Two cracks with coalesced interior plastic zones – The generalised Dugdale model approach

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Abstract. The problem investigated is of an elastic-perfectly plastic infinite plate containing two equal collinear and symmetrically situated straight cracks. The plate is subjected to loads at infinity inducing mode I type deformations at the rims of the cracks. Consequently, plastic zones are formed ahead of the tips of the cracks. The loads at infinity are increased to a limit such that the plastic zones formed at the neighbouring interior tips of the cracks get coalesced. The plastic zones developed at the tips of the cracks are closed by applying normal cohesive quadratically varying stress distribution over their rims. The opening of the cracks is consequently arrested. Complex variable technique is used in conjugation with Dugdale's hypothesis to obtain analytical solutions. Closed form analytical expressions are derived for calculating plastic zone size and crack opening displacement. An illustrative numerical example is discussed to study the qualitative behaviour of the loads required to arrest the cracks from opening with respect to parameters viz. crack length, plastic zone length and inter-crack distance. Crack opening displacement at the tip of the crack is also studied against these parameters.

Keywords. Plastic zone; mode I type deformation; opening mode deformation; rims of cracks; crack-opening displacement.

1. Introduction

Work on crack problems was first started by Inglis (1913). The complex variable formulation was given by Kolosov (1935). This was further developed by Muskhelishvili (1953) and was successfully applied by him to solve the crack problems for cracks of different geometry. The treatment of crack problems for fracture of brittle materials was given by Cherepanov (1974) using complex variable technique. Dugdale (1960) proposed an elastic-plastic model, called "strip yield model" for a sheet containing a slit. The sheet was subjected to tension at infinity giving rise to mode I type deformation at the rims of the slit which consequently opened forming the plastic zone ahead of the tips of the slit. This

opening was arrested by applying closing normal yield point stress to the rims of plastic zones. Smith (1974) gave a general theory based on the cohesive zone model for structures in the vicinity of a crack tip. The model was modified by Harrop (1978) for cases when plastic zones were closed by a cohesive normal parabolic stress distribution. The case of two interacting equal collinear straight cracks in an isotropic elastic unbounded plate was discussed by Viola (1983). Theocaris (1989) extended to "strip yield model" to the case when an infinitely elastic, perfectly plastic plate is weakened by two collinear straight cracks.

The present paper investigates the case of two equal, collinear, symmetrically-situated straight cracks weakening an infinite plate. The plate is subjected to uniform constant tension normal to the rims of the cracks, at the infinite boundary of the plate. Consequently, the cracks open forming plastic zones ahead of the tips of the cracks. The tension applied is increased to a limit such that the plastic zones at the two neighbouring interior tips of the cracks coalesce. The rims of plastic zones formed (at the interior and exterior tips of the cracks) are then subjected to normal cohesive stress distribution $t^2\sigma_{ye}$, where t is any point on any of the plastic zones and σ_{ye} is yield point stress. Thus, opening of the cracks is arrested.

2. Basic formulae

Stress components P_{xx} , P_{yy} and P_{xy} and displacement components u_x and u_y may be expressed in terms of two complex potentials $\phi(z)$ and $\Omega(z)$, developed by Muskhelishvili (1953), as

$$P_{yy} - iP_{xy} = \phi(z) + \Omega(\bar{z}) - (z - \bar{z})\phi'(z), \quad (1)$$

$$2\mu(u_{x,x} + iu_{y,x}) = \kappa\phi(z) - \Omega(\bar{z}) - (z - \bar{z})\phi'(z). \quad (2)$$

A bar over a function denotes its complex conjugate. A dash after a function denotes differentiation with respect to the argument, while a comma after a function signifies partial differentiation with respect to the subscript following it. The elastic constant μ denotes shear modulus and $\kappa = (3 - 4\nu)$ for the plane strain case and $\kappa = (1 - \nu)/(1 + \nu)$ for the plane stress case, ν being Poisson's ratio.

Consider a homogeneous, isotropic elastic infinite plate in the xy plane containing n straight cracks L_i ($i = 1, 2, \dots, n$) lying on the real x -axis. Dual problems of linear relationship are obtained using (1), when the rims of the cracks are acted upon by stresses $P_{yy}^\pm$ and $P_{xy}^\pm$. The two Hilbert problems so obtained may be written as

$$\phi^+(t) + \Omega^-(t) = P_{yy}^+ - iP_{xy}^+, \quad \text{on } L \quad (3)$$

$$\phi^-(t) + \Omega^+(t) = P_{yy}^- - iP_{xy}^-, \quad (4)$$

where $L = \sum_{i=1}^n L_i$, under the assumption $\lim_{y \rightarrow 0} \{y\phi'(t + iy)\} = 0$.

