Magneto-thermo-visco-elastic waves in an initially stressed conducting layer

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Abstract. The aim of this paper is to investigate magneto-thermo-visco-elastic surface waves in electrically and thermally conducting layers involving time rates of strain and stress of order n , the media being under an initial stress in the nature of hydrostatic tension or compression. The theory of magneto-thermo-visco-elastic surface waves in the conducting medium involving strain rate and stress rate of n th order is derived under initial stress. This theory is then employed to obtain wave velocity equations in specific cases. Results obtained in the above cases reduce to well-known classical results when additional fields are absent.

Keywords. Magneto-thermo-visco-elasticity; visco-elasticity; surface waves; conducting layer; hydrostatic tension compression.

1. Introduction

In recent years, the interaction of Maxwell's electromagnetic fields with the deformation and propagation of waves in conducting solids has been receiving greater attention (for example, Knopoff 1955, Banos 1956, Chadwick 1957, Ewing *et al* 1957, Willson 1963, De & Sengupta 1971, etc). Owing to its theoretical and practical importance in different branches of science, technology and engineering, Yu & Tang (1966) thoroughly studied the dilatational and rotational waves in magneto-elastic initially stressed conductors. They considered three particular cases of initial stress, viz. (a) hydrostatic tension or compression, (b) uniaxial tension or compression and (c) uniform shear stress. De & Sengupta (1971, 1972) investigated magneto-elastic waves and disturbances in initially stressed conducting media. Recently accommodation of thermal fields in the above kind of interaction is receiving greater attention from many investigators (Nowacki 1962; Paria 1962; Willson 1963; Bakshi 1970; Acharya & Sengupta 1978). The effect of viscosity of first order of the conducting media with interaction of the magnetic field has been studied by Das *et al*

(1994, pp 243–255). A limited discussion can also be found in the works of some applied scientists (Knopoff 1955; Flugge 1967; Roy & Sengupta 1985; Das & Sengupta 1990). However, in all the previous cases these effects have been studied either separately or in particular types of problems.

In the present paper, the authors have investigated and reviewed the joint effect of magnetic field, thermal field and initial stress in the form of hydrostatic tension or compression. The viscous nature of the media has been characterised by strain and stress rates of order n , and the waves are propagated in such layers. It is assumed in the present case that heat is radiated from the free boundary surfaces of the layers, the maximum temperature difference across these surfaces always being small (Carslaw & Jaeger 1959; Chadwick 1960). In particular, boundaries are assumed to be thermally insulated (Nowacki 1962).

The practical importance of the investigation presented here is that we may apply our results to the surface waves on Earth. The classical theories of propagation of waves in an elastic layer are very restricted and simplified. However, if the material of the Earth is supposed to be initially stressed and conducting, the above classical theories need proper modification. The inner parts of the Earth are under considerable stress from the weight of the matter resting on its surface. This causes a certain amount of deformation and so the Earth is initially stressed. We may suppose that the initial equilibrium stress is approximately of hydrostatic nature. Moreover, the Earth has its own magnetic field and a thermal field also inside it. Besides these, the material medium of the Earth may be considered visco-elastic in general, in some places satisfying strain and stress rates of order n .

Variations of the wave velocity in a layer due to the above fields are sometimes not of practical importance if the concerned fields are sufficiently weak. However, if the concerned fields and the corresponding effects are significant then the variations must be taken into account. These criteria are considered in the discussion part of the problem in the present paper. It is believed that the present problem has not been investigated earlier.

2. Basic equations

In the absence of displacement current, Maxwell's equations for homogeneous isotropic material (strictly speaking, the material will not remain homogeneous and isotropic when it is subjected to initial stress, magnetic field and thermal field; we, however, ignore such variations in the present investigation) may be written as follows:

$$\begin{aligned}\nabla \times \mathbf{H} &= \mathbf{J}, \quad \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t \\ \nabla \cdot \mathbf{B} &= 0, \quad \mathbf{B} = \mu_e \mathbf{H}\end{aligned}\tag{1}$$

and Ohm's law (Bakshi 1970) is

$$\mathbf{J} = \sigma [\mathbf{E} + (\partial \mathbf{u} / \partial t) \times \mathbf{B}] - k_0 \nabla T,\tag{2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and the magnetic intensity vectors. $\mathbf{B}$ denotes the magnetic induction vector, $\mathbf{J}$ the current density vector, μ_e the magnetic permeability, σ the electric conductivity, k_0 the linking electric field with temperature gradient, $\mathbf{u}$ the displacement vector, T the absolute temperature over initial absolute temperature T_0 and t denotes time.

