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Stability theory of differential/algebraic models of power systems

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Abstract. Lyapunov stability results are given for differential/algebraic models of power systems which include the effect of generator damping and nonlinear loads. The global dynamical structure of such a system is studied in terms of multivalued energy functions defined on so-called 'voltage causal regions' where voltage behaviour is predicted from angle behaviour. These regions are separated by 'impasse surfaces' related to singularity in the load flow equations.

Keywords. Power systems; stability; Lyapunov methods; nonlinear systems.

1. Introduction

Direct methods of transient stability assessment for power systems rely on simplified nonlinear equation models. Older models assumed impedance loads and used network reduction to derive a model as a set of coupled (swing) differential equations (Pai 1981). Bergen & Hill (1981) suggested using models where the loads and network structure were preserved. This approach leads naturally to models which are of differential/algebraic type. More recently such models have been used as a basis for voltage stability analysis (Kwatny *et al* 1986; Venkatasubramanian *et al* 1991). Thus the theoretical basis for direct methods using structure-preserving models is dependent on stability theory for differential/algebraic equations (Chiang & Fekih-Ahmed 1992). This paper provides further results in that direction.

Hill & Mareels (1990) have given some basic results on Lyapunov stability of differential/algebraic systems and used these to justify use of an energy function for undamped power systems. Hiskens & Hill (1989) have explored more practical aspects of using this energy function; this work identifies several theoretical extensions which should be made. Of these, the two considered here are as follows. First, the theory is easily extended to allow for generator damping. Second, an improved decomposition of the state-space is presented; so-called voltage causal regions are defined as open sets which are separated by 'impasse surfaces' of algebraic singularity and within which ordinary differential equation theory can be used. This helps to formulate practical algorithms for finding the region of transient stability. Further, there are useful tools for analysis of short-term voltage collapse.

2. Differential/algebraic power system model

2.1 *Model development*

The classical machine model is used in the development of the system model. Therefore the synchronous machines are represented by a constant voltage $|E_i|$ in series with transient reactance. This assumption corresponds to ignoring flux decay and having an exciter which is too slow to act in the transient period.

Consider now a network consisting of n_0 buses connected by transmission lines. At m of these buses there are generators. The buses which have load but no generation are labelled $i = 1, \ldots, n_0 - m$. The network is augmented with m fictitious buses representing the generator internal buses, in accordance with the classical machine model. They are labelled $i + m$ where i is the bus number of the corresponding generator bus. The total number of buses in the augmented system is therefore $n_0 + m := n$.

The network is assumed lossless, so all lines (including those corresponding to the machine transient reactances) are modelled as series reactances. The bus admittance matrix $\mathbf{Y}$ is therefore purely imaginary, with elements $Y_{ij} = jB_{ij}$.

Let the complex voltage at the ith bus be the (time varying) phasor $V_i = |V_i| \angle \delta_i$ where δ_i is the bus phase angle with respect to a synchronously rotating reference frame. Define $|\mathbf{V}| = [|V_1|, \ldots, |V_{n_0}|]^t$, where t denotes matrix transpose. The bus frequency deviation is given by $\omega_i = \dot{\delta}_i$.

Using machine reference angles, we take the nth bus as the reference. We use the internodal angles $\alpha_i := \delta_i - \delta_n$. Define $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_{n-1}]^t$ and $\boldsymbol{\omega}_g = [\omega_{n_0+1}, \ldots, \omega_n]^t$.

Let P_{bi} and Q_{bi} denote the total real and reactive power leaving the ith bus via transmission lines. Then

$$P_{bi}(\boldsymbol{\alpha}, |\mathbf{V}|) = \sum_{j=1}^{n} |V_i||V_j|B_{ij}\sin(\alpha_i - \alpha_j) \tag{1a}$$

$$Q_{bi}(\boldsymbol{\alpha}, |\mathbf{V}|) = -\sum_{j=1}^{n} |V_i||V_j|B_{ij}\cos(\alpha_i - \alpha_j). \tag{1b}$$

In these equations, we assume the substitution $|V_i| = |E_{i-n_0}|$, $i = n_0 + 1, \ldots, n$ has been made. Also we take $\alpha_n := 0$.

Now consider the modelling of loads. Denote the real and reactive power demand at the ith bus by P_{di} and Q_{di} respectively. In general these powers are functions of voltage and frequency. For the stability theory to be developed, the loads must be restricted to satisfy

$$P_{di} = P_{di}^0, \tag{2a}$$

$$Q_{di} = Q_{di}(|V_i|), \quad i = 1, \ldots, n_0. \tag{2b}$$

There are unresolved difficulties in allowing voltage dependent real loads. However, this restriction will be relaxed when considering the state space decomposition.

The last component of the model to be considered is the generator dynamics, given by the swing equations

$$M_j(\mathrm{d}\omega_{j+n_0}/\mathrm{d}t) + D_j\omega_{j+n_0} + P_{b,j+n_0}(\boldsymbol{\alpha}, |\mathbf{V}|) = P_{Mj}^0; \quad j = 1, \ldots, m, \tag{3}$$

where M_j are the inertia constants, D_j the generator damping constants, and P^0_{Mj} the mechanical input powers. We assume that $D_i \neq 0$ for at least one $i = 1, \ldots, m$. The usual assumptions of constant mechanical power, and the network being in a sinusoidal steady state are made.

Combining the power balance and swing equations gives the total system representation

$$M_{i-n_0}\dot{\omega}_i + D_{i-n_0}\omega_i + P_{bi}(\alpha, |\mathbf{V}|) = P^0_{M,i-n_0}, \qquad i = n_0+1, \ldots, n, \tag{4a}$$

$$P_{bi}(\alpha, |\mathbf{V}|) = -P^0_{di}, \qquad i = 1, \ldots, n_0, \tag{4b}$$

$$Q_{bi}(\alpha, |\mathbf{V}|) = -Q_{di}(|V_i|), \qquad i = 1, \ldots, n_0. \tag{4c}$$

Define $\tilde{\mathbf{P}}^t_b = [P_{b1}, \ldots, P_{b,n-1}] = [\mathbf{P}^t_l \mathbf{P}^t_g]$ where $\mathbf{P}_l, \mathbf{P}_g$ are n_0, $(m-1)$ vectors referring to loads, generators respectively; $\tilde{\mathbf{P}}^{0t} = [-\mathbf{P}^{0t}_d \tilde{\mathbf{P}}^{0t}_M]$ where $\tilde{\mathbf{P}}^0_M = [P^0_{M1}, \ldots, P^0_{M,m-1}]^t$; $\mathbf{Q}_b = [Q_{b1}, \ldots, Q_{bn_0}]^t$; and $\mathbf{Q} = [-Q_{d1}, \ldots, -Q_{dn_0}]^t$. Also set $P_n := P^0_{Mm}$.

From (1a)

$$\sum_{i=1}^{n} (P_{bi} - P^0_i) = -\sum_{i=1}^{n} P^0_i := -P_T. \tag{5}$$

It is convenient for this excess bus power to be distributed across generator powers in proportion to damping. Define

$$P_{M,j} = P^0_{M,j} - (P_T/D_T)D_j, \tag{6}$$

where $D_T := \sum_{i=1}^{m} D_i$ ($\neq 0$ by assumption).

