Effect of surface stresses on surface waves in elastic solids

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Abstract. This paper deals with the propagation of surface waves in homogeneous, elastic solid media whose free surfaces or interfaces of separation are capable of supporting their own stress fields. The general theory for the propagation of surface waves in a medium which supports surface stresses is first deduced, and then this theory is employed to investigate the particular cases of surface waves, viz. (a) Rayleigh waves, (b) Love waves and (c) Stoneley waves. It is seen that the Rayleigh waves become dispersive in nature; and, in case of low frequency with residual surface tension, a critical wavelength exists, below which the propagation of Rayleigh waves is not possible. This critical wavelength is directly proportional to the surface tension. Some numerical calculations have been made in the case of Love waves and conclusions have been drawn.

Keywords. Surface waves; effect of surface stresses; Rayleigh waves; Love waves; Stoneley waves; residual stress.

1. Introduction

Surface waves play an important role in the science of earthquakes. To match the actual situation in a given problem, three types of surface waves are generally introduced which are classified according to their nature of displacements (Dey & Sengupta 1978; Chandrasekharaiah 1986; Das *et al* 1992; Pal *et al* 1996). Again it is known that the physical properties of bodies in the neighbourhood of the surface are sensibly different from those in the interior. Thus the boundary surface may be regarded as a two-dimensional elastic continuum which adheres to the body without slipping (Gurtin & Murdoch 1976; Chandrasekharaiah 1987). Different authors (Plaster 1972; Chandrasekharaiah 1987) presented their research papers including surface stresses as their subject of discussion. In the

present paper, following a theory of surface stresses, the authors investigate surface wave propagation. The Rayleigh wave velocity equation has been obtained as a particular case. Following Chandrasekharaiah (1987), the effect of surface stresses on Rayleigh wave propagation has been discussed. Next we discuss Love wave propagation with surface stress on its free boundary, with some numerical computations and conclusions. Wave velocity equation for Stoneley waves has also been deduced.

2. Basic equations

Let us consider two homogeneous, isotropic, elastic solid media M_1 (lower medium) and M_2 (upper medium) with different material constants. We also consider that a two-dimensional elastic layer is present at the surface of separation of the two media. We take an orthogonal cartesian frame of axes $Ox_1x_2x_3$, O is taken on the interface and Ox_3 is positive in the downward direction. As in Bullen & Bolt (1985), here also the wave travels in the Ox_1 direction. The disturbances of the particles on lines parallel to Ox_2 vibrate in phase, which means all partial derivatives with respect to x_2 are zero.

The equation of motion in the absence of body forces is

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \text{grad div } \mathbf{u} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \quad (1)$$

In the above Navier's equation of classical elasticity $\mathbf{u}$ (u_1, u_2, u_3) is the displacement vector; λ, μ are Lamé elastic constants; ρ is mass density and t represents time. To investigate plane deformation parallel to the x_1x_3 plane we introduce the displacement potentials P, Q related to the displacement components $u_1(x_1, x_3, t)$ and $u_3(x_1, x_3, t)$ by the equations,

$$u_1 = \frac{\partial P}{\partial x_1} - \frac{\partial Q}{\partial x_3}, \quad u_3 = \frac{\partial P}{\partial x_3} + \frac{\partial Q}{\partial x_1}. \quad (2)$$

Introducing (2) in (1) we obtain,

$$\left(\nabla^2 - \frac{1}{a^2} \frac{\partial^2}{\partial t^2} \right) P = 0, \quad (3)$$

$$\left(\nabla^2 - \frac{1}{b^2} \frac{\partial^2}{\partial t^2} \right) Q = 0, \quad (4)$$

$$\left(\nabla^2 - \frac{1}{b^2} \frac{\partial^2}{\partial t^2} \right) u_2 = 0, \quad (5)$$

where $a = (\lambda + 2\mu/\rho)^{1/2}$ and $b = (\mu/\rho)^{1/2}$ are velocities of dilatation and distortion respectively. Befitting the actual situation of the problem, we seek solutions to (3)–(5) as

$$[P, Q, u_2] = [\bar{P}(x_3), \bar{Q}(x_3), \bar{u}_2(x_3)] e^{i(\eta x_1 - \zeta t)}. \quad (6)$$