The n th order stress strain relation for an isotropic visco-elastic medium under temperature field when both strain and stress rates are taken into account is (Flügge 1967; Nowacki 1962; Pal & Sengupta 1987)

$$D_\eta \tilde{\tau}_{jk} = 2D_\mu e_{jk} + [D_\lambda e - D_\beta T]\delta_{jk}, \quad (3)$$

where

$$\begin{aligned} D_\eta &= \eta_1 + \eta_2 \frac{\partial}{\partial t} + \eta_3 \frac{\partial^2}{\partial t^2} + \cdots + \eta_{n+1} \frac{\partial^n}{\partial t^n}, \\ D_\mu &= \mu_1 + \mu_2 \frac{\partial}{\partial t} + \mu_3 \frac{\partial^2}{\partial t^2} + \cdots + \mu_{n+1} \frac{\partial^n}{\partial t^n}, \\ D_\lambda &= \lambda_1 + \lambda_2 \frac{\partial}{\partial t} + \lambda_3 \frac{\partial^2}{\partial t^2} + \cdots + \lambda_{n+1} \frac{\partial^n}{\partial t^n}, \\ D_\beta &= \beta_1 + \beta_2 \frac{\partial}{\partial t} + \beta_3 \frac{\partial^2}{\partial t^2} + \cdots + \beta_{n+1} \frac{\partial^n}{\partial t^n}, \end{aligned} \quad (4)$$

and, $\eta_1, \lambda_1, \mu_1, \beta_1$ stand for classical physical parameters whereas $\eta_i, \lambda_i, \mu_i, \beta_i$ [$i = 2, 3, \dots, n+1$] denote the physical constants of order n . Also,

$$\begin{aligned} e_{jk} &= \frac{1}{2}(u_{j,k} + u_{k,j}), \\ \beta_i &= \alpha_t(3\lambda_i + 2\mu_i) \end{aligned}$$

α_t being the coefficient of linear thermal expansion.

The generalised Fourier Law of heat conduction (in the absence of a heat source) is given by (Paria 1962)

$$k\nabla^2 T = \rho c_v \frac{\partial T}{\partial t} + T_0 D_\beta \frac{\partial e}{\partial t} + \pi_0 \nabla \cdot \mathbf{J}, \quad (5)$$

where k is the coefficient of heat conduction, ρ denotes density of the material, C_v is the specific heat of the body at constant volume, π_0 is the coefficient linking the current velocity with that of heat flow and e is dilatation.

For electrically as well as thermally conducting charge-free thermoelastic solid under an initial stress in an electro-magnetic field, the equation of motion, based on the finite strain theory when the body is subjected to small perturbation is (Yu Tang 1966; Bakshi 1970)

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j} \left[\tau_{jk} \left(\delta_{jk} + \frac{\partial u_i}{\partial x_k} \right) \right] + (\mathbf{J} \times \mathbf{B})_i + F_i, \quad (6)$$

where

$$\begin{aligned} \tau_{jk} &= \tau_{jk}(x_1, x_2, x_3) + \tilde{\tau}_{jk}(x_1, x_2, x_3, t), \\ F_i &= F_i^0(x_1, x_2, x_3) + \tilde{F}_i(x_1, x_2, x_3, t), \\ H_i &= H_i(x_1, x_2, x_3) + \tilde{H}_i(x_1, x_2, x_3, t), \quad i, j, k = 1, 2, 3, \end{aligned}$$

in which the superscript '0' corresponds to the initial equilibrium state and $\tilde{\tau}_{jk}, \tilde{F}_i, \tilde{H}_i$ are the corresponding perturbed quantities with respect to the coordinates (x_1, x_2, x_3) and time t .

From (1) and (2) we have,

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{\mu_2} \nabla \times \left[\mu_e \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H} - \frac{1}{\sigma} \nabla \times \mathbf{H} \right]. \quad (7)$$

Again (6) may be written as

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j} \left[\tau_{jk} \left(\delta_{jk} + \frac{\partial u_i}{\partial x_k} \right) \right] + \mu_e [(\nabla \times \mathbf{H}) \times \mathbf{H}] + \mathbf{F}_i. \quad (8)$$

Equations (7) & (8) are formally identical to the corresponding equations obtained by Yu & Tang (1966) for the magneto-elastic case. Hence following the same process of approximation as Yu & Tang (1966) and Acharya & Sengupta (1978) we arrive at the same set of equations of motion for a perfectly conducting thermo-elastic solid under the hydrostatic tension or compression in a uniform magnetic field as follows:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = -p_0 \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu_e H_0 \left(\frac{\partial \tilde{H}_i}{\partial x_1} - \frac{\partial \tilde{H}_1}{\partial x_i} \right) + \frac{\partial \tilde{\tau}_{ij}}{\partial x_j}, \quad (9)$$

$$\tilde{H}_i = H_0 \left(\frac{\partial u_i}{\partial x_1} - \frac{\partial u_1}{\partial x_i} \right), \quad (10)$$

where $-p_0$ is hydrostatic tension or compression according as $p_0 < 0$ or $p_0 > 0$ and $\tilde{\tau}_{ij}$ is given by (3) and H_0 is the intensity of the uniform magnetic field parallel to the x_1 axis.