Define the modified real power vector $\tilde{\mathbf{P}}^t = [-\mathbf{P}^{0t}_d \tilde{\mathbf{P}}^t_M]$. Then from (4b), (5), (6)

$$\sum_{i=n_0+1}^{n} (P_{bi} - P_i) = 0.$$

So

$$P_{bn} - P_n = -\mathbf{l}^t_{n-1}(\mathbf{P}_g - \tilde{\mathbf{P}}_M),$$

where $\mathbf{l}_{n-1}$ is the $(n-1)$ vector with unity entries. Define $\mathbf{T}_g = [\mathbf{I}_{n-1} : -\mathbf{l}_{n-1}]$ where $\mathbf{I}_{n-1}$ is the $(n-1)$ identity matrix. Then (4) can be rewritten

$$\mathbf{M}_g\dot{\boldsymbol{\omega}}_g + \mathbf{D}_g\boldsymbol{\omega}_g + \mathbf{T}^t_g(\mathbf{P}_g(\alpha, |\mathbf{V}|) - \tilde{\mathbf{P}}_M) = \mathbf{0}, \tag{7a}$$

$$\mathbf{P}_l(\alpha, |\mathbf{V}|) + \mathbf{P}_d = \mathbf{0}, \tag{7b}$$

$$\mathbf{Q}_b(\alpha, |\mathbf{V}|) + \mathbf{Q}_d(|\mathbf{V}|) = \mathbf{0}, \tag{7c}$$

where $\mathbf{M}_g, \mathbf{D}_g$ are diagonal matrices of inertia, damping constants. Note that use of $\tilde{\mathbf{P}}_M$ requires a reference shift for $\dot{\boldsymbol{\omega}}_g$ so that (4a) remains valid. Partition α as $\alpha^t = [\alpha^t_l \alpha^t_g]$ so the loads can be identified.

Also define

$$\tilde{\mathbf{f}}_g(\alpha_g, \alpha_l, |\mathbf{V}|) := \mathbf{P}_g(\alpha_g, \alpha_l, |\mathbf{V}|) - \tilde{\mathbf{P}}_M, \tag{8a}$$

$$\mathbf{f}_l(\alpha_g, \alpha_l, |\mathbf{V}|) := \mathbf{P}_l(\alpha_g, \alpha_l, |\mathbf{V}|) + \mathbf{P}_d, \tag{8b}$$

$$\mathbf{g}(\alpha_g, \alpha_l, |\mathbf{V}|) := [|\mathbf{V}|]^{-1}(\mathbf{Q}_b(\alpha_g, \alpha_l, |\mathbf{V}|) + \mathbf{Q}_d(|\mathbf{V}|)), \tag{8c}$$

where [a] denotes [diag $\{a_i\}$] for vector a. Then (7) can be written

$$\dot{\omega}_g = -\mathbf{M}_g^{-1}\mathbf{D}_g\omega_g - \mathbf{M}_g^{-1}\mathbf{T}_g^t\tilde{\mathbf{f}}_g(\alpha_g, \alpha_l, |\mathbf{V}|), \tag{9a}$$

$$\dot{\alpha}_g = \mathbf{T}_g\omega_g, \tag{9b}$$

$$\mathbf{0} = \mathbf{f}_l(\alpha_g, \alpha_l, |\mathbf{V}|), \tag{10a}$$

$$\mathbf{0} = \mathbf{g}(\alpha_g, \alpha_l, |\mathbf{V}|). \tag{10b}$$

Equations (9), (10) describe the model on which all further results are based. We note it consists of a set of differential–algebraic (DA) equations. The system variables are clearly

$$\omega_g \in \mathbb{R}^m, \alpha_g \in \mathbb{R}^{m-1}, \alpha_l \in \mathbb{R}^{n_0} \quad \text{and} \quad |\mathbf{V}| \in \mathbb{R}_+^{n_0}.$$

It is easy to check that the bus power transformation (and associated frequency reference shift) implies the equilibrium points are given by $\omega_g = \mathbf{0}$ and

$$\tilde{\mathbf{f}}_g(\alpha_g, \alpha_l, |\mathbf{V}|) = \mathbf{0}, \tag{11a}$$

$$\mathbf{f}_l(\alpha_g, \alpha_l, |\mathbf{V}|) = \mathbf{0}, \tag{11b}$$

$$\mathbf{g}(\alpha_g, \alpha_l, |\mathbf{V}|) = \mathbf{0}. \tag{11c}$$

2.2 *Local ODE representation*

Here it is shown that the model is locally equivalent to a set of ordinary differential equations (ODE) for almost all operating states. The load bus variables $\alpha_l, |\mathbf{V}|$ are related to the generator angles α_g by the algebraic equations (10). In fact, (10) defines an $(m-1)$-manifold on which α_g can flow. Define the Jacobian

$$J_{ll} = \begin{bmatrix} \partial\mathbf{f}_l/\partial\alpha_l & \partial\mathbf{f}_l/\partial|\mathbf{V}| \\ \partial\mathbf{g}/\partial\alpha_l & \partial\mathbf{g}/\partial|\mathbf{V}| \end{bmatrix}. \tag{12}$$

Then, by the implicit function theorem (Fleming 1977), if $\det J_{ll} \neq 0$, locally the load bus variables can be written explicitly in terms of the generator angles as

$$\alpha_l = \Phi(\alpha_g), |\mathbf{V}| = \Psi(\alpha_g). \tag{13}$$

An equivalent differential equation form can therefore be obtained locally by substituting (13) into (9a). Setting

$$\mathbf{P}_g^*(\alpha_g) := \mathbf{P}_g(\alpha_g, \Phi(\alpha_g), \Psi(\alpha_g)),$$

gives the model

$$\dot{\omega}_g = -\mathbf{M}_g^{-1}\mathbf{D}_g\omega_g - \mathbf{M}_g^{-1}\mathbf{T}_g^t(\mathbf{P}_g^*(\alpha_g) - \tilde{\mathbf{P}}_M), \tag{14a}$$

$$\dot{\alpha}_g = \mathbf{T}_g\omega_g. \tag{14b}$$

Equations (14) define ordinary differential equations which are locally equivalent to the DA system.

This idea of local solvability will be extended later to solvability over disjoint regions.

3. Stability theory of differential/algebraic systems

This section develops a useful result on the stability of equilibria in general DA systems. The general topic of Lyapunov stability for such systems has been studied in Hill & Mareels (1990). The result required here is a LaSalle invariance version of an asymptotic stability criterion. This is easily developed using ideas given in Hill & Mareels (1990).

We consider DA in the general form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}), \tag{15a}$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{y}), \tag{15a}$$

with some compatible initial conditions, $(\mathbf{x}_0, \mathbf{y}_0)$, i.e. $\mathbf{0} = \mathbf{g}(\mathbf{x}_0, \mathbf{y}_0)$ where $\mathbf{f}: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\mathbf{g}: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$.
We assume throughout:

A1. $\mathbf{f}$ and $\mathbf{g}$ are twice continuously differentiable in some open connected set, $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, i.e. $\mathbf{f}, \mathbf{g} \in C^2(\Omega)$.