Inserting (6) in (3)–(5), we get

$$\begin{aligned}\frac{d^2 \bar{P}}{dx_3^2} - \eta^2 \left(1 - \frac{\zeta^2}{\eta^2 a^2}\right) \bar{P} &= 0, \\ \frac{d^2 \bar{Q}}{dx_3^2} - \eta^2 \left(1 - \frac{\zeta^2}{\eta^2 b^2}\right) \bar{Q} &= 0, \\ \frac{d^2 \bar{u}_2}{dx_3^2} - \eta^2 \left(1 - \frac{\zeta^2}{\eta^2 b^2}\right) \bar{u}_2 &= 0.\end{aligned}\quad (7)$$

Now, $\bar{P}(x_3)$, $\bar{Q}(x_3)$, $\bar{u}_2(x_3)$ describe surface waves and as such they must be vanishingly small as $x_3 \rightarrow \infty$. Hence the solution of (5) for the medium M_1 may be taken as

$$\begin{aligned}P &= A_1 \exp(-\eta m_1 x_3) e^{i(\eta x_1 - \zeta t)}, \\ Q &= A_2 \exp(-\eta m_2 x_3) e^{i(\eta x_1 - \zeta t)}, \\ u_2 &= A_3 \exp(-\eta m_2 x_3) e^{i(\eta x_1 - \zeta t)},\end{aligned}\quad (8)$$

where

$$\begin{aligned}m_1 &= (1 - r^2)^{1/2}, \quad m_2 = (1 - s^2)^{1/2}, \\ c &= \frac{\zeta}{\eta}, \quad r = \frac{c}{a}, \quad s = \frac{c}{b},\end{aligned}\quad (9)$$

wavelength $= 2\pi/\eta$, wave velocity $= \zeta/\eta$, ζ being a known constant while η is an unknown constant; and, A_1 , A_2 , A_3 are arbitrary constants.

Analogous solutions with primes (for the region $0 \leq x_3 < -\infty$) are

$$\begin{aligned}P' &= A'_1 \exp(\eta m'_1 x_3) e^{i(\eta x_1 - \zeta t)}, \\ Q' &= A'_2 \exp(\eta m'_2 x_3) e^{i(\eta x_1 - \zeta t)}, \\ u'_2 &= A'_3 \exp(\eta m'_2 x_3) e^{i(\eta x_1 - \zeta t)},\end{aligned}\quad (10)$$

are obtained for the upper medium M_2 of the material, where

$$\begin{aligned}m'_1 &= (1 - r'^2)^{1/2}, \quad m'_2 = (1 - s'^2)^{1/2}, \\ c &= \frac{\zeta}{\eta}, \quad r' = \frac{c}{a'}, \quad s' = \frac{c}{b'}.\end{aligned}\quad (11)$$

3. Boundary conditions and solutions

We assume that the plane $x_3 = 0$ is a material layer which adheres, without slipping, to the lower medium and is a two-dimensional elastic continuum. This layer is capable of supporting its own stress represented by surface stress tensor $\sum_{i\alpha}$, and obeys the law (Chandrasekharaiah 1987),

$$\sum_{i\alpha} = \begin{cases} \delta_{i\alpha} [\sigma + (\lambda_0 + \sigma) u_{\gamma,\gamma}] + \mu_0 u_{i,\alpha} + (\mu_0 - \sigma) u_{\alpha,i}, & \text{for } i, \alpha, \gamma = 1, 2, \\ \sigma u_{3,\alpha}, & \text{for } i = 3. \end{cases}\quad (12)$$