3. Formulation of the problem

Let us consider a homogeneous, isotropic, visco-elastic, conducting (both electrically and thermally) solid layer including strain and stress rates of finite thickness $2H$ under an initial hydrostatic tension or compression permeated by uniform magnetic and thermal fields. Let us introduce an orthogonal Cartesian coordinate system $Ox_1x_2x_3$, the origin O being any point on the middle plane of the infinite plate and Ox_3 being a line drawn vertically downwards. There exists a plane wave moving with a constant velocity C in the x_1 direction. From the nature of the problem we assume that all causes and effects depend on two space variables x_1 and x_3 and time t . So we assume displacement vector $\mathbf{u}$ in the form

$$\mathbf{u} = (u_1, 0, u_3), \quad u_i = u_i(x_1, x_3, t), \quad i = 1, 3.$$

We now introduce the displacement potentials $\Phi(x_1, x_3, t)$ and $\Psi(x_1, x_3, t)$ where,

$$\begin{aligned} u_1 &= \frac{\partial \Phi}{\partial x_1} - \frac{\partial \Psi}{\partial x_3}, \\ u_3 &= \frac{\partial \Phi}{\partial x_3} + \frac{\partial \Psi}{\partial x_1}. \end{aligned} \quad (11)$$

Hence we obtain,

$$\begin{aligned} \nabla^2 \Phi &= e, & \nabla^2 \Psi &= \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}, \\ \nabla^2 &\equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}, & e &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}. \end{aligned} \quad (12)$$

Now introducing (3) into (9) we obtain the displacement equations of motion for perfectly conducting visco-elastic medium of n th order under hydrostatic tension or compression in an uniform magnetic as well as thermal field as follows:

$$\begin{aligned}\rho D_\eta \frac{\partial^2 u_1}{\partial t^2} &= (D_\lambda + D_\mu) \frac{\partial e}{\partial x_1} + D_\mu \nabla^2 u_1 - p_0 D_\eta \nabla^2 u_1 - D_\beta \frac{\partial T}{\partial x_1}, \\ \rho D_\eta \frac{\partial^2 u_3}{\partial t^2} &= (D_\lambda + D_\mu) \frac{\partial e}{\partial x_3} + D_\mu \nabla^2 u_3 - p_0 D_\eta \nabla^2 u_3 \\ &\quad + K D_\eta \left(\frac{\partial^2 u_3}{\partial x_1^2} - \frac{\partial^2 u_1}{\partial x_1 \partial x_3} \right) - D_\beta \frac{\partial T}{\partial x_3},\end{aligned}\quad (13)$$

where

$$k = \mu_e H_0^2.$$

Now introducing (11) and (12) into (13) we get

$$\begin{aligned}\frac{\partial^2 \Phi}{\partial t^2} &= \left(M_L - \frac{p_0}{\rho} \right) \nabla^2 \Phi + \frac{K}{\rho} \cdot \frac{\partial^2 \Psi}{\partial x_1 \partial x_3} - \frac{M_T}{\rho} \cdot T, \\ \frac{\partial^2 \Psi}{\partial t^2} &= \left(M_S - \frac{p_0}{\rho} \right) \nabla^2 \Psi + \frac{K}{\rho} \cdot \frac{\partial^2 \Psi}{\partial x_1^2},\end{aligned}\quad (14)$$

in which.

$$\begin{aligned}M_L &= (D_\lambda + 2D_\mu) / (\rho D_\eta), \\ M_S &= D_\mu / (\rho D_\eta), \\ M_T &= D_\beta / D_\eta.\end{aligned}\quad (15)$$

Equation (5) in view of (1), (11) and (12) may be written as,

$$\nabla^2 T = \frac{\rho C_v}{k} - \frac{\partial T}{\partial t} + \frac{T_0}{k} \cdot D_\beta \frac{\partial}{\partial t} (\nabla^2 \Phi). \quad (16)$$

3.1 Boundary conditions

For the present problem the boundary surfaces ($x_3 = \pm H$) of the layer are free from stress, i.e.,

$$\tilde{\tau}_{13} = 0 = \tilde{\tau}_{33}, \quad \text{at } x_3 = \pm H. \quad (17)$$

By (11) & (12) we obtain the stress components from (3) as follows:

$$\begin{aligned}\tilde{\tau}_{13} &= \rho M_S \left[2 \frac{\partial^2 \Phi}{\partial x_1 \partial x_3} - \frac{\partial^2 \Psi}{\partial x_3^2} + \frac{\partial^2 \Psi}{\partial x_1^2} \right], \\ \tilde{\tau}_{33} &= \rho M_L \nabla^2 \Phi - 2\rho M_S \left[\frac{\partial^2 \Phi}{\partial x_1^2} - \frac{\partial^2 \Psi}{\partial x_1 \partial x_3} \right] - M_T T.\end{aligned}\quad (18)$$

Since the temperature difference across the free surface is always small, the linearised form of radiative condition is valid (Carslaw & Jaeger 1959; Chadwick 1960) on each of the boundaries $x_3 = \pm H$.