A2. The Jacobian of $\mathbf{g}$ with respect by $\mathbf{y}$ has constant full rank on Ω, i.e.

$$\operatorname{rank}(D_2\mathbf{g}(\mathbf{x}, \mathbf{y})) = m, \quad \forall (\mathbf{x}, \mathbf{y}) \in \Omega.$$

We use the following notations:

$\mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0)$, $\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0)$ are solutions of (15) as functions of time and initial conditions

$B_\varepsilon = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \|(\mathbf{x}, \mathbf{y})\| < \varepsilon\}$,

$G = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{g}(\mathbf{x}, \mathbf{y}) = 0\}$,

$\Omega_G = \Omega \cap G$,

$\bar{\Omega}$ = closure of Ω in $\mathbb{R}^n \times \mathbb{R}^m$,

$K = \{a : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, strictly increasing, $a(0) = 0\}$,

$\dot{V}_{(n)}$ = derivative of the function V with respect to time along the solution of the system with equations (n).

We now consider stability properties of equilibria of the general DA system (15). First, note that a local ODE description – exemplified by (14) – can be given.

It follows from the implicit function theorem and assumption A2 that given $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \Omega_G$ there is some neighbourhood $U \subset \mathbb{R}^n$ of $\bar{\mathbf{x}}$ and a unique twice differentiable function $\mathbf{u}: \mathbb{R}^n \to \mathbb{R}^m, \mathbf{u} \in C^2(U)$ such that

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \forall \mathbf{x} \in U \text{ and } (U \times \mathbf{u}(U))_G \subset \Omega_G,$$

with Jacobian

$$(D\mathbf{u})(\mathbf{x}) = -(D_2\mathbf{g})^{-1}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \cdot (D_1\mathbf{g})(\mathbf{x}, \mathbf{u}(\mathbf{x})), \ \forall \mathbf{x} \in U.$$

Let

$$\Lambda := U \times \mathbf{u}(U) \text{ and } \Lambda_G := (U \times \mathbf{u}(U))_G = (U \times \mathbf{u}(U)) \cap G.$$

Lemma 1. *In the neighbourhood* $\Lambda_G(\bar{\mathbf{x}}) \subset \Omega_G$, *the system* (15) *reduces to*

$$\dot{x} = \mathbf{f}(x, \mathbf{u}(x)). \tag{16}$$

We assume that the system (15) has a unique (isolated) equilibrium in Ω, which we regard to be the origin, without loss of generality.

A3. In Ω, $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ and $\mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$, iff $(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{0})$.

Remark. In order to satisfy assumption A3 it may be necessary to shrink the set Ω of assumptions A1 and A2 to a smaller subset.

When discussing stability in the DA system context we only consider stability with respect to perturbations which satisfy the algebraic constraints. (When using the reduced system representation (16), this feature has been accounted for.) However it should be noted that often the algebraic equations of the DA system are a model of some (perhaps unknown) underlying dynamic process. Stability in the DA sense does not imply stability of the system obtained by including the underlying dynamics.

We now present the formal definitions of stability of the trivial solution $(\mathbf{x}(t, \mathbf{0}, \mathbf{0}), \mathbf{y}(t, \mathbf{0}, \mathbf{0})) \equiv (\mathbf{0}, \mathbf{0})$ of the DA system (15).

DEFINITION 1
The trivial solution of (15) is called stable if given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $(\mathbf{x}_0, \mathbf{y}_0) \in \Omega_G \cap B_\delta$ then $(\mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0), \mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0)) \in \Omega_G \cap B_\varepsilon$, $\forall t \in \mathbb{R}_+$.

DEFINITION 2
The trivial solution of (15) is called asymptotically stable if it is stable and there exists $\eta > 0$ such that for all $(\mathbf{x}_0, \mathbf{y}_0) \in \Omega_G \cap B_\eta$ then

$$\lim_{t \to \infty} \| (\mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0), \mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0)) \| = (\mathbf{0}, \mathbf{0}).$$

It is straightforward to derive versions of the basic Lyapunov stability arguments for DA systems. Some basic results are given in Hill & Mareels (1990). In the latter power system analysis we need a LaSalle invariance type result.

Theorem 1. *Suppose there exists a $C^1(\Omega)$ function $V: \Omega \to \mathbb{R}_+$ such that V is positive definite and has negative semi-definite derivative on Ω_G, i.e.*

$$V(\mathbf{x}, \mathbf{y}) \geqslant a(\| (\mathbf{x}, \mathbf{y}) \|),$$
$$\dot{V}_{(15)} \leqslant 0,$$

on Ω_G for some $a \in K$. Let

$$S = \{(\mathbf{x}, \mathbf{y}) \in \Omega_G \cap \Lambda : \dot{V}_{(15)} = 0\} \tag{17}$$

and M be the largest invariant set within S. Further define

$$\alpha = \sup_{\gamma \in \mathbb{R}_+} \{\gamma : B_{\gamma G} \subset \Omega_G \cap \Lambda\},$$
$$V_\alpha^{-1} = \{(\mathbf{x}, \mathbf{y}) \in \Omega_G : V(\mathbf{x}, \mathbf{y}) \leqslant a(\alpha)\}.$$

Then the trivial solution $(0, 0)$ of the DA system is stable, $(\mathbf{x}(t), \mathbf{y}(t)) \to M \cap V_\alpha^{-1}$ as $t \to \infty$ and the domain of attraction contains V_α^{-1}.

Proof. Lemma 1 gives that there exists a neighbourhood $\Lambda(0)$ of the origin in which DA system (15) is equivalent to the ODE (16). Let $r \in \mathbb{R}_+$ be such that $B_{rG} \subset \Lambda \cap \Omega_G$. Within B_r, the corresponding arguments for ODE systems can be used (LaSalle 1976; Rouche *et al* 1977).

Remarks. (a) In general, solution of (15b) yields multiple values of $\mathbf{y}$ for each $\mathbf{x}$. For each branch, system (16) and $V(\mathbf{x}, \mathbf{u}(\mathbf{x}))$ are well-defined. However, on Ω, we must regard these as multi-valued. For instance, V may be represented by multiple surfaces. (b) Clearly, the stability result follows easily from one for the reduced system. However, the reduced system is not usually known. So stability conditions which work directly on functions $\mathbf{f}, \mathbf{g}$ in (15) are needed.

4. Stability result

In this and the following sections, we provide some basic methodology for determining large disturbance stability of the equilibria in the DA power system model. Emphasis will be given to new insights into the nature of energy surfaces in the presence of multiple equilibria and the statement of stability results for the differential-algebraic equation model.