Here λ_0 , μ_0 are the Lamé moduli of the material boundary and σ is the residual surface tension on the layer $x_3 = 0$. Following Gurtin & Murdoch (1976) we use the surface elasticity tensor. The forces on the bounding surface are governed by surface stress tensor $\sum_{i\alpha}$. Given a smooth oriented curve γ in $x_3 = 0$ with positive unit normal n_i , $\sum_{i\alpha} n_\alpha$ is the force per unit length of γ by that part of the boundary into which n_α is directed, upon that part of $x_3 = 0$ away from which n_α points. Dimensions of λ_0 , μ_0 , σ are the same (N/m).

The boundary conditions are

$$(i) \quad \tau_{i3} + \sum_{i\alpha} -\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \tau'_{i3}, \quad \text{on } x_3 = 0,$$

where ρ_0 is the mass per unit surface area (kg m^{-2}) of the layer (Gurtin & Murdoch 1976), and τ_{ij} and τ'_{ij} are the stress tensors in the interior of the media M_1 and M_2 respectively. The conventional stress tensor τ_{i3} has the unit of force per unit area and the new stress tensor $\sum_{i\alpha}$ has the unit of force per unit length. τ_{ij} obeys the law (Gurtin & Murdoch 1976),

$$\tau_{ij} = \lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i}). \quad (13)$$

The boundary conditions are

$$\begin{aligned} & \mu \left(2 \frac{\partial^2 P}{\partial x_1 \partial x_3} + \frac{\partial^2 Q}{\partial x_1^2} - \frac{\partial^2 Q}{\partial x_3^2} \right) + \left[(\lambda_0 + 2\mu_0) \frac{\partial^2}{\partial x_1^2} - \rho_0 \frac{\partial^2}{\partial t^2} \right] \left(\frac{\partial P}{\partial x_1} - \frac{\partial Q}{\partial x_3} \right) \\ &= \mu' \left(2 \frac{\partial^2 P'}{\partial x_1 \partial x_3} + \frac{\partial^2 Q'}{\partial x_1^2} - \frac{\partial^2 Q'}{\partial x_3^2} \right), \\ & 2\mu \left(\frac{\partial^2 Q}{\partial x_1 \partial x_3} - \frac{\partial^2 P}{\partial x_1^2} \right) + \frac{\mu}{b^2} \frac{\partial^2 P}{\partial t^2} + \left(\alpha \frac{\partial^2}{\partial x_1^2} - \rho_0 \frac{\partial^2}{\partial t^2} \right) \left(\frac{\partial P}{\partial x_3} + \frac{\partial Q}{\partial x_1} \right) \\ &= 2\mu' \left(\frac{\partial^2 Q'}{\partial x_1 \partial x_3} - \frac{\partial^2 P'}{\partial x_1^2} \right) + \frac{\mu'}{b'^2} \frac{\partial^2 P'}{\partial t^2} \left(\mu_0 \frac{\partial^2}{\partial x_1^2} - \rho_0 \frac{\partial^2}{\partial t^2} + \mu \frac{\partial}{\partial x_3^2} \right) u_2 \\ &= \mu' \frac{\partial^2 u'_2}{\partial x_3^2}. \end{aligned} \quad (14)$$

(ii) The displacement components must be continuous on the surface of separation $x_3 = 0$ at all places and times. Therefore

$$\begin{aligned} \frac{\partial P}{\partial x_1} - \frac{\partial Q}{\partial x_3} &= \frac{\partial P'}{\partial x_1} - \frac{\partial Q'}{\partial x_3}, \\ \frac{\partial P}{\partial x_3} + \frac{\partial Q}{\partial x_1} &= \frac{\partial P'}{\partial x_3} + \frac{\partial Q'}{\partial x_1}, \\ u_2 &= u'_2. \end{aligned} \quad (14a)$$