Clearly, at $x_3 = \pm H$ we have

$$\frac{\partial T}{\partial x_3} + hT = 0, \quad (19)$$

where h is Planck's constant.

4. Solution of the problem

For simple harmonic waves and corresponding temperature fluctuations, we seek the solutions of (14) and (16) in the following form:

$$\begin{aligned} \Phi(x_1, x_3, t) &= \Phi^*(x_3) \cdot \exp[i(\omega t - \eta x_1)], \\ \Psi(x_1, x_3, t) &= \Psi^*(x_3) \cdot \exp[i(\omega t - \eta x_1)], \\ T(x_1, x_3, t) &= T^*(x_3) \cdot \exp[i(\omega t - \eta x_1)]. \end{aligned} \quad (20)$$

Introducing (20) in (14) and (16) we obtain,

$$\begin{aligned} \left[\frac{d^2}{dx_3^2} - \eta^2 + \frac{\omega^2}{M_L^*} \right] \Phi^* &= \frac{i\eta k}{\rho M_L^*} \cdot \frac{d\Psi^*}{dx_3} + \frac{M_T^*}{\rho M_L^*} T^*, \\ \left[\frac{d^2}{dx_3^2} - \eta^2 + \frac{\omega^2}{M_S^*} \right] \Psi^* &= \frac{K\eta^2}{\rho M_S^*} \Psi^*, \\ \left[\frac{d^2}{dx_3^2} - \eta^2 - \frac{\rho C_v i \omega}{k} \right] T^* &= \frac{i\omega T_0 D_\beta^*}{k} \left(\frac{d^2}{dx_3^2} - \eta^2 \right) \Phi^*, \end{aligned} \quad (21)$$

where

$$\begin{aligned} M_L^* &= \frac{D_\lambda^* + 2D_\mu^*}{\rho D_\eta^*} - \frac{p_0}{\rho}, \\ M_S^* &= \frac{D_\mu^*}{\rho D_\eta^*} - \frac{p_0}{\rho}, \quad M_T^* = \frac{D_\beta^*}{D_\eta^*}, \end{aligned} \quad (22a)$$

and

$$\begin{aligned} D_\eta^* &= \sum_{j=0}^n \eta_{j+1} (i\omega)^j, & D_\lambda^* &= \sum_{j=0}^n \lambda_{j+1} (i\omega)^j, \\ D_\mu^* &= \sum_{j=0}^n \mu_{j+1} (i\omega)^j, & D_\beta^* &= \sum_{j=0}^n \beta_{j+1} (i\omega)^j. \end{aligned} \quad (22b)$$

Clearly, solutions of (21) must have exponential order. The solutions of (21) may be taken as,

$$\begin{aligned}\Phi^* &= A \sinh \zeta_1 x_3 + B \cosh \zeta_1 x_3 + c \sinh \zeta_2 x_3 + D \cosh \zeta_2 x_3 \\ &\quad + E \sinh \zeta_3 x_3 + F \cosh \zeta_3 x_3, \\ \Psi^* &= E_1 \sinh \zeta_3 x_3 + F_1 \cosh \zeta_3 x_3, \\ T^* &= A_2 \sinh \zeta_1 x_3 + B_2 \cosh \zeta_1 x_3 + C_2 \sinh \zeta_2 x_3 + D_2 \cosh \zeta_2 x_3 \\ &\quad + E_2 \sinh \zeta_3 x_3 + F_2 \cosh \zeta_3 x_3,\end{aligned}\quad (23)$$

where $\zeta_i^2 = \eta^2 - \delta_i^2$, $i = 1, 2, 3$. In which δ_1^2, δ_2^2 are the solutions of

$$\delta^4 - \delta^2 \left[\frac{\omega^2}{M_L^*} - \frac{i\omega\rho C_v}{k}(1 + \varepsilon) \right] - \frac{i\omega^3\rho C_v}{kM_L^*} = 0, \quad (24)$$

and

$$\delta_3^2 = \frac{\omega^2}{M_S^*} - \frac{K\eta^2}{\rho M_S^*}, \quad \varepsilon = \frac{T_0 M_T^* D_\beta^*}{\rho^2 C_v M_L^*},$$

and the constants in the solutions (23) are related by,

$$\begin{aligned}A_2 &= \gamma_1 A, & B_2 &= \gamma_1 B, & C_2 &= \gamma_2 C, & D_2 &= \gamma_2 D, \\ E_2 &= \gamma_3 E, & F_2 &= \gamma_3 F, & E &= m F_1, & F &= m E_1,\end{aligned}$$

where,

$$\begin{aligned}\gamma_j &= \frac{\rho M_L^*}{M_T^*} \left(\frac{\omega^2}{M_L^*} - \delta_j^2 \right), \quad j = 1, 2, \\ \gamma_3 &= (i\omega T_0 D_\beta^* \delta_3^2) / \left[k \left(\frac{\rho C_v i\omega}{k} + \delta_3^2 \right) \right], \\ m &= \left[\left(\frac{\rho C_v i\omega}{k} + \delta_3^2 \right) \frac{i\eta k}{M_T^*} (\eta^2 - \delta_3^2)^{1/2} \right] \div \\ &\quad \left[\frac{\rho M_L^*}{M_T^*} \left(\frac{\rho C_v i\omega}{k} + \delta_3^2 \right) \left(\frac{\omega^L}{M_L^*} - \delta_3^2 \right) - \frac{i\omega T_0 D_\beta^* \delta_3^2}{k} \right].\end{aligned}\quad (25)$$

In evaluating quantities like $(\eta^2 - \delta_3^2)^{1/2}$ the root with positive real part must be taken in each case.