4.1 *Energy function* (Lyapunov function candidate)

The development of energy functions for the DA model has been studied elsewhere (Narasimhamurthi & Musavi 1984; Hill & Chong 1989) using first integral and Lur'e problem analysis methods. Here we summarise from Hill & Chong (1989). A valid energy function is

$$V(\boldsymbol{\omega}_g, \mathbf{z}) = \tfrac{1}{2}\boldsymbol{\omega}_g^t \mathbf{P}_1(\mu)\boldsymbol{\omega}_g + \int_{\mathbf{z}_s}^{\mathbf{z}} \langle \mathbf{h}(\boldsymbol{\lambda}), \mathrm{d}\boldsymbol{\lambda} \rangle \tag{18}$$

where $\mathbf{z} = (\boldsymbol{\alpha}_g, \boldsymbol{\alpha}_l, |\mathbf{V}|)$, $\mathbf{h}(\mathbf{z}) = (\tilde{\mathbf{f}}_g(\mathbf{z}), \mathbf{f}_l(\mathbf{z}), \mathbf{g}(\mathbf{z}))$ and $\mathbf{z}_S$ denotes a stable equilibrium point. $\mathbf{P}_1$ is given by

$$\mathbf{P}_1(\mu) = q\mathbf{M}_g + \mu \mathbf{M}_g \mathbf{l}_{mm} \mathbf{M}_g, \tag{19}$$

where μ is a scalar and $\mathbf{l}_{pq}$ denotes a $p \times q$ matrix with all its elements equal to 1. (To simplify notation, the dimensions will be omitted.) The scalar μ is chosen to ensure $\mathbf{P}_1(\mu) > 0$. Note that $\mathbf{P}_1(0) = q\mathbf{M}_g$. The energy function (18) can be evaluated as

$$V(\boldsymbol{\omega}_g, \boldsymbol{\alpha}_g, \boldsymbol{\alpha}_l, |\mathbf{V}|) = \tfrac{1}{2}\boldsymbol{\omega}_g^t \mathbf{P}_1(\mu)\boldsymbol{\omega}_g - \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}\,(|V_i||V_j|\cos\alpha_{ij} - |V_i^S||V_j^S|\cos\alpha_{ij}^S)$$
$$- \int_{\boldsymbol{\alpha}^*}^{\boldsymbol{\alpha}} \tilde{\mathbf{P}}^t \mathrm{d}\boldsymbol{\alpha} + \sum_{i=1}^{n_Q} \int_{|V_i^S|}^{|V_i|} [Q_{di}(\varrho_i)/\varrho_i]\mathrm{d}\varrho_i. \tag{20}$$

Define the constraint manifold

$$G := \{\mathbf{z} : \mathbf{f}_l(\mathbf{z}) = \mathbf{0}, \mathbf{g}(\mathbf{z}) = \mathbf{0}\}.$$

Differentiating V on G gives

$$\dot{V} = \tfrac{1}{2}\boldsymbol{\omega}_g^t \mathbf{Z}_0(\mu)\boldsymbol{\omega}_g, \tag{21}$$

where

$$\mathbf{Z}_0(\mu) = -2q\mathbf{D}_g - \mu(\mathbf{M}_g\mathbf{1}\mathbf{D}_g + \mathbf{D}_g\mathbf{1}\mathbf{M}_g). \tag{22}$$

A further requirement on μ is to make $\mathbf{Z}_0(\mu) \leqslant 0$. Note that $\mathbf{Z}_0(0) \leqslant 0$.

Clearly $\mu = 0$ gives the simple kinetic energy function $\frac{1}{2}\boldsymbol{\omega}_g^T \mathbf{M}_g \boldsymbol{\omega}_g$. This remains a valid energy function for any (non-negative) values of damping $\mathbf{D}_g$. However, better estimates of stability regions can be obtained with a value of μ which is more closely related to the damping. This is considered in much more detail by Hill & Chong (1989) following the results for impedance load systems by Willems (1970). In the special case of zero or uniform damping, the kinetic energy term becomes the familiar

$$(1/2M_T)\sum_{i=n_0+1}^{n-1}\sum_{j=i+1}^{n}(\omega_i - \omega_j)^2 M_i M_j.$$

4.2 *Stability result*

In the development of large disturbance stability results, it is useful to establish the connection between small disturbance stability and asymptotic stability of equilibria.

Small disturbance stability refers to the stability of the linear system obtained by linearizing (9), (10). Hiskens (1990) shows that this linearization yields,

$$\begin{bmatrix}\Delta\dot{\boldsymbol{\alpha}}_g \\ \Delta\dot{\boldsymbol{\omega}}_g\end{bmatrix} = \left[\begin{array}{c|c} 0 & \mathbf{T}_g \\ \hline -\mathbf{M}_g^{-1}\mathbf{T}_g^t\mathbf{F} & -\mathbf{M}_g^{-1}\mathbf{D}_g \end{array}\right]\begin{bmatrix}\Delta\boldsymbol{\alpha}_g \\ \Delta\boldsymbol{\omega}_g\end{bmatrix} = \mathbf{A}\begin{bmatrix}\Delta\boldsymbol{\alpha}_g \\ \Delta\boldsymbol{\omega}_g\end{bmatrix}, \tag{23}$$

where

$$\mathbf{F} = \mathbf{J}_{gg} - \mathbf{J}_{gl}\mathbf{J}_{ll}^{-1}\mathbf{J}_{lg} \tag{24}$$

with $\mathbf{J}_{gg}$, $\mathbf{J}_{gl}$ and $\mathbf{J}_{lg}$ defined in appendix A.

It is interesting to compare the linearized system (23) with the reduced system (14). Notice that F is the Jacobian of $\mathbf{P}_g^*(\boldsymbol{\alpha}_g)$. Nonsingularity of $\mathbf{J}_{ll}$ ensured the existence of functions (13), and is also required for F to exist.

The operating point is small disturbance stable iff all eigenvalues of A lie in the open left half of the complex plane. It was shown by Hiskens (1990) that the eigenvalues of A satisfied that condition if F was positive definite. So we can say that an equilibrium point is small disturbance stable if $\mathbf{J}_{ll}$ is nonsingular and F is positive definite.

Theorem 2. *If an equilibrium point $\mathbf{z}_e$ is small disturbance stable, then it is asymptotically stable in the sense of definition* 2.

The proof of this is given in appendix A.

Similar results have been derived before (DeMarco & Bergen 1984), but have relied on singular perturbation results. In so doing they have placed conditions on the sign of $\mathbf{J}_{ll}$ which are not required here.

5. Global dynamical structure

In this section, we move beyond the local ODE equivalence given by lemma 1 to study the DA system as globally decomposed into multiple ODE systems on regions bounded by surfaces of algebraic singularity.

The assumption of constant real power load made at (2a) was necessary to ensure path independence of the energy function integral in (18). However, such an assumption is not required when considering structural aspects of DA systems. It shall be relaxed for this discussion, i.e., we will allow real power loads to take the form $P_{di} = P_{di}^0(|V_i|)$.

It can be seen from (8) and (12) that real power loads influence $\mathbf{J}_{ll}$. When real power loads are constant $\mathbf{J}_{ll}$ is symmetric, so its eigenvalues are all real. However voltage dependence of real power loads causes $\mathbf{J}_{ll}$ to be asymmetric. This can (and often does) cause $\mathbf{J}_{ll}$ to have complex eigenvalues.

5.1 *ODE decomposition*

Our first result will establish the ODE decomposition.

It is convenient to make the following assumption.

A4. For all $\mathbf{z}\in G$, all negative real eigenvalues of $\mathbf{J}_{ll}|_{\mathbf{z}}$ are distinct.

