equations (6) for Re = 15,000 and reveals the range where the rotating-disk flow becomes viscously absolutely unstable. **(b)** Eigenfunctions of the normal velocity component (h) with the same β values as in figure 1b are shown in the viscous rotating-disk boundary-layer flow at Re = 15,000.

$\beta = 0.135$, $\alpha = (0.217, -0.122)$, $\omega = -17.72$ and $\varepsilon = 31.84°$, respectively. Note that Lingwood (1995) calculated the critical Reynolds number of absolute instability as 510[1].

To make a comparison between the inviscid absolutely unstable region and the viscous absolutely unstable region, we have obtained the branch points at a sufficiently large Reynolds number, namely Re = 15,000. The viscous equations in this case result in an absolutely unstable region as shown in figure 2a (notice that $\bar{\omega} = \omega/\mathrm{Re}$). It can be seen by comparing this figure and figure 4 of Lingwood (1995) (as well as figure 1a) that the characteristics of the eigenvalues at the inviscid and at the viscous branch points are almost the same for the most part of the azimuthal wave number β range. The upper limit in both cases is almost identical, yielding a value of azimuthal wave number $\beta = 0.265$. On the

[1]Lingwood (private communication) has corrected her critical Reynolds number to 507.30, which is now very close to our value.

other hand, when the eigenvalues are sufficiently small, the agreement seems to vanish. The reason for this may be due to round-off errors generated in computing such long-wavelength viscous disturbances. The asymptotic far-field decay becomes much harder to enforce as the domain size needs to be drastically increased in this limit.

A notable feature common in both figures 1a and 2a is that instability waves having negative frequencies prevail in the most part of the absolutely unstable β range. Such travelling waves have also been identified in the theoretical stability calculations of Balakumar & Malik (1990), Faller (1991) and Turkyilmazoglu (1998) as well as in the recent experiments of Jarre & Chauve (1996). The theoretical study of Lingwood (1995) has been the first to prove the dominance of negative frequency disturbances in the domain of absolute instability. Afterwards, the relevance of these negative frequency perturbations with the absolute instability has been made clear through the experiment of Lingwood (1996) for the flow over a rotating-disk. It has been demonstrated in this experiment that the radial propagation of the trailing-edge of excited wave packets tends towards zero as the packet approaches the critical Reynolds number found in this investigation. According to the subsequent suggestion of Lingwood (1996), the above phenomenon in turn initiates an accumulation of energy, which consequently leads to the onset of transition from laminar flow to turbulence.

In figure 2b the evolution of the eigenfunction of normal velocity component (h) is shown again at $\mathrm{Re} = 15000$ for the same β values as in figure 1b. The significance of the results displayed in figure 2 is that they demonstrate the viscous absolutely unstable region match onto the inviscid region shown in figure 1.

The resonance mechanism of Benney & Gustavsson (1981) with modal coalescence can also be observed for three-dimensional disturbances. In fact, the existence of several modes in rotating-disk flow signifies modal coalescence. As mentioned in the introduction, when two coalescing modes originate from waves propagating in the same direction as in the case of a convectively unstable flow, then the corresponding branch point is not a pinch point, but only a double-pole of the dispersion relation. According to linear theory, such disturbances ultimately decay far downstream. However, the short-term algebraic growth associated with such a double-pole may be decisive and the corresponding potentially large amplitudes may initiate nonlinearity and so carry the whole system into the nonlinear stages. This may be particularly important provided that the coalescing modes are linearly neutral. Consequently, the corresponding local response, being of $O(1/\alpha_i)$, can be so large that it may initiate the nonlinear stages before the exponentially growing modes. Such a resonance case between coalescing modes occurring in the same α plane has been studied by Koch (1986). He investigated the direct spatial resonance in plane-Poiseiulle flow and Blasius boundary-layer flow, which are typical examples of convectively unstable flows.

In this paper we use the ideas of Koch (1986) and apply them to the viscous rotating-disk boundary-layer problem in order to investigate whether this flow is subject to direct spatial resonance instabilities. For this purpose, in figure 3a the coalescence of the two spatial branches which originate initially in the same half-α plane, is shown in the fourth portion. The branch point between the two branches happens to occur at $\omega_i = 0.025$, $\omega_r = 9.5$, $\alpha = \alpha_s = 0.201 - i0.0069$, $\beta = 0.014$ and $\mathrm{Re} = 500$ respectively. The ω contours are taken parallel to the real ω-axis and the values of ω_i are 5.45, 2.17, 0.298, 0.025 and 0 respectively. The first three parts of figure 3a correspond to a contour above all values of the temporal eigenvalues, the fourth one to a contour passing at the absolute frequency and the fifth one to the spatial eigenvalue spectrum. Wave number branches in this case with