Hence, (20) and (23) together give the solution of Φ, Ψ, T .

Now employing the boundary conditions (17) and (19) to (20) and (23) we get

$$\begin{aligned}A\xi_1 p_1 - B\alpha_1 p_1 + c\xi_2 p_2 - D\alpha_2 p_2 - E_1\xi_3 p_3 + F_1\alpha_3 p_3 &= 0, \\ A\alpha_1 q_1 - B\xi_1 q_1 + c\alpha_2 q_2 - D\xi_2 q_2 - E_1\alpha_3 q_3 + F_1\xi_3 q_3 &= 0, \\ An_1 p_1 - Bv_1 p_1 + cn_2 p_2 - Dv_2 p_2 - E_1n_3 p_3 + F_1v_3 p_3 &= 0, \\ Av_1 q_1 - Bn_1 q_1 + cv_2 q_2 - Dn_2 q_2 - E_1v_3 q_3 + F_1n_3 q_3 &= 0, \\ As_1 p_1 - Br_1 p_1 + cs_2 p_2 - Dr_2 p_2 - E_1s_3 p_3 + F_1r_3 p_3 &= 0, \\ Ar_1 q_1 - Bs_1 q_1 + cr_2 q_2 - Ds_2 q_2 - E_1r_3 q_3 + F_1s_3 q_3 &= 0,\end{aligned}\quad (26)$$

where,

$$p_j = \sinh \zeta_j H, \quad q_j = \cosh \xi_j H, \quad j = 1, 2, 3, \quad (27)$$

and

$$\begin{aligned} \xi_1 = 0 = \xi_2, \quad \xi_3 = \eta^2 + \zeta_3^2 + zim\eta\xi_3, \\ \alpha_j = zim\eta\xi_j \quad (j = 1, 2), \quad \alpha_3 = 0, \\ \nu_1 = 0 = \nu_2, \quad n_3 = 0, \\ \nu_3 = m\rho \left(M_L^* + \frac{P_0}{\rho} \right) (\eta^2 - \zeta_3^2) \\ \quad - 2\rho \left(M_S^* + \frac{P_0}{\rho} \right) (m\eta^2 - i\eta\zeta_3) + m\zeta_3 M_T^*, \\ n_j = \rho \left(M_L^* + \frac{P_0}{\rho} \right) (\eta^2 - \zeta_j^2) + 2\rho \left(M_S^* + \frac{P_0}{\rho} \right) \eta^2 + \gamma_j M_T^*, \quad (j = 1, 2), \\ s_j = -h\gamma_j, \quad r_j = \gamma_j\zeta_j, \quad (j = 1, 2), \\ s_3 = m\zeta_3\gamma_3, \quad r_3 = -mh\gamma_3. \end{aligned} \quad (28)$$

Now eliminating A, B, C, D, E_1, F_1 from (26) we have the following determinantal equation:

$$\Delta = \begin{vmatrix} \xi_1 & \alpha_1 & \xi_2 \frac{y_2}{y_1} & \alpha_2 \frac{y_2}{y_1} & \xi_3 \frac{y_3}{y_1} & \alpha_3 \frac{y_3}{y_1} \\ n_1 \frac{y_1}{y_2} & \nu_1 \frac{y_1}{y_2} & n_2 & \nu_2 & n_3 \frac{y_3}{y_2} & \nu_3 \frac{y_3}{y_2} \\ s_1 \frac{y_1}{y_3} & r_1 \frac{y_1}{y_3} & s_2 \frac{y_2}{y_3} & r_2 \frac{y_2}{y_3} & s_3 & r_3 \\ \alpha_1 & \xi_1 & \alpha_2 & \xi_2 & \alpha_3 & \xi_3 \\ \nu_1 & n_1 & \nu_2 & n_2 & \nu_3 & n_3 \\ r_1 & s_1 & r_2 & s_2 & r_3 & s_3 \end{vmatrix} \quad (29)$$

where,

$$y_j = \tanh \zeta_j H, \quad j = 1, 2, 3. \quad (30)$$

5. Particular cases

Case 1: Let us consider the particular case in which symmetry and asymmetry of stresses and displacement components are taken into consideration and for which we take,

$$\begin{aligned} \Phi_1 &= (B \cosh \zeta_1 x_3 + D \cosh \zeta_2 x_3 + mE_1 \cosh \zeta_3 x_3) \times \exp[i(\omega t - \eta x_1)], \\ \Psi_1 &= (E_1 \sinh \zeta_3 x_3) \times \exp[i(\omega t - \eta x_1)], \\ T_1 &= (\gamma_1 B \cosh \zeta_1 x_3 + \gamma_2 D \cosh \zeta_2 x_3 + m\gamma_3 E_1 \cosh \zeta_3 x_3) \\ &\quad \times \exp[i(\omega t - \eta x_1)]. \end{aligned} \quad (31)$$