This assumption eliminates the possibility that as a path over the constraint manifold is traversed, negative real eigenvalues of $\mathbf{J}_{ll}$ could merge, then split from the real axis as a complex pair. Under this assumption, the only way that the number of negative real eigenvalues of $\mathbf{J}_{ll}$ can change is by a real eigenvalue crossing the imaginary axis, i.e., $\mathbf{J}_{ll}$ going singular. The assumption is valid for most realistic power systems. However rare circumstances can be found where it is not true (Hiskens 1990). The consequences of such behaviour are explored later.

Open sets C_l which lie within the constraint manifold can now be defined as

$$C_l = \{\mathbf{z}\in G: \det \mathbf{J}_{ll}|_{\mathbf{z}} \neq 0, \mathbf{J}_{ll}|_{\mathbf{z}} \text{ has } l \text{ negative real eigenvalues}\}. \tag{25}$$

These sets may not necessarily be connected. Partition each C_l into its connected components $C_{l1}, \ldots C_{lk}$, i.e.

$$C_l = \bigcup_{i=1}^{k} C_{li} \text{ and } C_{li} \cap C_{lj} = \phi,\ i \neq j,\ i,j = 1 \ldots k.$$

From (13), it can be seen that the functions ϕ, ψ, which enable the establishment of the equivalent ODE representations, are functions of generator angles only. Therefore it is convenient to project the sets C_{li} onto their generator angle components, as

$$A_{li} = \{\boldsymbol{\alpha}_g : (\boldsymbol{\alpha}_g, \boldsymbol{\alpha}_l, |\mathbf{V}|) \in C_{li}\} \tag{26}$$

A5. Each C_{li} is simply connected.

This assumption is difficult to check. However, in the context of power systems, extensive studies have not revealed a counter-example.

Theorem 3. *Assume each C_{li} is simply connected. On each C_{li}, the set G is represented by unique continuous functions $\phi_{li}: A_{li} \to \mathbb{R}^{no}, \psi_{li}: A_{li} \to \mathbb{R}^{no}$ such that $\boldsymbol{\alpha}_l = \phi_{li}(\boldsymbol{\alpha}_g)$, $|\mathbf{V}| = \psi_{li}(\boldsymbol{\alpha}_g)$. The DA system (9), (10) is equivalent to the local ODE representation (14).*

The proof is given in appendix B.

Remarks. (a) It is easy to generate examples of systems where the sets C_l are not connected (Hiskens 1990).

(b) The sizes of sets C_{li} depend greatly on the load model parameters. Conditions can be given for ensuring $C_l, l \neq 0$ are empty (Hiskens & Hill 1989; Hiskens 1990).
(c) This result sharpens the one given earlier by Hiskens & Hill (1989).

The boundaries of the sets C_{li} are referred to as "impasse surfaces" – a term borrowed from circuit theory (Hasler & Neirynck 1986).

The regions C_{li} are referred to as voltage causal regions. Within any C_{li}, the load bus voltages and angles are continuously dependent on the generator angles. If trajectories meet an impasse surface, voltage behaviour can no longer be predicted from the DA model.

The impasse surface I is given by

$$I = \{\mathbf{z} \in G : \det \mathbf{J}_{ll}|_{\mathbf{z}} = 0\}. \tag{27}$$

Note that

$$G = \left(\bigcup_{i=0}^{2n_0} C_i \right) \cup I.$$

Define augmented algebraic constraint function

$$\mathbf{i}(\mathbf{z}) = (\det \mathbf{J}_{ll}, \mathbf{f}_l, \mathbf{g}).$$

Fact. Suppose rank $D\mathbf{i} = 2n_0 + 1$ at a point p in I. Then in a neighbourhood of p, I is a differentiable $(m-2)$-manifold.

From this fact we can build a picture of I as composed of intersecting differentiable $(m-2)$-manifolds. On each of these manifolds, $\mathbf{J}_{ll}$ has exactly one zero eigenvalue. They intersect at lower dimensional manifolds where 2 or more eigenvalues are zero and rank $D\mathbf{i} < 2n_0 + 1$. It remains to determine whether $D\mathbf{i}$ has full rank at all non-intersection points, i.e. does some $(m-2)$-manifold segment have a boundary?

The following example examines the constraint manifold structure in the context of power systems.

Example 1. Consider the network shown in figure 1. For simplicity we will not augment this network by buses representing fictitious generator internal buses. The dynamic variables are therefore $\alpha_{g1}, \alpha_{g2}, \omega_{g1}, \omega_{g2}, \omega_{g3}$. The algebraic variables are

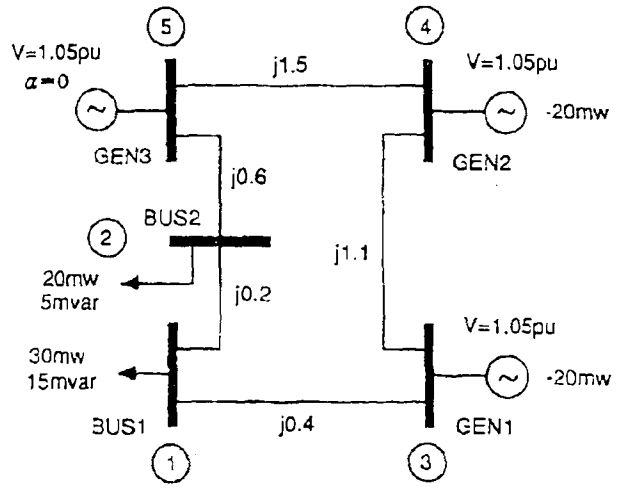

Figure 1. Power system network.

