Viscous–inviscid interaction schemes for external aerodynamics

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Abstract. For aerofoils a calculation, which involves the coupling of the external inviscid flow with the viscous flow in the boundary layer and the wake, still provides a worthwhile alternative to the solution of the ‘time-averaged’ Navier–Stokes equations. Classical viscous–inviscid interaction methods which can be extended to include flows with separations and significant pressure gradients across the boundary layer are described. Basic theoretical principles of interactive methods in two dimensions are discussed. The extension of the classical methods leads to generalisations of the concept of displacement thickness and the momentum integral equation. The boundary conditions for the equivalent inviscid flow (EIF) are also described and these also include the effect of normal pressure gradients. An integral method based on the lag-entrainment method for the calculation of the turbulent boundary layer is described. The correlations associated with the method are extended to include separated flow. Two methods of solving the boundary-layer equations through a separation region are described: the inverse method and the quasi-simultaneous method. Principles of techniques for coupling the flows are described and the properties of the direct, fully inverse, semi-inverse and quasi-simultaneous methods are discussed. Results from a method for incompressible flow about a stalled aerofoil, a method for compressible flow about a high-lift aerofoil and a method for compressible flow about a transonic aerofoil are compared with experimental results. The current situation regarding the development of viscous–inviscid interaction methods is briefly summarized and future possibilities are considered.

Keywords. Viscous–inviscid interaction schemes; external aerodynamics; time-averaged Navier–Stokes equations.

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1. Introduction

In the external aerodynamics of aircraft the representation of the complete flowfield by a solution of the time-averaged form of the Navier–Stokes equations is still not practicable for the aircraft designer despite the impressive advances in algorithm development and computer architecture. In many problems of practical importance the effects of viscosity and turbulence are confined to thin layers close to the ‘wetted surfaces’ and their wakes: whilst the flow outside these regions can be accurately described by the Euler equations of inviscid flow. The method of ‘viscous–inviscid interaction’ (vii) relies on these physical features and matches separate calculations for the external inviscid flow and the viscous shear layers iteratively to provide a composite solution for the complete flowfield: the subject has been reviewed by several authors (Le Balleur 1980; Lock & Firmin 1981; Lock & Williams 1987). In this work the method of viscous/inviscid interactions is extended to cover flows where there is a strong interaction between inviscid and viscous flows. This strong interaction is usually characterised by significant pressure gradients normal to the surface and the possibility of flow separation.

For computational efficiency the outer inviscid flow is usually extended over the region occupied by the viscous flow to form the ‘equivalent inviscid flow’ (EIF). An extended definition of the displacement thickness can be derived for the shear layers and this is used to define an inner boundary condition for the equivalent inviscid flow.

The approximations governing the shear layers are extended from the classical assumption that the pressure variation across the boundary layer is negligible. For two-dimensional flow it is indicated how these higher order effects can be represented approximately but adequately in an integral formulation of the shear-layer equations.

The classical direct methods of calculating the boundary layer are ill-conditioned at separation. However the ill-conditioning is associated with the method of solution rather than being an inherent property of the equations as demonstrated for laminar boundary layers by Catherall & Mangler (1966). The boundary-layer equations can be solved through separation by an inverse method.

Although the boundary-layer solution can be extended into regions where there is a strong interaction between the inviscid and viscous flows (such as separated flow) it is necessary to refine the iterative schemes for matching the inviscid and viscous solutions. In the classical method the inviscid calculation of the equivalent inviscid flow provides improved boundary conditions for the shear layer calculation: in turn the shear layer calculation provides new boundary conditions for the equivalent inviscid flow. These two calculations are repeated alternatively with the intention of obtaining a converged solution. It is demonstrated that this process is only likely to converge for attached flow and then only with the assistance of under-relaxation. However the semi-inverse coupling scheme is able to produce solutions for attached and separated flow. In this scheme the inverse solution of the boundary-layer equations is linked with a direct solution of the inviscid flow.

The semi-inverse method tends to converge slowly and a study of the coupling methods suggests alternative procedures with improved convergence characteristics. In particular the ‘quasi-simultaneous’ scheme is simple to implement and has good convergence characteristics.

Examples of calculations with all the coupling schemes will be compared with experiments for single and multiple aerofoils at subsonic and transonic speeds.

2. Physical background

The effects of viscosity are most noticeable in transonic flow although significant effects are present in subsonic flow. A qualitative description of some situations occurring in the flow over wings will indicate where these effects are most pronounced: in particular we concentrate on situations where there is a 'strong interaction' between the inviscid and viscous flows.

The flow over a typical lifting aerofoil at transonic speed is illustrated in figure 1. The boundary layer on the upper surface develops through a stronger adverse pressure gradient than the boundary layer on the lower surface, so that even for attached flow

Figure 1. Viscous flow over an aerofoil at transonic speeds.

the boundary layer at the trailing edge is thicker on the upper surface than on the lower surface. The displacement effect associated with the boundary layers at the trailing edge has the effect of reducing the camber of the aerofoil and thus reduces the lift. For modern sections with rear loading the reduction in lift from the inviscid value can be as much as 25% even at flight Reynolds numbers.

In the regions marked (A) and (B) there is likely to be a strong interaction between the inviscid and viscous flows. In both the cases of a shock-wave/boundary-layer

Figure 2. Viscous flow over a multiple-element aerofoil at low speed.

interaction (A), and a trailing edge (B), there are significant pressure gradients in the normal (as well as the streamwise) direction. If the flow separates ahead of the trailing edge the effect of the Reynolds normal stresses will become important.

The flow about a typical multiple-element aerofoil is indicated in figure 2 and there are several regions in which higher order effects will be important including the curved wakes, the separation bubbles in the coves, rear separation on the flap and the interaction between the wakes from upstream elements with the boundary layers on downstream elements.

The developments in the theory of viscous/inviscid interaction for two-dimensional flow described in the next sections allow these effects to be included in an approximate, yet adequate manner. For three-dimensional flows the theory is not so well developed. There are no adequate approximations for any of the important higher order effects and it has proved difficult to couple inverse solutions with inviscid methods.

3. Basic theoretical principles of interactive methods

The representation of the displacing effect of the shear layer in the outer inviscid flow is the first requirement of a VII method. In this section a generalisation of the displacement effect allows the treatment of flows with significant pressure gradients across the boundary layer and this includes separated flows. In the same manner the momentum integral equations are extended to include the effect of pressure gradients normal to the surface. The remainder of the paper develops the theory using integral boundary layer methods. The approximations that are developed could be used with a finite difference method instead of solving the full normal momentum equation.