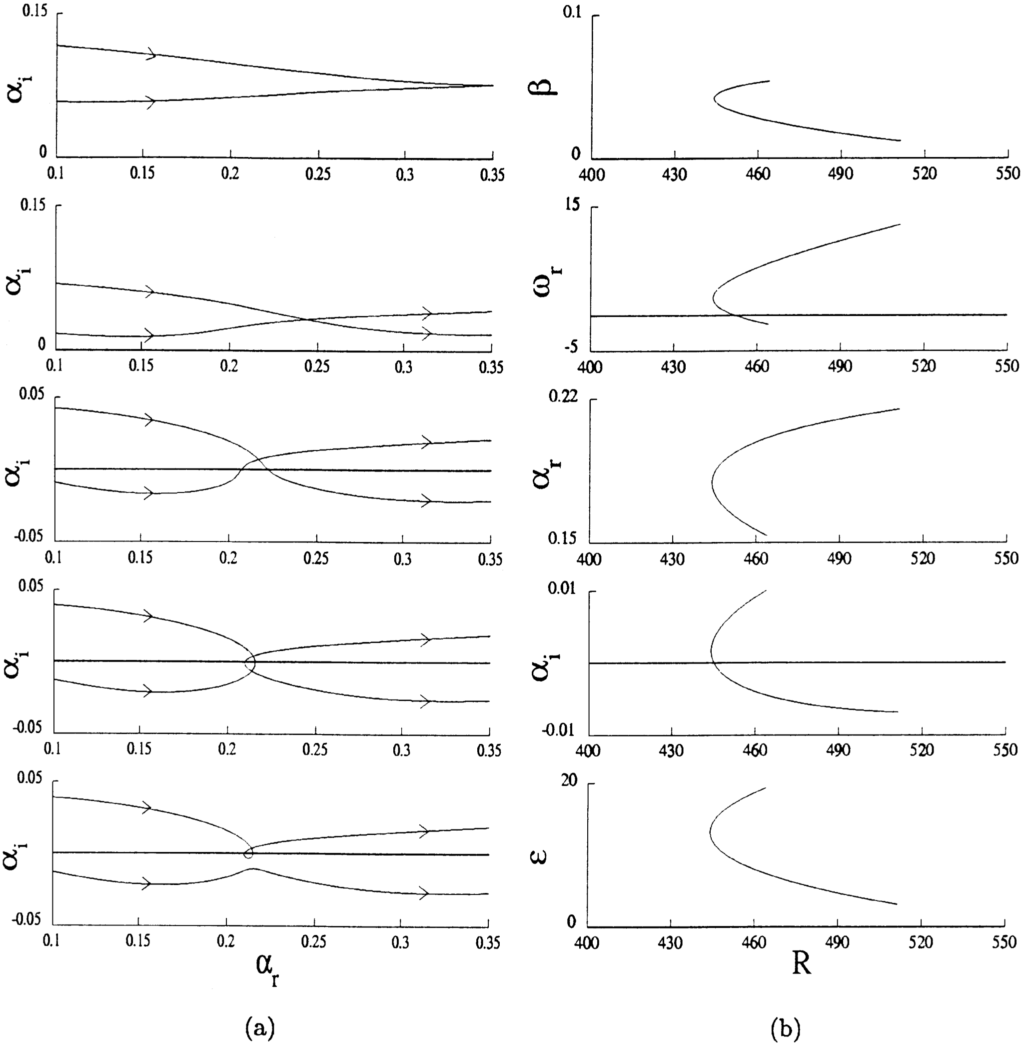

Figure 3. **(a)** The progression of the two spatial branches at Re $= 500$ and $\beta = 0.014$ in the α plane is given in the viscous rotating-disk boundary-layer flow. A typical example of an ordinary branch point, since the pinching requirements are apparently violated. Graphs are for ω_i = 5.45, 2.17, 0.298, 0.025 and 0 respectively. Branch point occurs at $\omega = (9.5,\ 0.025)$, $\alpha = (0.201,\ -0.0069)$ (fourth portion). The direction of the arrows indicates the increasing frequency. **(b)** Neutral branch points of the two emerging branches from the same α plane are shown.

large values of ω_i play no part in the formation of a saddle point at α_s. This is why the branch point does not constitute an absolute instability.

We have made use of the above (α_s, ω) point in our neutral[2] branch point solver and kept track of such points. Figure 3b shows such neutral branch point curves, as a function of Reynolds number. It can be seen in the fourth figure that in the vicinity of a Reynolds number of 445 there occurs direct spatial resonance, in fact it starts at this Reynolds

[2]By neutral it is meant those eigenvalues α and ω which are real and also satisfy the zero group velocity.