Superscripts $+$ and $-$ indicate the limiting value of the function when any point t on the crack, other than end points, is approached from the positive y -plane ($y > 0$) and the negative y -plane ($y < 0$), respectively.

The solutions of (3) and (4) for complex potentials $\phi(z)$ and $\Omega(z)$ may be written, as

$$\phi(z) = \phi_0(z) + (P_n(z)/X(z)), \quad (5)$$

$$\Omega(z) = \Omega_0(z) + (P_n(z)/X(z)), \quad (6)$$

where

$$\phi_0(z) = \int_L \frac{p(t)X(t)}{t-z} dt + \int_L \frac{q(t)X(t)}{t-z} dt - \frac{1}{2}\sigma_\infty, \quad (7)$$

$$\Omega_0(z) = \int_L \frac{p(t)X(t)}{t-z} dt - \int_L \frac{q(t)X(t)}{t-z} dt + \frac{1}{2}\sigma_\infty, \quad (8)$$

σ_∞ being the tension applied at infinite boundary,

$$p(t) = \frac{1}{2}[P_{yy}^+ + P_{yy}^-] - \frac{i}{2}[P_{xy}^+ + P_{xy}^-], \quad (9)$$

$$q(t) = \frac{1}{2}[P_{yy}^+ - P_{yy}^-] - \frac{i}{2}[P_{xy}^+ - P_{xy}^-], \quad (10)$$

and

$$X(z) = \prod_{i=1}^n [(z - a_i)(z - b_i)]^{1/2}, \quad (11)$$

a_i and b_i are the end points of the crack L_i and

$$P_n(z) = C_0 z^n + C_1 z^{n-1} + \dots + C_n. \quad (12)$$

The constants C_i ($i = 1, 2, \dots, n$) are determined by the condition of single-valuedness of displacement at the rims of the crack and C_0 is determined from the boundary condition at infinity. Stress intensity factor, $K_I (= K_1 - iK_2)$, at the crack tip $z = z_1$ may be calculated, as given by Cherepanov (1974), from

$$K_I = K_1 - iK_2 = 2\sqrt{2\pi} \lim_{z \rightarrow z_1} \{(z - z_1)^{1/2} \phi(z)\}, \quad (13)$$

where $\phi(z)$ is obtained from (5).

3. Formulation of the problem

A homogeneous, isotropic, elastic-perfectly plastic infinite plate, in the xy -plane, is weakened by two equal, collinear and symmetrically situated straight cracks L_1 and L_2 . These cracks L_1 and L_2 occupy the ligaments $[-b, -a]$ and $[a, b]$ on the real x -axis as depicted in figure 1. The plate is subjected to uniform uniaxial tension, σ_∞ , parallel to the y -axis, at infinite boundary. Consequently, the cracks L_1 and L_2 open forming the plastic zones at the tips $-b, -a, a$ and b . The prescribed tension at infinity is increased to the limit where the plastic zones developed at the tips $-a$ and a get coalesced. Thus the entire ligament $[-a, a]$ forms the interior plastic zone, Γ_1 . The plastic zone Γ_2 ahead of the crack tip $-b$ occupies the ligament $[-c, -b]$ and the zone Γ_3 at the tip b occupies the ligament $[b, c]$. Each of the plastic zones Γ_i ($i = 1, 2, 3$) is closed by cohesive normal stress distribution $t^2\sigma_{ye}$, consequently arresting the opening of the cracks. Yield point stress is denoted by σ_{ye} and t is any point on any of the plastic zones.

4. Solution of the problem

The solution of the problem stated in § 3 is obtained by superposing the solution of two component problems contributing towards the stress singularity at the tips of the cracks.

Figure 1. Configuration.

These two problems are appropriately derived from the problem stated in § 3. These are termed as *problem I* and *problem II*.

4.1 Problem I

The configuration of the problem I is: an infinite, homogeneous, isotropic and elastic-perfectly plastic infinite plate, in the xy -plane, contains a stress-free straight crack R_1 ($= \Gamma_2 U L_1 U \Gamma_1 U L_2 U \Gamma_3$) occupying ligament $[-c, c]$ of the real x -axis. Uniform tension σ_∞ , parallel to y -direction, is applied at infinite boundary of the plate. The complex potentials $\phi_1(z)$, $\Omega_1(z)$ for this case may directly be written using Muskhelishvili (1953), as

$$\begin{aligned}\phi_1(z) &= 0.5\sigma_\infty z(z^2 - c^2)^{-1/2} \\ &= \Omega_1(z).\end{aligned}\tag{14}$$

The subscript 1 denotes that the potentials refer to problem I.