In the viscous-inviscid interaction technique, separate solutions for the inviscid flow external to the shear layer and the viscous flow in the shear layer are combined iteratively to form a composite solution which is continuous across the matching surface between the inviscid and viscous flows. The solution for the outer inviscid flow must take account of the displacing effect of the inner viscous flow by the specification of an appropriate inner boundary condition. There are several positions at which the matching surface can be placed. The most obvious approach places the matching surface at the outer edge of the shear layer which can be conveniently defined by δ , the thickness of the shear layer. At this surface the boundary conditions for both the inner and outer regions are the continuity of magnitude and direction of the velocity fields. However the thickness of the shear layer will not be known until the final solution has been obtained, thus the matching surface will vary as the iterations proceed and this leads to real computational difficulties. This difficulty is overcome if the matching surface is taken as the wall. It is now necessary to construct an equivalent inviscid flow which extends to the wall but accounts for the displacing effect of the shear layer by the application of suitable boundary conditions on the matching surface.

3.1 Generalised definition of the displacement thickness

The inner boundary conditions for the EIF are determined by taking the difference of the continuity equations for the EIF and the real viscous flow (RVF) (see Le Balleur 1980). The derivation is illustrated by considering the two-dimensional flow with an orthogonal curvilinear coordinate (s, z) system in which s and z are distances parallel and normal to the wall (or some convenient line in the wake) and U and W are the corresponding velocity components. The curvature of the surface is κ_w , taken to be

positive if the wall is concave outwards; so the metrics of this coordinate system are 1 and $1 - \kappa_w z$, with the suffix *w* representing conditions at the wall. The difference between the continuity equations for the RVF and the EIF (indicated by the suffix *i*) gives the equation of the form

$$\frac{\partial}{\partial s}(\rho_i U_i - \rho U) + \frac{\partial}{\partial z}[(1 - \kappa_w z)(\rho_i W_i - \rho W)] = 0, \quad (1)$$

where ρ is the density. Integrating (1) across the boundary layer, from $z = 0$ to δ gives

$$\begin{aligned} \rho_{iw} W_{iw} &= \int_0^\delta \frac{\partial}{\partial s}(\rho_i U_i - \rho U) dz \\ &= \frac{d}{ds} \int_0^\delta (\rho_i U_i - \rho U) dz. \end{aligned} \quad (2)$$

If the displacement thickness, δ^* is defined by

$$\delta^* = (1/\rho_{iw} U_{iw}) \int_0^\delta (\rho_i U_i - \rho U) dz, \quad (3)$$

then

$$W_{iw} = \frac{1}{\rho_{iw}} \frac{d}{ds}(\rho_{iw} U_{iw} \delta^*). \quad (4)$$

It is convenient to define the non-dimensional quantity

$$S = \frac{W_{iw}}{U_{iw}} = \frac{1}{\rho_{iw} U_{iw}} \frac{d}{ds}(\rho_{iw} U_{iw} \delta^*). \quad (5)$$

S is called the ‘equivalent source strength’ and is the tangent of the angle made by the inviscid flow streamline at the wall with the surface $z = 0$.

3.2 Streamwise momentum integral equation

The derivation of the generalised form of the momentum integral equation follows similar lines to determination of the displacement effect in the previous section. The derivation is described in detail in Lock & Williams (1987) and follows the work of East (1981) and Le Balleur (1981); only a short summary is reproduced with an emphasis on the implementation of the method. The difference between the streamwise momentum equations, in the (s, z) coordinate system, for the RVF and EIF is integrated across the boundary layer from 0 to δ . The resulting equation is nominally exact and takes the form

$$\begin{aligned} \frac{d\theta}{ds} + (H + 2 - M_{iw}^2) \frac{\theta}{U_{iw}} \frac{dU_{iw}}{ds} - \frac{1}{2} C_f = \kappa_w^{(1)} (\Theta + \delta \bar{C}_\tau^{(2)}) + \\ + \frac{1}{\rho_{iw} U_{iw}^2} \frac{d}{ds} \left[- \int_0^\delta (p_i \ominus p) dz + \int \rho u'^2 dz \right] + M_{iw}'^2 \theta \frac{d}{ds} (\frac{1}{2} S^2), \end{aligned} \quad (6)$$

where θ is a generalised definition of the momentum thickness given by

$$\theta = (1/\rho_{iw} U_{iw}^2) \int_0^\delta [\rho_i U_i^2 - \rho U^2 - U_{iw}(\rho_i U_i - \rho U)] dz,$$

$H = \delta^*/\theta$ is the shape factor, u' and w' are fluctuating velocity components, C_f is the skin friction coefficient and $\bar{C}_\tau$ is the mean Reynolds shear stress coefficient.

It is instructive to note the differences between (6) and the first-order momentum integral equation of von Karman, which has the form

$$\frac{d\theta}{ds}(H + 2 - M_e^2) \frac{\theta}{U_e} \frac{dU_e}{ds} - \frac{1}{2}C_f = 0 \quad (7)$$

where the suffix e denotes values at the outer edge of the boundary layer. In the next paragraph some suitable approximations are given for the second-order terms that appear on the right hand side of (6). However on the left hand side U_e is replaced by U_{iw} . In first-order theory these are assumed to be the same but in higher order theory U_{iw} is not the same as U_e (for details see §3.3). Hence, the velocities from the EIF should be used to calculate the development of the shear layer rather than the values measured in the experiment. For boundary layers approaching separation the pressure gradient in the EIF is normally more adverse than the measured pressure gradient: this helps to explain the observed tendency of first-order boundary layer methods failing to predict separation correctly when a measured pressure distribution is used.

An integral method relies on some empiricism so East (1981) and Lock & Firmin (1981) have used empiricism and intuition to derive approximations to the terms on the right hand side of (6). The approximate equation takes the form

$$\begin{aligned} \frac{d\theta}{ds} + (H + 2 - M_{iw}^2) \frac{\theta}{U_{iw}} \frac{dU_{iw}}{ds} = \frac{1}{2}C_f + \kappa_w \left\{ (\theta + \delta^*) \frac{d\delta^*}{ds} + 0.03\delta \frac{(\bar{H} - 1)^2}{\bar{H}} \right\} \\ + \frac{1}{\rho_{iw} U_{iw}^2} \frac{d}{ds} [\rho_{iw} U_{iw}^2 \{ -\frac{1}{2}\kappa^* (\theta + \delta^*)^2 + \theta F(\bar{H}) \}] + M_{iw}^2 \theta \frac{d}{ds} (\frac{1}{2}S^2), \quad (8) \end{aligned}$$

where the terms have retained their numbering so that the origins in (6) can be more easily identified and $F(\bar{H}) = 0.07(\bar{H} - 1)/\bar{H}$.

It is worth noting that the main justification for including the second-order terms even in an approximate form is the improvement in the prediction of overall lift and drag (see Ashill *et al* 1987, and §6.3). However it is worthwhile noting that in the method of Ashill *et al* (1987) the terms ①, ②, and ③ were justifiably omitted, and it was found that the effect of the pressure integral term ③ was usually negligible; so that the normal stress integral ④ is probably the most important under normal circumstances.