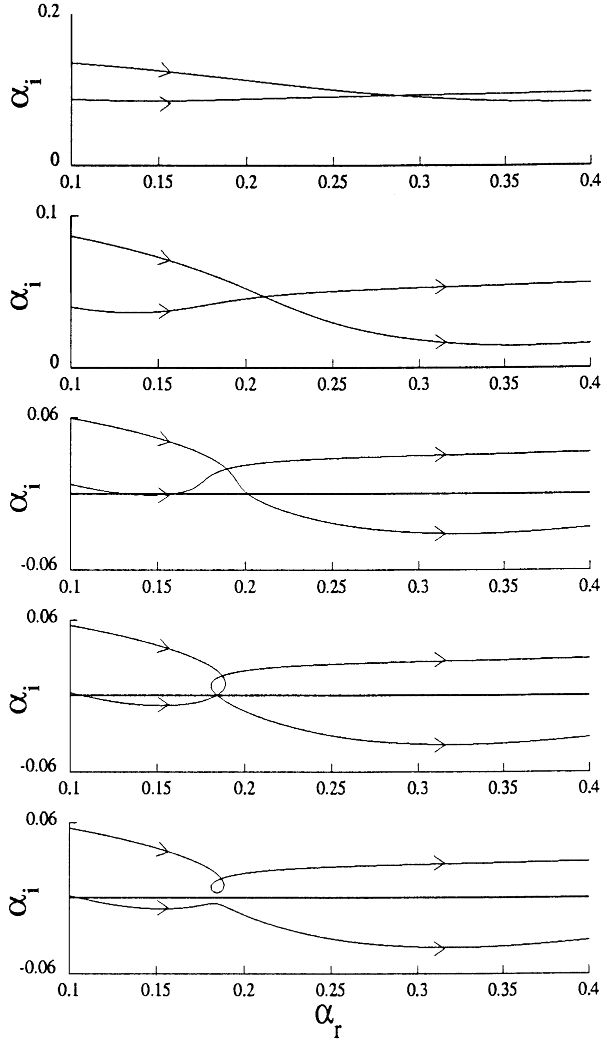

Figure 4. The progression of the two spatial branches at Re $= 445$ and $\beta = 0.0387$ in the α plane is given in the viscous rotating-disk boundary-layer flow. Graphs are for ω_i = 5, 2, 0.25, 0 and -0.02 respectively. Branch point is at $\alpha = (0.184,\ 0)$, $\omega = (3.25,\ 0)$ (fourth portion). A direct spatial resonance occurs between the two spatial branches since the corresponding α and ω are neutral. The direction of the arrows indicates the increasing frequency.

number, where α_i becomes almost zero. In order to support this further, in figure 4 we plot two spatial branches, both emerging from the upper half-α plane. Here, the modal coalescence parameters are $\beta = 0.0387$, $\alpha = (0.184, 0.0)$ and $\omega = (3.25, 0)$ respectively. This point is clearly a bifurcation point with a real ω from which several branches emerge. An indication of the achievable amplitude amplification at this resonance point can be evaluated as the inverse of α_i, for which α_i is practically zero. In fact, to better visualise the impact of such resonance, it is quite sufficient to evaluate some numerical estimates of amplitudes in the neighbouring values of Reynolds number 445. To illustrate this, at

Re $= 460$ the amplitude is approximately an order of magnitude $O(100)$, at Re $= 455$ $O(300)$, at Re $= 450$ $O(700)$ and ultimately as the critical Reynolds number Re $= 445$ is approached, the resonance between the two spatial modes yields a numerically large amplitude of order $O(10^5)$. Therefore, keeping in mind such a huge amplitude amplification leading to algebraic growth, it is suggested that, unlike plane-Poiseiulle flow or Blasius boundary-layer flow, the direct spatial resonance could be physically relevant in the rotating-disk boundary-layer flow. At a Reynolds number of 445 the flow is still in the laminar region. It should also be mentioned that we did not encounter any other branch point apart from the above point which seems to be the only point leading to direct spatial resonance.

We should bear in mind that the investigation presented here presumes a local instability analysis with parallel flow approximation. However, it is widely known that the non-parallel effects are particularly operative near low Reynolds numbers with a destabilizing feature by enlarging the unstable region for growing boundary layers, see for example, Gaster (1974), Fasel & Konzelmann (1990), Bertolotti *et al* (1992) and Malik & Li (1992) amongst others. The effect of non-parallelism, on the other hand, on the absolute instability mechanism is not known yet (an issue currently under investigation by the authors) and to fully understand this effect the complete linearized Navier-Stokes equations should be solved taking into account the downstream/upstream variations in the radial direction. The critical Reynolds number at which the first resonance event leading to algebraic growth takes place (Re $= 445$) has been found to be comparably smaller than the one for the onset of absolute instability (Re $= 507$). Therefore, it can be conjectured in the case of considering the non-parallel terms, even if the non-parallelism influences the absolute instability character of the flow by shifting the Reynolds number for the onset of the absolute instability to a turbulent regime, that the onset of direct spatial resonance will still persist to occur below the critical Reynolds number for the underlying flow to be transitional (approximately Re $= 515$, see Malik *et al* 1981). Then, of course, the absolute instability route to transition diminishes and transition is most probably driven through the direct spatial resonance instability mechanism found in this investigation.