Applying boundary conditions (14) and (14a) to (8) and (10) the following system of equations is obtained,

$$\begin{aligned}
 (2m_1 + \eta F)A_1 - i(2 - s^2 + \eta m_2 F)A_2 \\
 + \frac{2\mu'}{\mu}m'_1A'_1 + i\frac{\mu'}{\mu}(2 - s'^2)A'_2 &= 0, \\
 (2 - s^2 + \eta m_1 H)A_1 - i(2m_2 + \eta H)A_2 \\
 - \frac{\mu'}{\mu}(2 - s'^2)A'_1 - i2\frac{\mu'}{\mu}m'_2A'_2 &= 0, \\
 A_1 - im_2A_2 - A'_1 - im'_2A'_2 &= 0, \\
 m_1A_1 - iA_2 + m'_1A'_1 + iA'_2 &= 0, \\
 \left(\mu_0 - \rho_0c^2 + \frac{\mu}{\eta}m_2\right)A_3 + \frac{\mu'}{\eta}m'_2A'_3 &= 0, \\
 A_3 - A'_3 &= 0,
 \end{aligned} \tag{15}$$

where we have taken

$$\begin{aligned}
 \frac{1}{\mu}(\lambda_0 + 2\mu_0 - \rho_0c^2) &= F, \\
 \frac{1}{\mu}(\sigma - \rho_0c^2) &= H.
 \end{aligned} \tag{16}$$

The last two equations of (15) yield

$$A_3 = A'_3 = 0,$$

which implies that there is no propagation of displacement u_2 . Eliminating the constants A_1, A_2, A'_1, A'_2 from (15) we obtain

$$\Delta = \det[a_{ij}] = 0, \quad i, j = 1, 2, 3, 4, \tag{17}$$

where

$$\begin{aligned}
 a_{11} &= 2m_1 + \eta F, \quad a_{12} = 2 - s^2 + \eta m_2 F, \\
 a_{13} &= 2\frac{\mu'}{\mu}m'_1, \quad a_{14} = \frac{\mu'}{\mu}(2 - s'^2), \\
 a_{21} &= 2 - s^2 + \eta m_1 H, \quad a_{22} = 2m_2 + \eta H, \\
 a_{23} &= -\frac{\mu'}{\mu}(2 - s'^2), \\
 a_{24} &= -2\frac{\mu'}{\mu}m'_2, \\
 a_{31} &= 1, \quad a_{32} = m_2, \quad a_{33} = -1, \quad a_{34} = -m'_2, \\
 a_{41} &= m_1, \quad a_{42} = 1, \quad a_{43} = m'_1, \quad a_{44} = 1.
 \end{aligned}$$

Equation (17) represents the wave velocity dispersion equation for interface waves in elastic solid media under the influence of surface stresses, where the plane of separation

$x_3 = 0$ is a material boundary. The equation depends on the particular value of η which indicates dispersive nature caused by the presence of surface stresses. If the surface of separation is free of surface stresses, $F = H = 0$; and at once the classical equation of general surface waves is obtained from (17).

4. Particular cases

4.1 Rayleigh waves

As a particular case of the general surface wave discussed in the previous article we may study Rayleigh waves under the influence of surface stresses in the light of the investigation done by Chandrasekharaiah (1987). Here the upper medium (M_2) is replaced by vacuum. Applying the boundary conditions

$$\begin{aligned}\tau_{13} + \sum_{1\alpha, \alpha} -\rho_0 \frac{\partial^2 u_1}{\partial t^2} &= 0, \\ \tau_{33} + \sum_{3\alpha, \alpha} -\rho_0 \frac{\partial^2 u_3}{\partial t^2} &= 0,\end{aligned}$$

we get

$$\begin{aligned}(2m_1 + \eta F)A_1 + (2 - s^2 + \eta m_2 F)A_2 &= 0, \\ (2 - s^2 + \eta m_1 H)A_1 + (2m_2 + \eta H)A_2 &= 0.\end{aligned}$$