3.3 The effect of normal pressure gradients

In first-order boundary-layer theory the pressure variation across the layer is assumed to be zero but for flows with curvature the normal pressure gradient across the layer is significant and must be accounted for in the calculation. The curvature of the flow can be caused either by (a) high curvature of the wall or (b) rapid thickening of the boundary layer as it approaches separation. In the first case the corrections due to curvature can be derived analytically but in the second case some approximation is required.

The normal pressure gradient has two distinct effects. In the first case it induces a difference between the wall pressure, p_{iw} , calculated in the equivalent inviscid flow and the pressure, p_w , in the real flow and leads to a formula for correcting the inviscid

results. In the second case it enables the evaluation of the 'pressure-difference' integral $\int_0^\delta (p_i - p) dz$ which occurs in the streamwise momentum integral equation, as indicated by term ③ in (8).

The correction to the wall pressure, $\Delta p_w = p_{iw} - p_w$, is given approximately by

$$\Delta p_w / (\rho_{iw} U_{iw}^2) = \kappa^* (\theta + \delta^*), \quad (9)$$

where $\kappa^* = \kappa_w + d^2 \delta^* / ds^2$ which is the curvature of the displacement surface and κ_w is the curvature of the wall. This relation was proposed independently by Collyer & Lock (Collyer 1977; Collyer & Lock 1979) and Le Balleur (1977). In computation it is usually more convenient and accurate to replace $d^2 \delta^* / ds^2$ by dS/ds . Later East (1981) was able to show that (9) represented a good approximation for flows approaching separation by integrating the difference of the complete normal momentum equations for the EIF and RVF across the boundary layer. If only the wall curvature is significant then (9) can be used with κ^* replaced by κ_w .

The variation of the pressure across the boundary layer is indicated in figure 3 for the two flows (RVF and EIF) by (9) for the case where κ^* is positive (corresponding to a concave displacement surface). The pressure variation across the boundary layer in the real flow is given to the same degree of approximation by

$$(p_w - p_e) / \rho_{iw} U_{iw}^2 \simeq \kappa^* (\delta - \theta - \delta^*), \quad (10)$$

where δ is the thickness of the boundary layer and the suffix e indicates the edge of the boundary layer. For a boundary layer near separation ($\bar{H} \simeq 3$, say), $\theta + \delta^*$ will

Figure 3. Variation of pressure across the shear layer.

Figure 4. Flow at the trailing edge in RVF and EIF.

be about $\delta/2$ so $(p_w - p_e)/\rho_{iw} U_{iw}^2$ is approximately $\kappa^* \delta/2$, and this is of the same order as $(p_{iw} - p_w)/\rho_{iw} U_{iw}^2$. Thus if the variation of pressure across the boundary layer in the real flow is significant, then the difference in pressure between the RVF and EIF is also significant.

It is also worth noting that the wall pressure in the real flow is approximately equal to that in the EIF at $z = \theta + \delta^*$; this fact may sometimes be useful in establishing a 'Kutta' condition at the trailing edge. It is instructive to consider the difference in the pressures at the 'wall' in the RVF and EIF for the attached flow near a trailing edge. The pressures are sketched in figure 4 and it has been assumed that the curvature of the displacement surface is negligibly small on the lower surface. The adverse pressure gradient in the EIF near the trailing edge on the upper surface is more severe than that in the real flow. This helps to improve the prediction of separation and explains the failure of first-order boundary-layer methods to predict separation when a measured pressure distribution is used.

3.4 Matching conditions in the wake

The boundary conditions on the wake in the EIF are similar to the conditions on the aerofoil but are more difficult to apply as the position of the wake is generally not known in advance and forms part of the iterative solution. The displacement and curvature effects lead to discontinuities in both the normal and tangential components of the velocity on the dividing stream surface of the wake.

The jump conditions can be obtained by an analysis similar to that in §3.1 and 3.3. The position of the matching surface is assumed to be close to the dividing stream surface in the real wake. In most calculations it is not convenient to adapt the matching surface to align with the dividing stream surface since the basic computational grid will require modification at each iteration. Lock & Williams (1987) have attempted to quantify the error induced by satisfying the wake boundary conditions at positions other than the dividing stream surface and conclude that these are only significant in high lift conditions. The flow in the near wake is indicated in figure 5 and the

Figure 5. EIF for trailing edge and wake.

pressure variation across the wake is illustrated in figure 6. Using the suffix W to denote values on the wake surface, at points D and C on the upper (subscript u) and lower (subscript l) sides of the wake the normal components of velocity in the EIF are given by

$$W_{iu} = \frac{1}{\rho_{iu}} \frac{d}{ds} (\rho_{iu} U_{iu} \delta_u^*) = U_{iu} S_u,$$

$$W_{il} = -\frac{1}{\rho_{il}} \frac{d}{ds} (\rho_{il} U_{il} \delta_l^*) = -U_{il} S_l, \quad (11)$$

with W positive as z increases. A jump ΔW_{iw} is therefore required in the EIF at BC in the component of velocity normal to the wake surface, given by

$$\Delta W_{iw} \equiv W_{iu} - W_{il} = \sum_{u,l} \frac{1}{\rho_i} \frac{d}{ds} (\rho_i U_i \delta^*) = \sum_{u,l} U_i S_i, \quad (12)$$

where the sign Σ means that the contribution of the upper and lower parts of the

Figure 6. Pressure across wake in EIF and RVF.

wake are to be added. It is possible to simplify (12) by using mean values, but this is unsatisfactory for asymmetric wakes at high lift for example and the full equation is usually as convenient since the two parts of the wake are often calculated separately.

The wake curvature effect is derived by a double application of (9) to give,

$$\begin{aligned} p_w - p_{iu} &= -\kappa_u^* \rho_{iu} U_{iu}^2 (\delta_u^* + \theta_u), \\ p_w - p_{il} &= +\kappa_l^* \rho_{il} U_{il}^2 (\delta_l^* + \theta_l), \end{aligned} \quad (13)$$

where

$$\kappa_u^* = \kappa_w + (dS_u/ds)$$

and

$$\kappa_l^* = \kappa_w + (dS_l/ds).$$

Here the curvature of the wake (κ_w) or the displacement surfaces is reckoned positive if concave upwards.

It follows that a jump is needed in the EIF across the wake surface in order to match the variation of pressure across the wake in the real flow, given by

$$\Delta p_{iw} = p_{iu} - p_{il} = \sum_{u,l} \rho_i U_i^2 \kappa^* (\theta + \delta^*). \quad (14)$$

For computational purposes it is usually more convenient to replace the jump in pressure by one in the streamwise component of velocity. Since Δp_{iw} and ΔU_{iw} can be regarded as small, the linearised form of Bernoulli's equation can be used to give

$$\Delta U_{iw} = - \sum_{u,l} U_i \kappa^* (\theta + \delta^*), \quad (15)$$

where again the summation is taken over the upper and lower parts of the wake. The approximation

$$\Delta U_{iw} = - \bar{U}_{iw} \Sigma \kappa^* (\theta + \delta^*) \quad (15a)$$

will usually be satisfactory where $\bar{U}_{iw} = \frac{1}{2}(U_{iu} + U_{il})$.