6. Conclusions

In this paper we have applied the method of finding branch points in the light of Briggs–Bers criterion to the three-dimensional rotating-disk boundary layer. Using the singularities in the dispersion relationship that occur when the modes associated with the waves propagating in the same or opposite directions coalesce, the instability regime in the flow has been identified. As shown by Huerre & Monkewitz (1985), the calculated region of local absolute instability for the mixing layer problem may give rise to a self-excited global mode that may contaminate the entire mean flow field, which is also possible for the rotating-disk flow. The results we obtain suggest that rotating-disk boundary-layer flow is subject to both convective and absolute instabilities in some regions of the eigenvalues. Starting from Re $= 445$, a direct spatial resonance between the two families of eigenmodes takes place first. After this, above a certain critical Reynolds number, the rotating-disk boundary layer becomes locally absolutely unstable.

The investigation highlights the direct spatial resonance instability characteristics of the three-dimensional boundary-layer flow due to a rotating-disk. Both the inviscid Rayleigh equation and the full sixth-order viscous equations have been solved to determine the resonant nature of the flow. Two different numerical schemes have been employed to solve these equations. The singularities in the dispersion relation have been studied and several results have been obtained. First, the coalescing modes of spatial and temporal instability waves originate from the waves propagating in the opposite directions. This type of branch point singularity which is related to the absolute instability has been found in a specific range of parameter space. Therefore, we can conclude that, above a certain critical Reynolds number, which has been found to be 507.40 in this study, the flow over the rotating-disk becomes absolutely unstable, causing the disturbances at a fixed radial point in space to grow to large amplitudes in time. Otherwise, below the critical point the flow is convectively unstable, showing that growing disturbances travel downstream and eventually leave the flow undisturbed.

The second main result from this work is that the coalescing modes which originate in the same α wave number plane form a second-order singularity. This kind of singularity has been found to occur in the laminar regime of the flow at about $\mathrm{Re} = 445$. It can be inferred from the existence of such a singularity leading to the direct spatial resonance of the modes that this mechanism may cause a locally algebraic growth and consequently the initiation of nonlinearity. How important this is for the rotating-disk flow remains to be investigated further. It should also be noted that this sort of resonance event may persist to a larger Reynolds number in the case of inclusion of the non-parallel terms to the instability calculations, where the flow would still remain to be laminar, even though the same may not hold for the absolute instability.

The recent work by Cooper & Carpenter (1997b) suggests that the presence of wall compliance suppresses one of the coalescing eigenmodes postponing the absolute instability at least to a higher Reynolds number. It also shows that complete suppression of the absolute instability is possible beyond a critical level of wall compliance, thus removing a major route to transition in the rotating-disk boundary-layer flow. On the other hand, the wall compliance does not seem to have much effect on the spatial instability eigenfamilies, strengthening the probability that the route to transition could be owing to the resonance of two spatially developing eigenfamilies on the same wave number plane.

Furthermore, the results presented here for $O(1)$ Reynolds numbers have been shown to match with the inviscid solutions at large Reynolds numbers. As the long-wavelength limit is approached, the instability waves become neutrally stable (a rigorous asymptotic model verifies the occurrence of such region, see Turkyilmazoglu & Gajjar 2000a).

The work presented here may be extended in many ways. For example, one could compare the neutral stability and growth rates for both the stationary and non-stationary crossflow vortices with the asymptotic theory of Gajjar (1994) in the case of high Reynolds numbers ($\mathrm{Re} \gg 1$). As we have mentioned before, the analysis presented here is a local analysis in which the non-parallel effects were omitted. How non-parallelism affects the absolute instability regime determined from this investigation requires further work and moreover requires the numerical treatment of the full linearized stability equations outlined in this paper. Therefore, a more global analysis which makes use of non-parallel effects should also be considered. This can be done through the development of a linearized unsteady Navier–Stokes solver from which the true behaviour of the crossflow field can be extracted. The initial conditions to the solver can be supplemented by the eigenfunctions obtained from the spectral solution of the linear stability equations. With the aid of this solver some other aspects of the boundary-layer instability such as receptivity, and nonlinearity could also be assessed.

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