Eliminating the constants A_1, A_2 the Rayleigh wave velocity equation may be obtained as

$$\begin{vmatrix} 2m_1 + \eta F & 2 - s^2 + \eta m_2 F \\ 2 - s^2 + \eta m_1 H & 2m_2 + \eta H \end{vmatrix} = 0.$$

The determinantal equation on simplification gives

$$(1 - m_1 m_2) F H \eta^2 + (m_1 H + m_2 F) s^2 \eta + 4m_1 m_2 - (2 - s^2)^2 = 0. \quad (18)$$

This equation is in agreement with the corresponding equation obtained by Chandrasekharaiah (1987) in the absence of voids. Again in the case of conventional stress-free boundary, (18) becomes

$$4m_1 m_2 - (2 - s^2)^2 = 0. \quad (19)$$

When surface stresses $\sum_{i\alpha}$ arise from surface tension, we have as a special case the conditions $\lambda_0 = \mu_0 = \rho_0 = 0$. In such a situation $F = 0$, $H = \sigma/\mu$ and hence the equation (18) becomes

$$\eta \sigma s^2 (1 - s^2 \tau_0^2)^{1/2} = \mu [(2 - s^2)^2 - 4(1 - s^2)^{1/2} (1 - s^2 \tau_0^2)^{1/2}], \quad (20)$$

where

$$\tau_0^2 = b^2/a^2. \quad (21)$$

This equation is in agreement with the corresponding equation obtained by Chandrasekharaiah (1987) in the absence of voids. In the absence of residual surface tension, we take $\sigma = 0$ to obtain the classical Rayleigh wave velocity equation (19). In this case, (20) has one root $s = s_0$ in the open interval $(0, 1)$. If $\sigma \neq 0$, (20) has a real root in the interval $(s_0, 1)$, provided $\eta < \eta_c$ where $\eta_c = \mu/[\sigma(1 - \tau_0^2)^{1/2}]$. Again if $\eta = \eta_c$ we see that $s = 1$ is a root of (20). Lastly, the said equation has no real root if $\eta > \eta_c$. Thus we have a critical wavelength, $L_c = 2\pi/\eta_c$, below which Rayleigh waves cannot propagate at all under the influence of surface stresses. The relation,

$$(\mu/2\pi)L_c = \sigma(1 - (b^2/a^2))^{1/2},$$

reveals that critical wavelength varies directly with residual surface tension which is a factor of surface stress.

4.2 Love waves

In order to consider Love waves we shall assume that the upper medium (M_2) is of finite thickness H and the lower medium (M_1) is semi-infinite. Let the origin O of the orthogonal cartesian frame $Ox_1x_2x_3$ be on the interface, Ox_1 and Ox_3 are along the interface and vertically downwards into M_1 respectively. For Love wave propagation along x_1 axis (see figure 1),

$$\begin{aligned} u_1 &= u_3 = 0, \\ u_2 &= u_2(x_1, x_3, t). \end{aligned}$$

Here equations of motion are

$$\begin{aligned} \left(\nabla^2 - \frac{1}{b^2} \frac{\partial^2}{\partial t^2} \right) u_2 &= 0, \\ \left(\nabla^2 - \frac{1}{b'^2} \frac{\partial^2}{\partial t^2} \right) u'_2 &= 0. \end{aligned} \quad (22)$$

Figure 1. Love waves and their propagation.

The solutions of (22) are

$$\begin{aligned} u_2 &= A_1 \exp(-\eta m_2 x_3) e^{i(\eta x_1 - \zeta t)}, \\ u'_2 &= \{A_2 \exp(-\eta m'_2 x_3) + A_3 \exp(\eta m'_2 x_3)\} e^{i(\eta x_1 - \zeta t)}, \end{aligned} \quad (23)$$

where

$$\begin{aligned} m_2 &= (1 - s^2)^{1/2}, \\ m'_2 &= (1 - s'^2)^{1/2}, \\ C &= \frac{\zeta}{\eta}, \quad s = \frac{C}{b}, \quad s' = \frac{C}{b'}. \end{aligned} \quad (24)$$