3.5 Summary of boundary conditions for the EIF

It is useful to summarise the boundary conditions on the surface of the aerofoil and its wake, that are needed for the calculation of the equivalent inviscid flow. These are indicated in figure 7. Note that on the aerofoil the streamline curvature effect and the normal pressure gradients that it produces give rise to a correction which has to be applied to the pressure calculated in the EIF only after the iterative process has converged; but in the wake it leads to an important part of the boundary conditions for the EIF which is essential if the lift is to be determined correctly. As shown in figure 4, the resulting corrected pressure distribution should then be smooth and single valued on the two sides of the wake line and in particular at the trailing edge.

Far downstream in the wake the jump in U_i (and also W_i) approaches zero, and this condition in effect replaces the 'Kutta condition' for purely inviscid flow.

3.6 Local linearisation of the inviscid flow

In the calculation of the development of the boundary layer by a quasi-simultaneous method a local linearisation of the inviscid flow of the form

$$(dU/ds) = E_1 + E_2 S, \quad (16)$$

Figure 7. Boundary conditions and corrections for EIF.

is required. This can be derived from the continuity equation for any form of discretisation and is found to depend on primarily geometric quantities. The derivation of the formula is illustrated by using a discretisation with the dependent variables specified at the vertices of quadrilateral cells. A typical boundary cell is indicated in figure 8 with the corners numbered 1 $\rightarrow$ 4 and corners 1 and 2 on the boundary. The continuity equation, with κ_w set to zero,

$$\frac{\partial}{\partial s}(\rho U) + \frac{\partial}{\partial z}(\rho W) = 0, \quad (17)$$

is discretised using central difference approximations so that the velocity gradient at the mid-point of the side along the boundary, $\partial/\partial s(\rho U)_{12}$, is given by

$$\frac{\partial}{\partial s}(\rho U)_{12} + \frac{\rho_3 U_3 - \rho_4 U_4}{s_{34}} + \frac{\rho_3 W_3 - \rho_2 W_2}{s_{23}} + \frac{\rho_4 W_4 - \rho_1 W_1}{s_{14}} = 0, \quad (18)$$

where s_{ij} is the length of the side ij . Using the relationships $S = W/U$ and

$$\frac{1}{\rho U} \frac{\partial}{\partial s}(\rho U) = (1 - M_{iw}^2) \frac{1}{U} \frac{dU}{ds},$$

Figure 8. Geometry of discretisation scheme at aerofoil surface.

(17) can be reduced to the form

$$\frac{1}{U} \frac{dU}{ds} = E_1 + E_2 S \quad (19)$$

where

$$E_1 = -\frac{1}{\rho_{12} U_{12}} \frac{\{\rho_3 U_3 - \rho_4 U_4\}}{s_{34}} - \frac{1}{\rho_{12} U_{12}} \frac{\{\rho_4 W_4 - \rho_1 W_1\}}{s_{14}} - \frac{\rho_3 W_3}{\rho_{12} U_{12} s_{23}}$$

and

$$E_2 = 1/[(1 - M_{iw}^2) s_{23}].$$

The coefficient E_2 can be easily determined from geometric properties of the grid, s_{23} , and the local Mach number. The coefficient E_1 involves more computation and it is conveniently determined by considering subsequent iterations in the matching of the viscous and inviscid flows. If the quantities in the inviscid flow at the n th level are indicated by the superscript n then the linearisation of the inviscid flow takes the form

$$(1/U^n)(dU^n/ds) = E_1 + E_2 S^n.$$

If the number of iterations is sufficiently large that E_1 and E_2 can be considered constant the linearisation for the $(n+1)$ th iteration takes the form

$$(1/U^{n+1})(dU^{n+1}/ds) = E_1 + E_2 S^{n+1}.$$

The difference of these two equations yields

$$(1/U^{n+1})(dU^{n+1}/ds) = (1/U^n)(dU^n/ds) - E_2 S^n + E_2 S^{n+1}, \quad (20)$$

so that E_1 is given by $(1/U^n)(dU^n/ds) - E_2 S^n$ which can be calculated from known quantities from the previous iteration.

4. Integral methods for the turbulent boundary layer

As indicated in §3 it is relatively easy to account for the second-order effects in an approximate manner in an integral method. However an integral method requires additional closure equations and these are usually provided by the concept of entrainment as introduced by Head (1958). In this section the 'lag-entrainment' method of Green *et al* (1973) is described for direct and inverse formulations.

4.1 The entrainment equation

The entrainment coefficient, C_E , is defined as the (non-dimensional) rate at which fluid from the external inviscid flow enters the outer edge of the boundary layer. Thus

$$C_E = \frac{1}{\rho_e U_e} \frac{d}{ds} \int_0^\delta \rho U dz. \quad (21)$$

If the entrainment shape parameter H_1 is defined by

$$H_1 = (\bar{\delta} - \delta^*)/\theta = (1/\rho_{iw} U_{iw} \theta) \int_0^\delta \rho U dz, \quad (22)$$

with

$$\bar{\delta} = (1/\rho_{iw} U_{iw}) \int_0^{\delta} \rho_i U_i dz,$$

then the entrainment coefficient becomes

$$C_E = \frac{1}{\rho_e U_e} \frac{d}{ds} (H_1 \rho_{iw} U_{iw} \theta). \quad (23)$$

For the discussion of the structure of the direct method it is useful to expand the differential in (23) to give

$$C_E = H_1 (d\theta/ds) + H_1 (1 - M^2) [(\theta/U)] (dU/ds) + [\theta (dH_1/dH) (dH/ds)], \quad (24)$$

where the difference between $\rho_e U_e$ and $\rho_{iw} U_{iw}$ is neglected. For compressible flows it is convenient to use the transformed shape parameter $\bar{H}$ which is given by

$$\bar{H} = (1/\rho_{iw} U_{iw}) \int (U_i - U) \rho dz / \theta, \quad (25a)$$

and related to H by

$$(H + 1) = (\bar{H} + 1)(1 + 0.5(\gamma - 1)M^2), \quad (25b)$$

where γ is usually taken to be 1.4.

For direct methods (24) is combined with the momentum equation (8), with the right hand side set equal to zero to clarify the presentation, to give

$$\theta (d\bar{H}/ds) = (d\bar{H}/dH_1) [C_E - H_1 \{ \frac{1}{2} C_f - (H + 1)(\theta/U)(dU/ds) \}]. \quad (26)$$

The momentum equation (8) and the entrainment equation (26) are a system of ordinary differential equations and in the direct formulation the values of $\bar{H}$ and θ are calculated from specified values of the velocity gradient dU/ds .