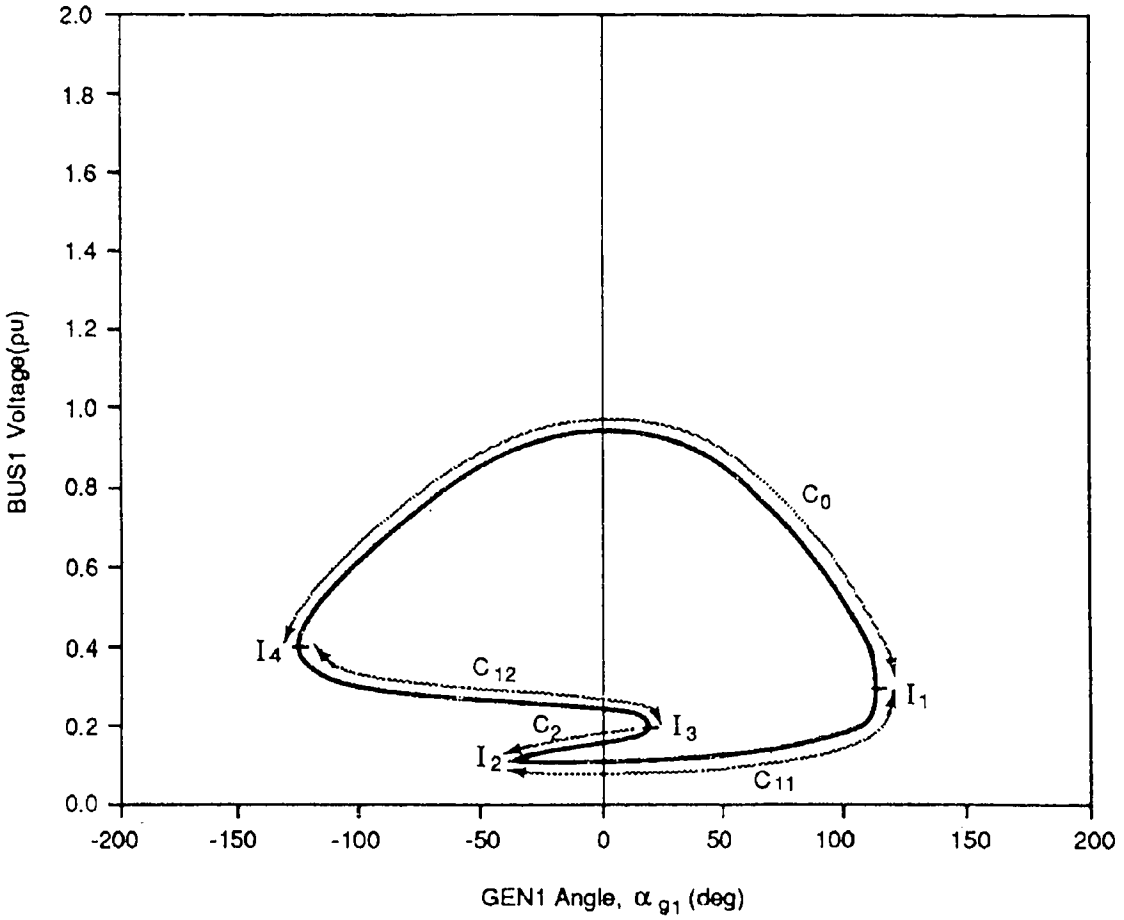

Figure 2. Constraint manifold projection, $(\alpha_{g1}, |V_1|$ space).

$\alpha_{l1}, \alpha_{l2}, |V_1|, |V_2|$. The constraint mainfold has dimension 2. Note though that because the load bus network is connected between generator GEN1 and the reference bus GEN3, the load bus variables depend only on α_{g1}. They are independent of α_{g2}. Therefore the constraint manifold has a cylindrical form in the sense that the same shape is maintained at all values of α_{g2}. Projections of a slice through this 2-manifold are shown in figure 2. The impasse surface appears in these projections as four points I_1, I_2, I_3, I_4, separating causal regions C_0, C_{11}, C_{12}, C_2. The impasse surface segments do not intersect. □

The impasse surface, which divides the constraint manifold into causal regions, is defined at (27) as the set of points where $\mathbf{J}_{ll}$ is singular. It is therefore composed of points at which at least one *real* eigenvalue of $\mathbf{J}_{ll}$ is zero. Note then that complex eigenvalues have no influence on the structure of the impasse surface. Hence, the causal region decomposition is not influenced at all by complex eigenvalue behaviour. (Because of its non-zero imaginary part, a complex eigenvalue can have zero real part at a causal point.)

However, it can be observed that along a path traversing the constraint manifold, the imaginary part of a complex pair may become zero, leaving a pair of repeated real eigenvalues. These real eigenvalues can then of course influence the decomposition in the usual way. Whilst not affecting the causal region structure, this behaviour does cause difficulties with the indexing scheme established for the regions in (25). As an

example, if the imaginary part of a complex pair, which was in the left half plane, became zero, two negative real eigenvalues would be created. Points in the same causal region would have causal indices differing by two. However, because the indexing scheme is no more than a convenient way of identifying regions, this effect is not considered important. Certainly such eigenvalue behaviour in no way affects the analysis of DA systems.

Thus the global structure of DA is established: the constraint set G consists of disjoint open sets C_{li} which are separated by the impasse surface and within which the dynamics are given by a local ODE description.

5.2 *Conditions for global voltage causality*

The above presents a complicated general picture for the dynamical structure of DA systems. For the power systems case, variations of the load indices cause significant structural changes to the causal region/impasse surface decomposition of the constraint manifold. It is possible to find load indices which ensure global voltage causality, i.e., $\det \mathbf{J}_{ll} \neq 0$ at all points on the constraint manifold. Then the DA model which employed those indices could be globally reduced to a unique set of differential equations, valid at all points on the constraint manifold. Results for special cases are available (Hiskens & Hill 1989; Hiskens 1990).

6. Stability assessment

In this section the implications of the global structure on the energy function picture are studied.

6.1 *Estimate of stability region*

In theorem 3, local solvability of (10) was extended to solvability over voltage causal regions. The same concept can be used to extend the region of validity of the local representation of V. An estimate for the region of attraction for a stable equilibrium point, $\mathbf{z}_S$, of the DA model can then be determined. Let the number of negative eigenvalues of $\mathbf{J}_{ll}|_{\mathbf{z}_S}$ be l. By theorem 3 there exist unique continuous functions $\alpha_l = \phi_{ll}(\alpha_g), |\mathbf{V}| = \psi_{ll}(\alpha_g)$ such that over a voltage causal region C_{li}, V at (18) can be written

$$V_{li}(\omega_g, \alpha_g) = V(\omega_g, \mathbf{z})|_{C_{li}}$$
$$= \tfrac{1}{2}\omega_g^t \mathbf{P}_1(\mu)\omega_g + \int_{\alpha_g^S}^{\alpha_g} \langle \tilde{\mathbf{f}}_g(\alpha_g, \phi_{li}(\alpha_g), \psi_{li}(\alpha_g)), \mathrm{d}\alpha_g \rangle. \tag{28}$$

Define the sets

$$R_{li}^k = \{(\omega_g, \alpha_g) |\ V_{li}(\omega_g, \alpha_g) \leqslant k\},$$
$$S_{li}^k = \{(\alpha_g, \alpha_l, |\mathbf{V}|) : (\omega_g, \alpha_g) \in R_{li}^k, \alpha_l = \phi_{li}(\alpha_g), |\mathbf{V}| = \psi_{li}(\alpha_g)\}.$$

Note that the elements of S_{li}^k are simply points in $(\alpha_g, \omega_l, |\mathbf{V}|)$-space which correspond to elements of R_{li}^k (i.e., points in (ω_g, α_g)-space).

An estimate of the stability region is obtained via the following theorem.

Theorem 4. *Let $\xi_S = (\mathbf{0}, \mathbf{z}_S)$ be an asymptotically stable equilibrium point of the DA model. Then, for all $k > 0$ such that R_{li}^k is bounded and $S_{li}^k \subseteq C_{li}$ any trajectory $s(t, \mathbf{x}_0)$ with initial conditions $\mathbf{x}_0 \in R_{li}^k$ has the following properties:*

(i) $\mathbf{s}(t, \mathbf{x}_0) \in R_{li}^k$, *for all* $t \geqslant 0$ (*i.e. R_{li}^k is invariant with respect to the DA model*).
(ii) $\mathbf{s}(t, \mathbf{x}_0) \to \mathbf{x}_s$ *as* $t \to \infty$.

The proof of this theorem is given in appendix A.