The boundary conditions are:

- (i) u_2 is continuous on the boundary surface of separation $x_3 = 0$;
- (ii) τ_{23} is continuous on the boundary surface of separation $x_3 = 0$;
- (iii) since the boundary surface $x_3 = -H$ is free of external loads

we have (Gurtin & Murdoch 1974–75)

$$\tau'_{23} + \sum_{2\alpha, \alpha} = \rho_0 \frac{\partial^2 u'_2}{\partial t^2} \quad \text{for } x_3 = -H.$$

Applying the above boundary conditions to (23) and then eliminating the constants A_1 , A_2 , A_3 we get the wave velocity equation for Love waves under the influence of surface stresses in determinantal form as,

$$\begin{vmatrix} 1 & -1 & -1 \\ \mu m_2 & -\mu' m'_2 & \mu' m'_2 \\ 0 \left\{ \frac{\mu' m'_2}{\eta} + (\mu_0 - \rho_0 c^2) \right\} & e^{\eta m'_2 H} & - \left\{ \frac{\mu' m'_2}{\eta} - (\mu_0 - \rho_0 c^2) \right\} e^{-\eta m'_2 H} \end{vmatrix} = 0.$$

This determinantal equation on simplification gives

$$\begin{aligned} \eta H &= \frac{1}{\{(c/b')^2 - 1\}^{1/2}} \\ &\times \tan^{-1} \left[\frac{\{(c^2/b^2) - 1\}^{1/2} \{(\mu/\mu')[1 - (c^2/b^2)]^{1/2} + (\eta/\mu')(\mu_0 - \rho_0 c^2)\}}{(c^2/b^2) - 1 - (\mu/\mu')[1 - (c^2/b^2)]^{1/2} \cdot (\eta/\mu')(\mu_0 - \rho_0 c^2)} \right]. \end{aligned} \quad (25)$$

when the surface stress effect is neglected ($\mu_0 = \rho_0 = 0$), (25) reduces to

$$\eta H = \frac{1}{\{(c^2/b^2) - 1\}^{1/2}} \tan^{-1} \left[\frac{(\mu'/\mu)\{1 - c^2/b^2\}^{1/2}}{\{(c^2/b^2) - 1\}^{1/2}} \right], \quad (26)$$

which is the classical Love wave velocity equation. From (25) we observe that Love wave propagation is possible if

$$1 < (c/b') < [(\mu/\rho)/(\mu'/\rho')]^{1/2}. \quad (27)$$

Table 1. Values of ηH .

η	c/b'								
	1.048	1.096	1.144	1.192	1.240	1.288	1.336	1.384	1.432
1	6.933	4.081	2.858	2.147	1.671	1.325	1.057	0.838	0.648
2	7.444	4.576	3.333	2.602	2.104	1.733	1.439	1.193	0.973
3	7.623	4.757	3.517	2.788	2.291	1.922	1.628	1.381	1.159
0*	2.975	1.630	1.078	0.771	0.571	0.428	0.317	0.225	0.139

* Denotes the classical case where the surface stresses are not present

4.3 Numerical calculation of Love waves

For numerical calculations we take

$$\mu = 3.00 \times 10^6 \text{ N/cm}^2, \quad \mu' = 5.00 \times 10^6 \text{ N/cm}^2, \quad \mu_0 = 6.47 \times 10^6 \text{ N/cm},$$

$$\rho = 2.72 \text{ g/cm}^3, \quad \rho' = 9.89 \text{ g/cm}^3, \quad \rho_0 = 3.40 \text{ g/cm}^2.$$

Table 1 gives the values of ηH for different values of C/b' taking $\eta = 1, 2, 3$. It also gives the values of ηH for different values of C/b' in the absence of surface stresses (denoted by 0 in the first column).