The method is completed by the determination of the parameters C_E , H_1 and C_f . In Head's (1958) original method the parameters C_E and H_1 are related by an algebraic relationship, but this gives poor results for boundary layers which are not in equilibrium. For equilibrium boundary layers the shape of the mean velocity profile is invariant in the streamwise direction, so that H and H_1 are constant. The lag-entrainment method as developed by Green *et al* (1973) is more accurate for boundary layers departing strongly from equilibrium and C_E is now determined from an ordinary differential equation. The differential equation is derived from considerations of the streamwise variation of the maximum shear stress $C_{\tau \max}$, derived from the turbulent kinetic energy equation and related empirically to C_E : the equation has the form

$$\theta \frac{dC_E}{ds} = F \left\{ (C_{\tau_{eq}} - C_{\tau}), \left[\left(\frac{\theta}{U_{iw}} \frac{dU_{iw}}{ds} \right)_{EQ} - \frac{\theta}{U_{iw}} \frac{dU_{iw}}{ds} \right] \right\}, \quad (27)$$

where the suffix EQ indicates the boundary layer in equilibrium conditions. The lag-entrainment equation is solved together with the continuity and entrainment equations: it does not alter the structure of the system and can be placed on one side in discussions about inverse methods.

4.2 The H, H_1 relationship

The parameter H_1 can be defined by an empirical relationship or by direct reference to a family of velocity profiles. In the original lag-entrainment method H_1 was defined by

$$H_1 = 3.15 + 1.72/(H - 1) - 0.01(H - 1)^2. \quad (28)$$

However this fails to account for the behaviour of the boundary layer at separation. It seems advisable to use correlations derived from families of velocity profiles, but experimental results indicate that the profile families are dependent on the pressure gradient as well as the skin friction and the momentum thickness. A general profile has been given by Cross (1980) and a better correlation for flows with separation occurring as a result of a relatively mild adverse gradient is provided by the direct integration of the relevant member of Cross's family of velocity profiles (Cross 1980). Lock (1986) proposed a simple expression for H_1 given by

$$H_1 = 2 + 1.5 \left(\frac{1.12}{H - 1} \right)^{1.093} + 0.5 \left(\frac{H - 1}{1.12} \right)^{1.093}, \quad \text{for } H < 4,$$

$$H_1 = 4 + (H - 4)/3, \quad \text{for } 4 < H < 12. \quad (29)$$

The H, H_1 curve is illustrated in figure 9 and it contains a minimum at $H \simeq 2.7$ which corresponds to a value of H which is normally associated with the separation of the boundary layer. For compressible flow Green (1979) showed that H should be replaced by the transformed shape parameter $\bar{H}$ [see (25)].

For separation occurring as a result of very strong pressure gradients other members of the profile family are required and formulae similar to (28) are derived. It is unlikely

Figure 9. H, H_1 correlation.

that there is a unique relationship between H and H_1 but a more systematic approach may be produced if the velocity gradient is introduced as an extra parameter.

4.3 The skin friction coefficient, C_f

The skin friction coefficient is determined as a function of H and Re . Following Green *et al* (1973) the skin friction in a general incompressible flow is related to the skin friction on a flat plate by the empirical expression

$$[(C_f/C_{f0}) + 0.5][(H/H_0) - 0.4] = 0.9, \quad (30)$$

where the suffix zero indicates values in zero pressure gradient. The quantities C_{f0} and H_0 are given by the empirical correlations

$$C_{f0} = [0.01013/(\log 10Re - 1.02)] - 0.00075$$

and $1 - 1/H_0 = 6.55(\frac{1}{2}C_{f0})^{1/2}$.

The empirical formulae have little basis in separated flows and the development of the boundary layer is not strongly dependent on the value of the skin friction: it is adequate to set the skin friction to a small negative value, say -0.00001 , for separated flow.

4.4 Solution of the boundary-layer equations at separation

The structure of the direct form of the integral boundary layer equations at separation can be studied by examining the momentum integral, (7), and entrainment equation, (26); the lag-entrainment equation provides only a value of C_E and does not alter the structure. With the second-order terms and the suffixes omitted for clarity, these become

$$(d\theta/ds) = \frac{1}{2}C_f - (H + 2 - M^2)(\theta/U)(dU/ds) \quad (31)$$

and

$$H_1(d\theta/ds) + \theta(dH_1/dH)(dH/ds) = C_E - H_1(1 - M^2)(\theta/U)(dU/ds), \quad (32)$$

which are two simultaneous equations for $d\theta/ds$ and dH/ds for specified dU/ds . These equations have a unique solution if the determinant $(\theta dH_1/dH)$ of the system is not zero. The determinant of the system becomes zero at $H \simeq 2.7$ and the derivative dH/ds will be infinite unless the equations are compatible. The equations will only be compatible if

$$(H + 1)(\theta/U)(dU/ds) = \frac{1}{2}C_f - (C_E/H_1), \quad (33)$$

which requires the velocity gradient to take a special value. In a direct method the velocity gradient is unlikely to achieve this special value and the equations will be ill-conditioned at separation. A general scheme for solving these equations through separation can be derived by using the inverse formulation or resorting to the inviscid flow to determine the compatible value of the velocity gradient. In the inverse formulation a new dependent variable, such as the rate of growth of the displacement surface, is introduced and the velocity gradient is determined by the specification of this new dependent variable. In a quasi-simultaneous method the direct boundary layer equations are solved in conjunction with a linearised form of the continuity equation for the inviscid flow. In the next two sections these approaches are described

in more detail and then the problem of coupling these methods to an inviscid method is discussed.

4.5 Inverse methods

In the inverse formulation the boundary-layer equations are used to determine the velocity gradient from some property of the boundary layer: various schemes have used the rate of growth of the displacement surface, the skin friction or the shape parameter. If the boundary-layer solution is to be coupled to the inviscid flow then it is most convenient to use a non-dimensional form of the rate of growth of displacement surface, the source strength S , given by

$$S = W_{iw}/U_{iw} = \frac{1}{\rho_{iw} U_{iw}} \frac{d}{ds} (\rho_{iw} U_{iw} \delta^*). \quad (34)$$

For the development of an inverse method it is useful to express S in terms of dU/ds and θ . By expanding (34) the source strength can be written as

$$S = H(d\theta/ds) + \theta(dH/ds) + H(1 - M^2)(\theta/U)(dU/ds).$$

The expression for S is reduced to the required form by using (7) & (26) and the differential of (25b) (see Lock & Williams (1987) for details),

$$S = F_2(dU/ds) + F_1, \quad (35)$$

where

$$F_2 = [\theta(H + 1)/UH'_1][H_1 - HH'_1 + 0.2M^2(H_1 + 2H'_1)]$$

and

$$F_1 = [C_E(1 + 0.2M^2) - \frac{1}{2}C_f(H_1 - HH'_1 + 0.2M^2H_1)]/H'_1.$$

Equation (35) is of fundamental importance in the theory of coupling the inviscid and viscous solutions which is described in § 5.