Remarks. (a) As k is increased, a value will be attained where one of the two conditions on R_{li}^k, S_{li}^k breaks down. Either

(i) R_{li}^k becomes unbounded, i.e. V_{li} is no longer locally positive definite; or
(ii) $S_{li}^k \not\subseteq C_{li}$, i.e. there are points in S_{li}^k for which the local model is no longer valid.

These phenomena are consistent with definitions of power system stability (Hiskens & Hill 1989). The limit placed on k by (i) ensures that all points in R_{li}^k are attracted to the stable equilibrium point, $\mathbf{z}_s$, i.e., if k was allowed to increase, then for some $\mathbf{x}_0 \in R_{li}^k$, $\mathbf{s}(t, \mathbf{x}_0) \not\to \mathbf{x}_s$, as $t \to \infty$. This is angle instability because the dynamic variables, i.e. generator frequencies and angles, do not tend towards the stable equilibrium point. (Note though that the impasse surface could be encountered as the system proceeded along this unstable trajectory. In that case voltage causality would be lost as a consequence of the initial angle instability.) The limit placed by (ii) ensures that the local model and energy function are valid for all points in S_{li}^k. In this case, if k was to increase, then for some $\mathbf{z}_0 \in S_{li}^k$, $\det \mathbf{J}_{ll} = 0$, i.e., lack of voltage causality.

(b) Let the largest value of k satisfying theorem 4 be k_{crit}. This value could be used in the traditional way as the critical value of energy able to be attained by the disturbed system with stability still guaranteed. This of course is likely to be quite conservative. A practical algorithm will employ information on fault location (Pai 1981).

(c) A result similar to this, but requiring all eigenvalues of $\mathbf{J}_{ll}$ to be positive has been derived by DeMarco & Bergen (1984). Singular perturbation results were used in that case.

6.2 *Multiple energy function sheets*

If the energy function (18) is treated in the usual way as the sum of kinetic and potential energy terms, then it is only the potential energy term which is dependent on the set C_{li}. The local potential energy functions are functions of $\boldsymbol{\alpha}_g$ only, and so can be conceptualized as $(m-1)$-hypersurfaces (or sheets) in $\boldsymbol{\alpha}_g$-space. (Recall the potential energy well concept in energy function methods, Pai 1981).

For each region C_{li} defined by theorem 3, a unique local potential energy function exists, each one a sheet in $\boldsymbol{\alpha}_g$-space. It is not difficult to imagine therefore how it is possible to have a number of asymptotically stable equilibria. (Those sheets with a locally positive definite section must have an asymptotically stable equilibrium point at the lowest point of that section.) Note that not all sheets need contain equilibria however.

All the PE sheets join on the impasse surface. The sheets can be thought of as approaching each other infinitesimally closely at the impasse surface.

While not hard to illustrate by example (Hiskens & Hill 1989), a complete theoretical

discussion of these issues remains to be studied. Nevertheless it is already clear that DA models change the traditional view of large disturbance stability substantially. For instance, the phenomenon of short-term voltage collapse can perhaps be explained in terms of "jumps" between different energy levels (Hiskens & Hill 1989).

Appendix A. Proofs of stability results

Proof of theorem 2

Define the Jacobian

$$\mathbf{J} = \partial \mathbf{h}/\partial \mathbf{z} = \left[\begin{array}{c|c} \partial \tilde{\mathbf{f}}_g/\partial \boldsymbol{\alpha}_g & \partial \tilde{\mathbf{f}}_g/\partial \boldsymbol{\alpha}_l \quad \partial \tilde{\mathbf{f}}_g/\partial |\mathbf{V}| \\ \hline \partial \mathbf{f}_l/\partial \boldsymbol{\alpha}_g & \\ \partial \mathbf{g}/\partial \boldsymbol{\alpha}_g & \mathbf{J}_{ll} \end{array}\right] = \begin{bmatrix} \mathbf{J}_{gg} & \mathbf{J}_{gl} \\ \mathbf{J}_{lg} & \mathbf{J}_{ll} \end{bmatrix},$$

where $\mathbf{h}$ is defined after (18). Let $\xi = (\omega_g, \mathbf{z}) = (\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (\omega_g, \boldsymbol{\alpha}_g)$ contains the dynamic states and $\mathbf{y} = (\boldsymbol{\alpha}_l, |\mathbf{V}|)$ contains the algebraic states.

If $\mathbf{z}_e$ is small disturbance stable, then $\det \mathbf{J}_{ll}|_{\mathbf{z}_e} \neq 0$ and $\mathbf{F}|_{\mathbf{z}_e} = \mathbf{J}_{gg} - \mathbf{J}_{gl}\mathbf{J}_{ll}^{-1}\mathbf{J}_{lg}|_{\mathbf{z}_e}$ is positive definite.

Observe that $\tilde{\mathbf{f}}_g(\mathbf{z}_e) = \mathbf{0}$. Hence $(\partial V/\partial \xi)|_{\xi_e} = 0$, and a Taylor expansion of $V(\xi)$ about ξ_e yields,

$$V(\xi) = (\xi - \xi_e)^t \begin{bmatrix} \mathbf{P}_1(\mu) & 0 \\ 0 & \mathbf{J}_{\mathbf{z}_e} \end{bmatrix} (\xi - \xi_e) + 0(\|\xi - \xi_e\|^3), \tag{A1}$$

where $0(\cdot)$ represents higher order terms.

Now note that

$$\begin{bmatrix} \mathrm{d}\mathbf{f}_l \\ \mathrm{d}\mathbf{g} \end{bmatrix} = \mathbf{J}_{lg}\mathrm{d}\boldsymbol{\alpha}_g + \mathbf{J}_{ll}\begin{bmatrix} \mathrm{d}\boldsymbol{\alpha}_l \\ \mathrm{d}|\mathbf{V}| \end{bmatrix} = \mathbf{0}.$$

So

$$\begin{bmatrix} \mathrm{d}\boldsymbol{\alpha}_l \\ \mathrm{d}|\mathbf{V}| \end{bmatrix} = \mathbf{J}_{ll}^{-1}\mathbf{J}_{lg}\mathrm{d}\boldsymbol{\alpha}_g.$$

By hypothesis $\mathbf{J}_{ll}|_{\mathbf{z}_e}$ is nonsingular, so

$$\int_{(\boldsymbol{\alpha}_{le}, |\mathbf{V}|_e)}^{(\boldsymbol{\alpha}_l, |\mathbf{V}|)} \mathrm{d}\begin{bmatrix} \boldsymbol{\alpha}_l \\ |\mathbf{V}| \end{bmatrix} = \int_{\boldsymbol{\alpha}_{ge}}^{\boldsymbol{\alpha}_g} -(\mathbf{J}_{ll}^{-1}\mathbf{J}_{lg})|_{\mathbf{z}_e}\mathrm{d}\boldsymbol{\alpha}_g,$$

i.e.,

$$\begin{bmatrix} \boldsymbol{\alpha}_l \\ |\mathbf{V}| \end{bmatrix} - \begin{bmatrix} \boldsymbol{\alpha}_{le} \\ |\mathbf{V}|_e \end{bmatrix} = -(J_{ll}^{-1}\mathbf{J}_{lg})|_{\mathbf{z}_e}(\boldsymbol{\alpha}_g - \boldsymbol{\alpha}_{ge}). \tag{A2}$$