Figure 2. Variation of normalised load ratio versus crack length.

Opening mode stress intensity factor, K_1^I , for this problem, at the crack tip $z = c$ is calculated by substituting $\phi_1(z)$ for $\phi(z)$ from (14) into (13) and is given by Cherepanov (1974).

$$K_1^I = \sigma_\infty (\pi c)^{1/2} \quad (15)$$

4.2 Problem II

A homogeneous, isotropic and elastic-perfectly plastic infinite plate in the xy -plane contains two equal collinear symmetrically situated straight cracks $L_1: [-b, -a]$ and $L_2: [a, b]$ lying on the x -axis (refer figure 1). The cracks L_1 and L_2 possess the plastic zones $\Gamma_1, \Gamma_2, \Gamma_3$ ahead of the tips of the cracks. These plastic zones $\Gamma_1, \Gamma_2, \Gamma_3$ occupy the region $[-a, a]$, $[-c, -b]$ and $[b, c]$ respectively. The boundary conditions of the problem are the following.

- (i) The unbounded plate is stress-free at infinite boundary.
- (ii) The rims of the plastic zones Γ_i , ($i = 1, 2, 3$) are subjected to tensile stress $P_{yy}^\pm = t^2 \sigma_{ye}$ and $P_{xy}^\pm = 0 = P_{xx}^\pm$, where t is any point on any of the plastic zones and σ_{ye} is the yield point stress.
- (iii) Rims of the cracks L_1 and L_2 are stress-free.
- (iv) The displacements are single-valued at the tips of the cracks.

Figure 3. Normalised load ratio versus inter-crack distance. Crack length is one unit.

Dual Hilbert problems are obtained in terms of potentials $\phi_2(z)$ and $\Omega_2(z)$ using (1) and boundary conditions (ii) and (iii) as

$$[\phi_2(t) + \Omega_2(t)]^+ + [\phi_2(t) + \Omega_2(t)]^- = t^2 \sigma_{ye}, \quad (16)$$

$$[\phi_2(t) - \Omega_2(t)]^+ - [\phi_2(t) - \Omega_2(t)]^- = 0, \quad \text{on } \Gamma = U_{i=1}^3 \Gamma_i. \quad (17)$$

The subscript 2 indicates that the functions refer to the problem II.

Solutions of (16) and (17), are written using (5) to (12) and boundary conditions (i) and (iv), as

$$\begin{aligned} \phi_2(z) &= \frac{\sigma_{ye} z}{\pi i X(z)} \left[\left(\frac{2c^2}{3} - z^2 \right) A_1 - iz X(z) A_2 \right] \\ &= \Omega_2(z), \end{aligned} \quad (18)$$

$$A_1 = (\pi/2) + \sin^{-1}(a/c) - \sin^{-1}(b/c),$$

$$A_2 = \tan^{-1} \frac{z(c^2 - a^2)^{1/2}}{a(z^2 - c^2)^{1/2}} - \tan^{-1} \frac{z(c^2 - b^2)^{1/2}}{b(z^2 - c^2)^{1/2}} - \frac{\pi}{2},$$

where,

$$X(z) = (c^2 - z^2)^{1/2}. \quad (19)$$

Figure 4. Variation of normalised crack opening displacement versus crack length at the interior tip of the crack.

Opening mode stress intensity factor for problem II, K_I^{II} , is obtained substituting the value of $\phi_2(z)$ for $\phi(z)$ from (18) into (13) and simplifying, one gets

$$K_I^{\text{II}} = -\frac{2\sigma_{ye}c^2}{3} \left(\frac{c}{\pi} \right)^{1/2}. \quad (20)$$

5. Plastic zone size and crack opening displacement

Plastic zone size for original problem (stated in § 3) is obtained by superposing two stress intensity factors obtained for problem I and problem II. Using the fact that the stresses remain finite at every point of the plate leads to the nonlinear equation.

$$\frac{c^2}{3} [1 - (2/\pi) \{ \sin^{-1}(b/c) - \sin^{-1}(a/c) \}] - \sigma_{\infty}/\sigma_{ye} = 0, \quad (21)$$

for determining c when values of $\sigma_{\infty}/\sigma_{ye}$, a and b are prescribed. Finally the plastic zone length is calculated by $|c - b|$.