The boundary-layer equations can now be expressed in inverse form in which S is specified and dU/ds 'unknown'. The three basic equations (7), (26) and (35) are written in matrix form

$$\begin{bmatrix} 1 & H + 2 - M^2 & 0 \\ H_1 & H_1(1 - M^2) & H_1 \\ H & H(1 - M^2) + 0.2M^2(H + 1) & 1 + 0.2M^2 \end{bmatrix} \begin{bmatrix} d\theta/ds \\ (\theta/U)(dU/ds) \\ \theta(d\bar{H}/ds) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}C_f \\ C_E \\ S \end{bmatrix}, \quad (36)$$

and these can be solved for $(\theta/U)dU/ds$, $d\bar{H}/ds$ and $d\theta/ds$ to give

$$(H + 1)D(\theta/U)(dU/ds) = H'_1 S + \frac{1}{2}C_f\{(1 + 0.2M^2)H_1 - HH'_1\} - C_E(1 + 0.2M^2), \quad (37)$$

$$D\theta(d\bar{H}/ds) = H_1 S - 0.2M^2 H_1 C_f - C_E(H - 0.4M^2) \quad (38)$$

and

$$(H + 1)D(d\theta/ds) = (H + 2 - M^2)\{-H'_1 S + C_E(1 + 0.2M^2)\} - \frac{1}{2}C_f\{H_1 - HH'_1 - 0.2M^2(4H_1 + 2H'_1 - 3HH'_1)\}, \quad (39)$$

where

$$D = H_1 - HH'_1 + 0.2M^2(H_1 + 2H'_1) = H_1(1 + 0.2M^2) - H'_1(H - 0.4M^2).$$

It can be shown that D is never zero so that the inverse equations are non-singular. Equation (37) is a re-arrangement of (35). In practice it is convenient not to use (39), but to determine $(\theta/U)dU/ds$ from (37) and use the original form of the momentum equations (6) or (7), to obtain $d\theta/ds$.

4.6 Quasi-simultaneous methods

Veldman (1980) drew attention to the strong simultaneous character of the coupling between inviscid and viscous flows close to separation and the importance of retaining these characteristics in the coupling scheme. The mechanics and properties of the coupling scheme are described in §5 and in this section the details of solving the boundary-layer and inviscid equations in a simultaneous manner are presented. In §3.6 a local linearisation of the inviscid flow was given in the direct form

$$dU^I/ds = E_1 + E_2 S^I. \quad (16)$$

An equation of the same nature also in direct form, (35), has been derived for the boundary layer in the previous section from the momentum and continuity equations. For the subsequent analysis it is more convenient to express this relationship in the inverse form

$$dU^V/ds = G_1 + G_2 S^V. \quad (40)$$

In the calculation of the development of the boundary layer, (16) and (40) are solved simultaneously for dU/ds and S at each new streamwise position to give

$$S = (E_1 - G_1)/(G_2 - E_2) \quad (41)$$

and

$$dU/ds = (E_1 G_2 - E_2 G_1)/(G_2 - E_2). \quad (42)$$

These values are then used to determine the derivatives of the momentum thickness and transformed shape parameter and the ordinary differential equations are integrated by, say, a Runge–Kutta scheme. In particular the momentum thickness θ is determined from the momentum equation, (31), and the transformed shape parameter $\bar{H}$ from the continuity/entrainment equation, (26). If the flow is attached and dH/dH_1 is less than zero then the entrainment equation can be used in the form given in (26); if the boundary layer passes through separation and dH/dH_1 takes the value zero then the form of the entrainment equation given in (38) can be used. Alternatively a simpler procedure can be defined by rewriting the definition of the source strength given in (34) as

$$d\delta^*/ds = S - (1 - M^2)(\delta^*/U)(dU/ds) \quad (43)$$

and this equation can be used to determine δ^* for both attached and separated flow. The shape parameter is then given by the algebraic formula $H = \delta^*/\theta$.

5. Coupling methods

The VII technique requires the solution of the nonlinear equations for the inviscid and viscous flows with the boundary conditions on the matching surface satisfied simultaneously by both flows. This simultaneous solution of two nonlinear boundary-value problems is normally accomplished by an iterative procedure. Thwaites (1960) describes the classical approach for the calculation of attached flow about an aerofoil. The velocity gradient derived from an inviscid flow without any representation of viscous effects is used to calculate the development of the boundary layer. The rate of growth of the boundary layer is used to provide new boundary conditions for calculating the first inviscid approximation to the viscous flow, as described in §3. This process is then repeated with the intention of providing better approximations to the overall flow.

For a flow with a weak interaction between the viscous and inviscid parts, this classical iterative procedure provides an adequate estimate of the viscous effects. However, in most practical flows, the interaction is not weak and the iterative scheme rarely converges without assistance: for flows involving separation the iterations are normally divergent. In §3 the extension of the VII technique to flows including separation has been outlined and the challenge is to provide an iterative scheme that provides converged solutions for flows with strong interaction.

Two classes of coupling scheme which are successful for separated flow have been developed. In the first class the boundary layer is calculated in an inverse manner and in this manner the problem with ill-conditioning of the equations at separation is overcome. The inverse solutions of the boundary-layer equations have been coupled with inverse solution of the inviscid flow by Calvert (1982) for internal flows whereas the coupling to a direct inviscid method has proved more useful for external flows (Carter 1979; Le Balleur 1983; Williams 1985). However these methods have some disadvantages: they do not converge rapidly and require the development of inverse methods at least for the viscous flow. In the second class of methods the direct form of the inviscid and viscous flows are solved simultaneously. For supersonic external flow Lees & Reeves (1964) simultaneously marched solutions for the inviscid and viscous flows, relying on the hyperbolic nature of the inviscid flow equations. More recently the approach has been extended to capture the elliptic behaviour of subsonic regions of transonic flow by Brune *et al* (1974), Gilmer & Bristow (1981), and Drela *et al* (1986). Veldman (1980) simplified this approach significantly by introducing a local linearisation of the inviscid flow and solving this simultaneously with the boundary-layer equations: the quasi-simultaneous approach.

In this section a range of techniques for coupling the flows is described and their convergence properties indicated by a unified form of presentation.