We observe that the Love wave velocity equation is independent of σ and depends solely upon μ_0 and ρ_0 . It is also observed that for a particular value of η , wave velocity decreases with increase of ηH . Again for a particular wave velocity, ηH increases with increase of η . For comparison we give the values of ηH in table 1 for the classical case (denoted by 0) where the surface stresses are not present.

4.4 Stoneley waves

Stoneley waves are a generalized form of Rayleigh waves propagating along the common boundary of M_1 and M_2 . Equation (17) represents the wave velocity equation for Stoneley waves in elastic media with surface stresses. This equation after a little manipulation may be obtained in the form

$$\eta^2 CD(1 - ST)(1 - S'T') + s^2 \left\{ (CT + DS)(1 - S'T') + \frac{\mu'}{\mu} \tau_2^2 (CT' + DS')(1 - ST) \right\} \eta - \Delta(s) = 0, \quad (28)$$

where

$$\begin{aligned} \Delta(s) = & (1 - S'T') \left\{ (2 - s^2)^2 - 4ST - 4\frac{\mu'}{\mu} (1 - s^2 - ST) \right\} \\ & + (1 - ST) \left\{ (2 - \tau_2^2 s^2)^2 - 4S'T' - 4\frac{\mu'}{\mu} (1 - \tau_2^2 s^2 - S'T') \right\} \\ & - \tau_2^2 s^4 \frac{\mu'}{\mu} (2 + ST' + S'T), \end{aligned} \quad (29)$$

and

$$\begin{aligned}
 s &= c/b, \quad C = \sigma^*/\rho b^2, \quad \sigma^* = \sigma - \rho_0 c^2, \\
 D &= \frac{\Gamma^*}{\rho b^2}, \quad \Gamma^* = \Gamma - \rho_0 c^2, \quad \Gamma = \lambda_0 + 2\mu_0, \\
 S &= (1 - s^2)^{1/2}, \quad T = (1 - \tau_1^2 s^2)^{1/2}, \\
 S' &= (1 - \tau_2^2 s^2)^{1/2}, \quad T' = (1 - \tau_3^2 s^2)^{1/2}, \\
 \tau_1 &= b/a, \quad \tau_2 = b/b', \quad \tau_3 = b/a'.
 \end{aligned} \tag{30}$$

For the classical case, i.e., when surface stress is absent, we take $\sigma = \Gamma = \rho_0 = 0$, so that $C = D = 0$, (28) reduces to $\Delta(s) = 0$ which is equivalent to the classical Stoneley wave velocity equation. It is observed that the Stoneley waves under the influence of surface stress are dispersive in contrast with the classical situation for which waves of all frequencies propagate uniformly at the same speed. The equation $\Delta(s) = 0$ was solved numerically by Koppe (1948) who concluded that the velocity of the interface wave falls between the values of the velocity of Rayleigh waves and of the transverse waves in the medium with greater acoustic density. Taking $\mu' \rightarrow 0$ in (28) we get the Rayleigh wave velocity equation under the influence of surface stress as

$$\eta^2 C D (1 - ST) + s^2 (CT + DS) \eta - g(s) = 0, \tag{32}$$

where

$$g(s) = (2 - s^2) - 4ST. \tag{33}$$

This result is in agreement with the result obtained in (18) and also that by Murdoch (1976). Further, when there is no surface stress, this equation reduces to the classical Rayleigh wave velocity equation $g(s) = 0$. This equation has one real root $s = s_0$, in the interval $(0, 1)$. If $s = s_1$ satisfies the equation $\Delta(s) = 0$, then $s_0 < s_1 < b \times$ transverse wave velocity in the medium with greater acoustical density. From (28) when $CD \neq 0$ we have

$$\begin{aligned}
 \eta &= \left\{ -s^2 \left[(CT + DS)(1 - S'T') + \frac{\mu'}{\mu} \tau_2^2 (CT' + DS')(1 - ST) \right] \right. \\
 &\quad \pm \left[s^4 \left[(CT + DS)(1 - S'T') + \frac{\mu'}{\mu} \tau_2^2 (CT' + DS')(1 - ST) \right]^2 \right. \\
 &\quad \left. \left. + 4\Delta(s)CD(1 - ST)(1 - S'T') \right]^{1/2} \right\} \div \{2CD(1 - ST)(1 - S'T')\}.
 \end{aligned} \tag{34}$$