Introducing (31) in (11) and (18) we see that the displacements u_1 and u_3 and the stress component $\tilde{\tau}_{33}$ are symmetric with respect to the plane $x_3 = 0$, while the stress component $\tilde{\tau}_{13}$ is antisymmetric with respect to the same plane. On the boundary, we have

$$\begin{aligned}\tilde{\tau}_{13} &= 0, \\ \tilde{\tau}_{33} &= 0,\end{aligned}\tag{32}$$

in the plane $x_3 = \pm H$.

Further, we assume that boundaries are thermally insulated, so that in this particular case we have $h = 0$ (Nowacki 1962) and hence

$$\partial T_1 / \partial x_3 = 0, \quad \text{on } x_3 = \pm H.\tag{33}$$

Applying boundary conditions (32) and (33) into (31) we get,

$$\begin{aligned}B\alpha_1 p_1 + D\alpha_2 p_2 + E_1 \xi_3 p_3 &= 0, \\ Bn_1 q_1 + Dn_2 q_2 + E_1 v_3 q_3 &= 0, \\ Br_1 p_1 + Dr_2 p_2 + E_1 s_3 p_3 &= 0,\end{aligned}\tag{34}$$

where all the notations are defined in (25), (27), (28).

Eliminating B, D, E_1 from (34) we get the following equation after a little simplification,

$$\frac{\tanh \zeta_1 H}{\tanh \zeta_2 H} = -L \left/ \left[P + Q \frac{\tanh \zeta_2 H}{\tanh \zeta_3 H} \right] \right.\tag{35}$$

where,

$$\begin{aligned}L &= n_1(r_2 \xi_3 - \alpha_2 s_3), \\ P &= n_2(\alpha_1 s_3 - r_1 \xi_3), \\ Q &= v_3(r_1 \alpha_2 - r_2 \alpha_1).\end{aligned}\tag{36}$$

Case 2: We now consider a special case of displacement potential and temperature field as explained in the following section and for which we take

$$\begin{aligned}\Phi_2 &= (A \sinh \zeta_1 x_3 + C \sinh \zeta_2 x_3 + m F_1 \sinh \zeta_3 x_3) \times \exp[i(\omega t - \eta x_1)] \\ \Psi_2 &= (F_1 \cosh \zeta_3 x_3) \times \exp[i(\omega t - \eta x_1)] \\ T_2 &= (\gamma_1 A \sinh \zeta_1 x_3 + \gamma_2 C \sinh \zeta_2 x_3 + m \gamma_3 F_1 \sinh \zeta_3 x_3) \\ &\quad \times \exp[i(\omega t - \eta x_1)].\end{aligned}\tag{37}$$

Introducing (37) in (11) and (18) we see that the displacement components u_1 and u_3 and the stress components $\tilde{\tau}_{33}$ are anti-symmetric with respect to the plane $x_3 = 0$, whereas $\tilde{\tau}_{13}$ is symmetric with respect to the same plane.

Proceeding in the same way as in case 1, we obtain the following wave-velocity transcendental equation.

$$\frac{\tanh \zeta_1 H}{\tanh \zeta_2 H} = - \left[P + Q \cdot \frac{\tanh \zeta_3 H}{\tanh \zeta_2 H} \right] / L,\tag{38}$$

where L, P, Q are the same as in (36).

6. Various situations and discussions

The determinantal equation (29) represents the wave velocity equation of the waves propagated in a magneto-thermo-visco-elastic conducting layer including strain and stress rates of order n permeated by a uniform magnetic field parallel to the x_1 -axis and a uniform thermal field as well as under the hydrostatic tension or compression. It is easy to see that this wave-propagation essentially depends on the magnetic and thermal fields and also on the initial stress and visco-elastic nature of the material medium.