5.1 A functional approach to coupling

An alternative description of the coupling process has been given by Brune *et al* (1974), Meauze (1984), and Delery & Marvin (1986), and has been extended by Lock & Williams (1987). It is useful to derive equations in the inviscid and viscous regions (represented by superscripts I and V) for the variables arising in the boundary conditions on their common boundary. For the inviscid flow the surface velocity gradient dU^I/ds is determined by the normal velocity at the surface and the shape of the aerofoil. The normal velocity is related to the source strength defined in (5),

so in general for the inviscid flow,

$$dU^I/ds = E(S^I). \quad (44)$$

In the boundary layer, its rate of growth, which can be characterised by S , is dependent on the surface velocity gradient, thus

$$S^V = F(dU^V/ds), \quad (45)$$

or in its inverse form

$$dU^V/ds = G(S^V). \quad (46)$$

A matched solution requires that the variables S and dU/ds simultaneously satisfy these equations on the matching boundary, thus

$$S^I = S^V \text{ and } dU^I/ds = dU^V/ds. \quad (47)$$

If a single point on the matching surface is considered, then this idea is illustrated in figure 10. Equations (44) and (46) are represented schematically by the curves whilst the matched solution is given by the intersection of the curves: the matching procedure must determine this point.

The properties of different coupling procedures are studied by examining local linearisations of (44) and (46), which are for the inviscid flow

$$dU^I/ds = E_1 + E_2 S^I, \quad (16)$$

and for the viscous flow,

$$dU^V/ds = G_1 + G_2 S^V. \quad (40)$$

Figure 10. Functional view of relationship between viscous and inviscid flows.

Figure 11. G_2 as a function of H .

For the inviscid flow the value of the coefficients E_1 and E_2 are determined by the numerical approximation used to describe the particular boundary conditions to the inviscid flow.

If the boundary-layer equations are represented by an integral method then an analytic form is given to (40) and the precise form of one curve in figure 10 can be defined. From (35) the coefficients G_1 and G_2 are given for incompressible flow by

$$G_1 = [U/(H+1)D\theta] [\frac{1}{2}C_f(H_1 - H'_1 H) - C_E] \quad (40a)$$

and

$$G_2 = H'_1 U/(H+1)D\theta. \quad (40b)$$

The correlation between shape parameters (H, H_1) given in §4.2 contains a minimum at approximately $H = 2.7$ and at this value of H the boundary layer is close to separation. As indicated in figure 11, G_2 changes sign at this minimum, being negative for attached flows and positive for separated flows.

The local form of the linearised viscous relationship is indicated in the three diagrams at the top of figure 12. For attached flow (figure 12a) the slope of the viscous curve is large and negative; for separating flow (figure 12b) the slope is zero; whilst for separated flow (figure 12c) the slope is positive and small. These diagrams summarise some of the known characteristics of boundary layers developing in adverse gradients as given by East *et al* (1977). For attached flow, with G_2 large and negative, quite large adverse velocity gradients result in small rates of growth of the boundary layer and solutions should be determined from the direct equation (35). On approaching separation the coefficient G_2 is close to zero and minute changes in the velocity gradient result in very large changes in the rate of growth of the boundary layer. In other words, solution of the direct equation proves to be ill-conditioned, which is analogous to the 'Goldstein singularity' for laminar flow, and the inverse form, (40), must be used. For separated flows the velocity gradient is determined by the viscous rather than the inviscid flow.

5.2 Iterative procedures

5.2a Direct methods: The local shape of the viscous and inviscid curves determines the convergence properties of a particular method. A flow diagram for the classical

Figure 12. Direct method.

direct method is presented at the bottom of figure 12 and the iterative procedure is represented in diagrammatic form for a single point on the matching surface at the top of figure 12 for attached, separating and separated flow. Considering the case of attached flow (figure 12a), an initial guess for the non-dimensional source strength $S^{(0)}$, which in general need not be zero, leads to a first approximation to the velocity gradient dU^I/ds , from a solution of the direct form of the inviscid equations as indicated by point (1). This estimate of the velocity gradient is used to calculate the development of the boundary layer which gives a new estimate for the rate of growth of the boundary layer, $S^{(1)}$ at the point (2). The new estimate $S^{(1)}$ is used to recalculate the inviscid flow and the process is repeated. In general the scheme can be written as

$$S^{(n+1)} = F(E(S^{(n)})) \text{ or } S^{(n+1)} = K(S^{(n)}), \quad (48)$$

where K is a functional representation of an inviscid calculation followed by a viscous calculation and the superscript denotes the level of the iteration. As the diagram suggests, the iteration path cycles around the matching point slowly converging onto it.

From a simple geometric argument, if $G_2 < 0$ then the fixed-point operator $S = K(S)$ will converge without assistance if $|E_2/G_2| < 1$ and with assistance of under-relaxation if $|E_2/G_2| > 1$. If $G_2 \geq 0$ and $E_2/G_2 > 1$ then the scheme will diverge as indicated in the diagrams for separating and separated flows in figures 12b and 12c.

For attached flow (figure 12a), under-relaxation can be used to obtain convergence in situations for which $G_2 < 0$ but $|E_2/G_2| > 1$ (or to speed up convergence when the stronger inequality $|E_2/G_2| < 1$ is maintained). The under-relaxation formula is given by

$$S^{(n+1)} = S^{(n)} + \omega[K(S^{(n)}) - S^{(n)}], \quad (49)$$

as illustrated in figure 13 and by geometric argument convergence is obtained if $\omega < |G_2/E_2|$ and the optimum value of ω is $|G_2|/(E_2 + |G_2|)$, which for linear problems gives convergence in one iteration.

Figure 13. Under-relaxation for a direct method.

5.2b *Fully-inverse method*: Well-conditioned solutions for separated flows are obtained by solving the boundary-layer equations in inverse form. The inverse solution is most readily introduced into a coupling scheme by adopting a 'fully-inverse method' (FI). As illustrated in figure 14a, since both the inviscid and viscous flow are solved in inverse mode, the flow diagram is now traversed in the opposite direction. An initial estimate of the velocity gradient is used by the inverse inviscid method to determine a value for the source strength, which in turn is used to calculate the velocity

Figure 14. Fully-inverse method.

gradients induced by the boundary-layer equations and the process is repeated. As illustrated in figure 14a the method diverges for attached flow (if $|E_2/G_2| < 1$), but converges for separating and separated flows (figures 14b & c). Again by a geometric argument the scheme will converge if $|E_2/G_2| > 1$ and will converge with the assistance of under-relaxation if $|E_2/G_2| < 1$. These inequalities are complementary to the inequalities given for the direct method.

The fully-inverse scheme has been used successfully for the calculation of internal flows, such as a transonic compressor cascade as described by Calvert (1982). However it has not found favour in external flows for at least two reasons: it is more difficult to define an inverse form of the inviscid method and, as described by Melnik & Chow (1976), the scheme requires increasing under-relaxation as the extent of the external domain is increased.