If $C = 0$, i.e. in the absence of residual surface tension for a specific value of s we have from (28)

$$\eta = \Delta(s) / \left[s^2 \left\{ DS(1 - S'T') + \frac{\mu'}{\mu} \tau_2^2 DS'(1 - ST) \right\} \right]. \tag{35}$$

Again when $D = 0$ i.e. in the absence of surface tension at some value of s we have

$$\eta = \Delta(s) / \left[s^2 \left\{ CT(1 - S'T') + \frac{\mu'}{\mu} \tau_2^2 CT'(1 - ST) \right\} \right]. \quad (36)$$

If both C and D vanish at some value s' , we have $\Delta(s') = 0$ so that Stoneley waves exist. Otherwise there is no propagation of Stoneley waves. The case when $\sigma, \Gamma > \rho_0 b^2$, C and D will remain positive in $(0, 1)$. Since η is positive, $C < b$, $\tau_1, \tau_2, \tau_3 < 1$ we discard the solution of (34) which is negative for some s in $(0, 1)$. Thus we see that Stoneley wave velocity under the influence of surface stresses may be less than or greater than the classical Stoneley wave velocity but always remains greater than the Rayleigh wave velocity. Moreover, in the case when $0 < \sigma < \Gamma < \rho_0 b^2$, C and D will vanish exactly at one point {each on $(0, 1)$ }, say s_2 and s_3 on $(0, 1)$, so that $s_2^2 = \sigma/\rho_0 b^2$, $s_3^2 = \Gamma/\rho_0 b^2$ and hence $0 < s_2 < s_3 < 1$, where C is positive on $(0, s_1)$ and negative on $(s_1, 1)$ and D is positive on $(0, s_2)$ and negative on $(s_2, 1)$. Again when σ, Γ lie in $(0, \rho_0 b^2)$, $CT + DS$ and $CT' + DS'$ vanish at say s_4 and s_5 in (s_2, s_3) . We observe that $CT + DS$ is positive on $(0, s_4)$ and negative on $(s_4, 1)$ while $CT' + DS'$ is positive on $(0, s_5)$ and negative on $(s_5, 1)$. Thus the number of positive values of η which satisfy (28) for a particular value of s depends upon the location of s_1 related to s_2, s_3, s_4 , and s_5 , as well as on the value of $Q(s)$ and zeros of $Q(s)$ where

$$Q(s) = s^4 \left\{ (CT + DS)(1 - S'T') + \frac{\mu'}{\mu} \tau_2^2 (CT' + DS')(1 - ST) + 4 \Delta(s) CD(1 - ST)(1 - S'T') \right\}. \quad (37)$$

Further discussion in the light of Murdoch (1976) is not pursued in this paper due to its complicated nature and cumbersome calculation.

5. Conclusion

Stoneley waves under the influence of surface stresses may be discussed in a manner similar to Rayleigh waves. It is seen that the propagation of Stoneley waves under the influence of surface stresses is not always possible. Under certain restricted conditions such a propagation is possible. Moreover, Stoneley wave velocity under the influence of surface stresses may be less or greater than the classical Stoneley wave velocity, but always remains greater than the Rayleigh wave velocity. We point out here that Stoneley waves are dispersive in character in contrast with the classical situation. Rayleigh waves have been deduced as a particular case of Stoneley waves. In case of Love waves we have shown that the wave velocity equation does not depend on residual surface tension. Some numerical calculations highlight the effect of surface stresses on Love wave as described in the paper.

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