First, assuming a situation where the wavelength is very small in comparison with the thickness $2H$ of the layer, the quantities $\zeta_j H$ ($j = 1, 2, 3$) and ηH become so large that the ratios of the hyperbolic tangents in (29) are approximately unity and hence (29) reduces to

$$\Delta_1 \cdot \Delta_2 = 0,$$

i.e. either

$$\Delta_1 = 0 \quad \text{or} \quad \Delta_2 = 0, \quad (39)$$

where

$$\Delta_1 = \begin{vmatrix} \xi_1 + \alpha_1 & \xi_2 + \alpha_2 & \xi_3 + \alpha_3 \\ n_1 + \nu_1 & n_2 + \nu_2 & n_3 + \nu_3 \\ s_1 + \gamma_1 & s_2 + \gamma_2 & s_3 + \gamma_3 \end{vmatrix},$$

and

$$\Delta_2 = \begin{vmatrix} \xi_1 - \alpha_1 & \xi_2 - \alpha_2 & \xi_3 - \alpha_3 \\ n_1 - \nu_1 & n_2 - \nu_2 & n_3 - \nu_3 \\ s_1 - \gamma_1 & s_2 - \gamma_2 & s_3 - \gamma_3 \end{vmatrix}. \quad (40)$$

Here $\Delta_1 = 0$ represents the wave-velocity equation for the magneto-thermo-visco-elastic Rayleigh waves under the initial stress of hydrostatic tension or compression involving both strain rate and stress rate of order n , propagating in a semi-infinite medium having a plane horizontal boundary in the uppermost part of the solid. On the other hand, if the semi-infinite body has a plane boundary at its lowest part, then $\Delta_2 = 0$ is the corresponding wave-velocity equation. Although the thickness of the layer is assumed to be finite and large, in the case of Rayleigh waves it has to be considered that it is a semi-infinite medium having a plane boundary at the uppermost or at the lowest part and for this, the Rayleigh wave-velocity equation occurs twice, depending on the whether the free plane boundary is at the top or the bottom of the medium respectively. In the first case, the medium is extended to infinity at the lower part where $\Delta_1 = 0$ represents the type of wave propagating in the vicinity of the free upper plane boundary. The latter is just the reverse of this, where $\Delta_2 = 0$ signifies the nature of Rayleigh waves propagating in the vicinity of the free lower plane boundary when the medium is extended towards infinity at the top. It is evident from the mathematical form of the two equations $\Delta_1 = 0$, $\Delta_2 = 0$ that they are inter-changable simply by converting ζ_j to $-\zeta_j$ ($j = 1, 2, 3$), i.e. by changing the direction of the x_3 -axis. This explains the existence of two wave-velocity equations.

Let us first study the equation $\Delta_1 = 0$; $\Delta_2 = 0$ may then be treated in similar fashion. As the equation $\Delta_1 = 0$ contains ζ , the magneto-thermo-visco-elastic Rayleigh waves in initially stressed medium is subjected to dispersion of general waveform.

In the absence of initial stress and magnetic field ($p_0 = 0, k = 0$), the determinantal equation $\Delta_1 = 0$ reduces to

$$\begin{aligned} & \left(2 - \frac{c^2}{M_S^*}\right)^2 \left(\sigma_1^2 + \sigma_1\sigma_2 + \sigma_2^2 - 1 + \frac{c^L}{M_L^*}\right) - 4\sigma_1\sigma_2\sigma_3(\sigma_1 + \sigma_2) \\ &= \frac{h}{\eta} \left[\left(2 - \frac{c^L}{M_S^*}\right)^2 (\sigma_1 + \sigma_2) - 4\sigma_3 \left(\sigma_1\sigma_2 + 1 - \frac{c^2}{M_L^*}\right) \right], \end{aligned} \quad (41)$$

where $\sigma_j^2 = 1 - \delta_j^2/\eta^2$, $j = 1, 2, 3$, $c = \text{velocity of wave} = \omega/\eta$, and

$$M_L^* = \frac{D_\lambda^* + 2D_\mu^*}{\rho D_\eta^*}, \quad M_S^* = \frac{D_\mu^*}{\rho D_\eta^*}.$$

The equation (41) is the wave-velocity equation for the n th order thermo-visco-elastic Rayleigh waves (Das & Sengupta 1990). Moreover, if we neglect the effect of viscosity we get a result similar to (41), the only difference being,

$$M_L^* = \frac{\lambda_1 + 2\mu}{\rho}, \quad M_S^* = \frac{\mu_1}{\rho},$$

and this result is in agreement with the corresponding result obtained by Chadwick (1960) under thermo-elastic conditions.

If we neglect the effect of temperature and consider stress and strain rates of order n then we have from (40),

$$\Delta_1 = \begin{vmatrix} \xi_1 + \alpha_1 & \xi_3 + \alpha_3 \\ n_1 + l_1 & n_3 + v_3 \end{vmatrix} = 0.$$

On simplification this yields,

$$\begin{aligned} & \left(2 - \frac{c^L}{M_S^*}\right)^2 + \left(2 - \frac{c^L}{M_S^*}\right) \frac{K}{\rho M_S^*} - \frac{4\xi_1\xi_3}{\eta^2} \\ &+ \frac{2i\gamma_1}{\eta} (\xi_1 - \xi_3) \left[\left(\frac{\xi_1\xi_3}{\eta^2} + 1\right) \frac{M_L^*}{M_S^*} - 2 \right] = 0. \end{aligned} \quad (42)$$

This wave-velocity equation (42) corresponds to the n th order magneto-visco-elastic Rayleigh waves. We note that the equivalent result with $n = 1$ is in perfect agreement with the result obtained by Das *et al* (1994).