From (A1), $V(\xi)$ can be rewritten

$$V(\xi) = \begin{bmatrix} \omega_g - \omega_{ge} \\ \boldsymbol{\alpha}_g - \boldsymbol{\alpha}_{ge} \\ \boldsymbol{\alpha}_l - \boldsymbol{\alpha}_{le} \\ |\mathbf{V}| - |\mathbf{V}|_e \end{bmatrix}^t \begin{bmatrix} \mathbf{P}_1(\mu) & 0 & 0 \\ 0 & \mathbf{J}_{gg} & \mathbf{J}_{gl} \\ 0 & \mathbf{J}_{lg} & \mathbf{J}_{ll} \end{bmatrix} \begin{bmatrix} \omega_g - \omega_{ge} \\ \boldsymbol{\alpha}_g - \boldsymbol{\alpha}_{ge} \\ \boldsymbol{\alpha}_l - \boldsymbol{\alpha}_{le} \\ |\mathbf{V}| - |\mathbf{V}|_e \end{bmatrix} + 0(\cdot), \tag{A3}$$

where all partial derivatives are evaluated at $\mathbf{z}_e$. Combining (A2) and (A3) gives

$$V(\alpha) = (\omega_g - \omega_{ge})^t \mathbf{P}_1(\mu)(\omega_g - \omega_{ge}) + (\alpha_g - \alpha_{ge})^t (\mathbf{J}_{gg} - \mathbf{J}_g \mathbf{J}_{ll}^{-1} \mathbf{J}_{lg})|_{z_e} (\alpha_g - \alpha_{ge}) + 0(\cdot)$$

It is given that $\mathbf{P}_1(\mu) > 0$. Also, the assumption of small disturbance stability implies

$$(\mathbf{J}_{gg} - \mathbf{J}_{gl} \mathbf{J}_{ll}^{-1} \mathbf{J}_{lg})|_{z_e} > 0.$$

Thus $V(\xi_e + \zeta)$ is locally positive definite on some neighbourhood of ξ_e.

Define

$$R_k = \text{component of } \{\xi: V(\xi) \leqslant k\} \text{ containing } \xi_e.$$

Because $\det \mathbf{J}_{ll} \neq 0$ by hypothesis, there exists a neighbourhood of ξ_e such that the algebraic equations (10) are solvable. Thus in the whole state space, there is a neighbourhood $\Lambda(\mathbf{x}_e)$ as defined for lemma 1.

Now consider

$$S = \{\xi: \dot{V}_{DA} = 0\},$$

where $\dot{V}_{DA}$ is the derivative of V along trajectories of the DA model (9), (10). From (21),

$$\dot{V}_{DA} = \tfrac{1}{2} \omega_g^t \mathbf{Z}_0(\mu) \omega_g,$$

with $\mathbf{Z}_0(\mu)$ given by (22). Therefore, it is easy to see that S consists of all equilibria $(\mathbf{0}, \mathbf{z}_e)$ where $\mathbf{z}_e$ satisfies (11).

Because V is locally positive definite, it is clear that for some k, sufficiently small R_k, is bounded and $S \cap R_{k_1} = \{\xi_e\}$.

The result then follows from theorem 1.

Proof of theorem 4. Because $S_{li}^k \subseteq C_{li}$ the local energy function $V_{li}(\omega_g, \alpha_g)$ is valid for all $(\omega_g, \alpha_g) \in R_{li}^k$. As in the proof of theorem 2, differentiating V_{li} along trajectories of the reduced ODE system (14) gives

$$\dot{V}_{li}(\omega_g, \alpha_g) = \omega_g^t \mathbf{Z}_0(\mu) \omega_g$$

i.e., $\dot{V}_{li}(\omega_g, \alpha_g) \leqslant 0$.

Due to the construction of R_{li}^k and because $\dot{V}_{li}(\omega_g, \alpha_g) \leqslant 0$, all trajectories $\xi(t, \xi_0)$ of the DA model with initial conditions $\mathbf{x}_0 \in R_{li}^k$, $\mathbf{z}_0 \in S_{li}^k$ must be such that $\mathbf{x}(t, \xi_0)$ remains in R_{li}^k (property i).

Property ii follows from theorem 1 and property i.

Appendix B. Proof of theorem 3.

Because C_{li} is open, each point in C_{li} has a neighbourhood contained in C_{li}. Further, as C_{li} is simply connected, it is pathwise connected, i.e., between any two points in C_{li} there exists a path which is completely contained in C_{li}.

The definition of C_{li} implies that the implicit function theorem is valid at all points $(\alpha_g', \alpha_1', |\mathbf{V}|') \in C_{li}$. Therefore, within a neighbourhood of any point $(\alpha_g', \alpha_1', |\mathbf{V}|')$ there

exist unique continuous functions ϕ'_{li}, ψ'_{li} which relate $\alpha_l, |\mathbf{V}|$ to α_g. (Continuity follows from the fact that $\mathbf{f}_l, \mathbf{g}$ are continuous.)

Consider two arbitrary points $\mathbf{z}_1, \mathbf{z}_2 \in C_{li}$ sufficiently close that neighbourhoods of these points, $U_1, U_2 \subset C_{li}$, overlap, i.e. $U_1 \cap U_2 \neq \phi$. There exist on U_1 unique continuous functions ϕ^1_{li}, ψ^1_{li}, whilst on U_2 there exist unique continuous functions ϕ^2_{li}, ψ^2_{li}. Now consider a point $\mathbf{z}_3 = (\alpha^3_g, \alpha^3_l, |\mathbf{V}|^3) \in U_1 \cap U_2$. Because this point belongs to both neighbourhoods, it must satisfy ϕ^1_{li}, ψ^1_{li} and ϕ^2_{li}, ψ^2_{li}, i.e.

$$\alpha^3_l = \phi^1_{li}(\alpha^3_g) \text{ and } |\mathbf{V}|^3 = \psi^1_{li}(\alpha^3_g),$$

and

$$\alpha^3_l = \phi^2_{li}(\alpha^3_g) \text{ and } |\mathbf{V}|^3 = \psi^2_{li}(\alpha^3_g),$$

But ϕ^1_{li}, ψ^1_{li} and ϕ^2_{li}, ψ^2_{li} are unique at all points within their respective neighbourhoods. So,

$$\phi^1_{li} = \phi^2_{li}, \text{ and } \psi^1_{li} = \psi^2_{li}.$$

Because points $\mathbf{z}_1, \mathbf{z}_2$ are arbitrary, the above argument applies for any sufficiently close points. In particular, along any path completely contained in C_{li} every point has a neighbourhood which intersects neighbourhoods of other points on the path. A chain of points with overlapping neighbourhoods can be formed along the path. By applying the above argument at successive points along that chain and since C_{li} is simply connected, it can be concluded that functions ϕ_{li} and ψ_{li} must be unique along the whole path, including its end points.

But any two points in C_{li} can be joined by some path. Further, because C_{li} is simply connected, paths can be continuously transformed within C_{li}. Therefore ϕ_{li} and ψ_{li} are unique over C_{li}. □

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