The Dugdale model crack face opening displacement, $2u_y$, corresponding to the calculated plastic zone, is obtained following Harrop (1978), by the relation

$$u_y = \frac{2}{E} \text{Im} \left[\int \Phi(z) dz \right], \quad (22)$$

Figure 5. Variation of normalised crack opening displacement versus crack length at the exterior tip of the crack.

where E is Young's modulus, $\text{Im}[\]$ denotes the imaginary part of the quantity in the bracket. The complex potential $\Phi(z)$ is obtained by superposing the non-singular terms of $\phi_1(z)$ {from (14)} and $\phi_2(z)$ {from (18)}. Substituting the value of $\Phi(z)$ thus obtained in (22) and integrating, one obtains the crack-opening displacement (COD) at the exterior tip $z = b$ as

$$u_y = -(2\sigma_{ye}cb/\pi E)A_1\{b(1 - b^2/c^2)^{1/2} + c \sin^{-1}(b/c)\}, \quad (23)$$

and at the interior tip $z = a$ the COD is given by

$$u_y = -(2\sigma_{ye}ca/\pi E)A_1\{a(1 - a^2/c^2)^{1/2} + c \sin^{-1}(a/c)\}. \quad (24)$$

6. Illustrative example

A qualitative study has been carried out to find the load ratio required to prevent the cracks formed from opening, as the crack length and inter-crack distance (the distance between the two neighbouring tips of the two cracks) is increased.

Normalised load ratio has been plotted against increasing crack length in figure 2. Normalising factor of stress intensity factor (SIF) is 10. It is observed that for longer cracks greater load is required as expected to close the crack. Also for a fixed crack length,

Figure 6. Normalised crack opening displacement at interior tip with respect to inter-crack distance. The crack length is taken as one unit.

if plastic zone ratio (the ratio of outer plastic zones/inner plastic zone) is increased then a higher load still is required to close the crack. The two cracks were kept apart at a fixed distance.

Figure 3 depicts the variation of load ratio as inter-crack distance is increased. The crack length is unity. It may be noted that, for a fixed plastic zone ratio, as the cracks move apart greater load is required to close them. This is because the effect of the cracks on one another diminishes.

The variation of crack-opening displacement at the interior tip, $z = a$ of the crack L_2 , versus increase in crack length is depicted in figure 4. The normalising factor of COD is $(\sigma_{ye}/10E)$. The same variation is plotted at the exterior tip $z = b$ of the crack L_2 in figure 5. It may be noted as crack length increases the cracks open more at both the tips, for a fixed plastic zone ratio. But it is interesting to note the difference in the variation patterns of COD at the inner and outer tips of the crack. It may be noted that at the two interior tips of the COD, the pattern is such that the plastic zones get coalesced.

Figures 6 and 7 show the behaviour of the COD at the interior and exterior tips of the crack L_2 against the inter-crack distance. It is seen as the inter-crack distance is increased (i.e. the cracks are moved apart) the COD variation patterns at both the tips are similar. This is because of the fact that as the cracks are moved apart their effect on one another diminishes. The length of each crack is taken as unity.

Figure 7. Normalised crack opening displacement at exterior tip with respect to inter-crack distance. The crack length is one unit.

7. Conclusion

The Dugdale model (1960) was modified by Harrop (1978) for a single crack when a parabolic load is distributed over its plastic zones. The present paper modifies the extended Dugdale model given by Theocaris (1989) for two straight cracks in an elastic-perfectly plastic infinite plate. Here, the quadratically varying cohesive stress distribution closes the rims of the plastic zone.

The results obtained may be applied to a plate-type structure in which two similar straight cracks develop in the vicinity of each other and their propagation under pressure causes coalescence in the plastic zone, viz. aeroplane wings, turbine blades, stretched sheets etc.

The dimensions of the plate should be about five times larger than the total length of the two cracks and the inter-crack distance should be such that the stresses at the crack tips do not affect the stresses at the boundary of the plate.

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List of symbols

a, b, c	tips of the cracks;
E	Young's modulus;
K_I	stress intensity factor for mode I type deformations;
L_j ($j = 1, 2$)	cracks;
P_{ij} ($i, j = x, y$)	stress components;
$P_n(z), X(z)$	complex functions;
u_i ($i = x, y$)	displacement components;
$z = x + iy$	complex variable;
Γ_j ($j = 1, 2, 3$)	plastic zones;
μ	shear modulus;
ν	Poisson's ratio;
σ_{ye}	yield point stress;
σ_∞	tension applied at infinite boundary;
$\phi(z), \Omega(z), \Phi(z)$	complex potentials.

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