In addition to the assumption made in (42), if we omit the viscosity effect and the strain and stress rates we get,

$$\alpha_1 v_3 - n, \xi_3 = 0, \quad (43)$$

which is the magneto-elastic Rayleigh wave velocity equation under initial stress of hydrostatic tension or compression and is in good agreement with the result obtained by Acharya & Sengupta (1978).

In (41), when there are no viscosity or thermal effects, we expect the Rayleigh wave-velocity equation to hold for elastic media. In this case, our result obtained from (41) is of the form

$$\left(2 - \frac{C^L}{M_S^*}\right)^2 = 4 \left[\left(1 - \frac{C^L}{M_L^*}\right) \left(1 - \frac{C^L}{M_S^*}\right) \right]^{1/2}, \quad (44)$$

where $c = \omega/\eta$, $M_L^* = (\lambda_1 + 2\mu_1)/\rho$, $M_S^* = \mu_1/\rho$ and agrees with the classical result of Rayleigh (1885).

Next, we consider the case for large wavelengths relative to the thickness of the layer. Here the arguments of the hyperbolic functions in (29) becomes small and we make the assumption that

$$\tanh \zeta_j H \simeq \zeta_j H, \quad j = 1, 2, 3.$$

Hence, the corresponding wave-velocity equation may be obtained from (29) by taking $y_j = \zeta_j H$, $j = 1, 2, 3$.

For the particular case 1, if the length of the wave is large when compared with the thickness of the layer by the above approximation, we have from (35)

$$\frac{\zeta_1}{\zeta_2} = -L \left[P + Q \frac{\zeta_2}{\zeta_3} \right]. \quad (45)$$

If the layer is initially stress-free, the magnetic field as well as the thermal field are absent from (45) (setting $k = 0$, $p_0 = 0$, $\gamma_1, \gamma_3 \rightarrow 0$, $\sigma_2 \rightarrow \alpha$) we get,

$$C^L M_L^* = 4M_S^*(M_L^* - M_S^*). \quad (46)$$

Equation (46) is the classical equation determining the wave-velocity in a visco-elastic layer including both strain and stress rates of order n . If we neglect the viscosity effect, (46) determines the wave-velocity in the classical isotropic elastic layer (Lamb 1916; Bullen 1963).

Again, if length of the wave is very small compared to the thickness of the layer, the quantities $\zeta_j H$ ($j = 1, 2, 3$), ηH are large and the ratios of hyperbolic tangents in (35) approach infinity. Hence from (35) we get,

$$P + Q + L = 0, \quad (47)$$

which is the equation determining the wave-velocity of Rayleigh surface waves in a magneto-thermo-visco-elastic initially stressed conducting (thermally and electrically) layer whose boundary is thermally insulated and which involves the stress and strain rates of order n . This result is in agreement with the result $\Delta_1 = 0$ in (39) obtained for $h = 0$.

When $p_0 = 0$, $k = 0$, i.e. the initial stress and magnetic fields are absent, (47) produces the thermo-visco-elastic Rayleigh wave-velocity equation including strain and stress rates as,

$$\left(2 - \frac{C^L}{M_S^*}\right)^2 \left(\sigma_1^2 + \sigma_1 \sigma_2 + \sigma_2^2 - 1 + \frac{C^L}{M_L^*} \right) - 4\sigma_1 \sigma_2 \sigma_3 (\sigma_1 + \sigma_2) = 0. \quad (48)$$

When we neglect the viscosity effect also, a similar result is obtained for the Rayleigh wave-velocity equation in a thermal field and the reduced equation is in accordance with the result obtained by Chadwick (1960) for $h = 0$.

Now, we consider the particular case 2 with the assumption that the length of the waves is large compared to the thickness $2H$ of the layer. For this criterion we may approximate the hyperbolic tangents to the first term of their expansion into series. So we get from (38)

$$\frac{\xi_1}{\xi_2} = - \left[P + Q \frac{\xi_3}{\xi_2} \right] / L. \quad (49)$$

Here again, if we withdraw the magnetic field, the thermal field, and the initial stress, proceeding exactly in the same manner as done earlier we find from (49) that the wave velocity $c(= \omega/\eta)$ becomes zero. Hence, to obtain classical results parallel to those of Rayleigh (1885) and Lamb (1916) in visco-elastic media with stress and strain rates, the hyperbolic tangents have to be approximated up to the second term in their expansions. Classical results, due to Rayleigh (1885) and Lamb (1916) in elastic media, immediately follow from this equation on omitting the term due to viscosity.

At the end of the discussion, we consider the case when the wavelength is very small compared to the thickness of the layer. Here (38) reduces to the same form of (47) which gives the velocity of Rayleigh surface waves in magneto-thermo-visco-elastic initially stressed conducting layer, whose boundary is thermally insulated and which involves strain and stress rates.

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