5.2c Semi-inverse method: The inverse solution of the boundary-layer equations can be matched to a direct solution of the inviscid equations by the 'semi-inverse method' (si). The outline of the method is illustrated in figure 15d; the basic feature is that two estimates of the velocity gradient are given by solutions of the direct inviscid and inverse viscous methods. The difference in the velocity gradients is used in a correction formula to improve the inviscid and viscous calculations. In detail an initial estimate of the source strength, say $S^{(0)}$, is used to derive estimates of the velocity gradient in the inviscid and viscous flows at points (1) and (2) respectively. The correction formula is derived by simultaneously solving the linear approximations to the flows given by (16) and (40). After some algebra the new estimate for the source strength $S^{(n+1)}$ is given by

$$S^{(n+1)} - S^{(n)} = [1/(G_2 - E_2)] \{ (dU^I/ds) - (dU^V/ds) \}. \quad (50)$$

Figure 15. Semi-inverse method.

This is the same equation as given by Le Balleur (1981) and he fixed the value of E_2 by a Fourier stability analysis: G_2 is already fixed by (40b). The correction formula then takes the explicit form

$$S^{(\infty)} - S^{(n)} = \frac{U\beta}{U\pi/\Delta x - \beta G_2} \left\{ \frac{1}{U^V} \frac{dU^V}{ds} - \frac{1}{U^I} \frac{dU^I}{ds} \right\}, \quad (51)$$

where $\beta = (1 - M^2)^{1/2}$ and Δx is a typical length scale for the grid. The relaxation factor $(G_2 - E_2)^{-1}$ depends upon the state of the boundary layer and the discretisation in the inviscid flow field. The relaxation factor varies throughout the field of calculation but is calculated rather than estimated.

5.2d Fully-simultaneous method: Veldman (1980) drew attention to the strong simultaneous character of the coupling between the inviscid and viscous flows close to separation and the importance of retaining these characteristics in the coupling scheme. Thus in a fully-simultaneous method the matched solution is obtained by a simultaneous solution of the inviscid and viscous flows and this is usually calculated by a Newton method. Brune *et al* (1974) defined a fully simultaneous scheme based on the observation that small perturbations to linear equations (for the inviscid and viscous flows) are governed by linear equations. Their scheme has very good convergence properties but can be computationally expensive. More recently the fully simultaneous method has been applied to a wider variety of flows by Gilmer & Bristow (1981), and Drela *et al* (1986). Both approaches use an integral method for the boundary-layer equation, whilst for the inviscid flow the former uses a subsonic surface-singularity method and the latter an Euler method.

5.2e Quasi-simultaneous method: The simultaneous solution of the viscous and inviscid flows can be very involved and a simpler, quasi-simultaneous scheme is obtained by simultaneous solution of a simplified version of the equation for either flow with the complete equations for the other flow. The simplification must significantly reduce the computational complexity but also must accurately represent the local behaviour of the approximated flow. A simplification of the equations for the viscous flow has been chosen by a number of authors, including Ghose & Kline (1976), Moses *et al* (1978), Wai & Yoshihara (1980), and Cross (1986). The viscous flow is described by an integral method leading to equations of the form of (35) and (40) which are applied as boundary conditions for the inviscid flow.

The alternative approach has been strongly advocated by Veldman (1980), it has been used by Houwink & Veldman (1984), Cebeci *et al* (1986), Veldman & Lindhout (1986), and King & Williams (1988). Now the inviscid flow is described by a simplified equation, such as (16), which is solved simultaneously with the equations for the viscous flow as described in §4.6.

The second scheme is illustrated in figure 16 with an initial estimate of the source strength providing the linearisation of the inviscid flow which is represented by the tangent to the inviscid curve. The intersection of the tangent to the inviscid curve with the viscous curve provides the quasi-simultaneous solution of the viscous equations and the next estimate of the source strength and the process is repeated. As indicated by the diagrams the method converges for attached, separating and separated flows.

Figure 16. Quasi-simultaneous method.

6. Methods for two-dimensional flow

The assessment of the VII methods starts by considering two-dimensional incompressible flow about a single aerofoil where significant viscous effects are found when the flow separates. Then, a method for the compressible flow about multiple element aerofoils is described. There are significant viscous effects about the multiple element aerofoil even in attached flow but it is demonstrated that the theoretical method is able to predict the maximum lift. Finally methods for two-dimensional transonic flow are considered: the presence of the shock wave produces significant viscous corrections even for attached flow. Some consideration is given to the effect of including second order terms in the momentum equation for the boundary layer.

6.1 A method for two-dimensional incompressible aerofoils

The stall on the NACA 4412 aerofoil at a Reynolds number of the order of one million arises from a trailing-edge separation of the boundary layer which moves towards the leading edge slowly with increasing incidence. As a result the NACA 4412 aerofoil is an ideal section for detailed measurements of a trailing-edge separation and measurements have been made by Wadcock (1987) and Hastings & Williams (1987). The ability of the VII method to predict separated flow is demonstrated by comparison of the semi-inverse method developed by Williams (1985) with the experimental results of Hastings & Williams (1987).

In the experiment of Hastings a two-dimensional model with the NACA 4412 section was tested in the RAE 3-962 m $\times$ 2.743 m (13 ft $\times$ 9 ft) atmospheric low-speed wind tunnel. The chord of the model was 1.0 m and it was mounted horizontally spanning the

tunnel. The Reynolds and Mach numbers of the test were 4.17 million and 0.18 respectively. The occurrence of laminar separation bubbles was avoided by fixing the transition from laminar to turbulent boundary layer by placing single wires at $x/c = 0.113$ on the upper and lower surfaces respectively and adding distributed roughness ahead of the transition wires. Boundary-layer fences were placed along the chord at 0.85 m on either side of the mid-span to maintain reasonably steady and uniform separated flow over the mid-portion of the wing and these were effective up to an angle of incidence of 13° . The main set of data was taken at an incidence of 12.15° for which the aerofoil had almost attained its maximum lift coefficient and the separation was at about $x/c = 0.8$. The lift coefficient was calculated by integrating the static pressure distribution around the mid-span of the wing. The calculations were performed in free-air conditions so the experimental results have been corrected for the effect of tunnel constraint as described in Hastings & Williams (1987). At the experimental angle of incidence of 12.15° the corrected lift coefficient and incidence are 1.44 and 12.49° respectively. A laser Doppler anemometer was used to obtain mean velocity profiles at eight stations between $0.59 < x/c < 0.99$, whilst a pitot traverse probe was used at $x/c = 0.2$ and 0.4 .

The theoretical method of Williams (1985) is a semi-inverse coupling of a surface-singularity method for the inviscid flow (Newling 1977) with the lag-entrainment method for the turbulent boundary layer (see §4). The laminar portion of the boundary layer ahead of transition is calculated by a compressible version of the method due to Thwaites (1960). The method allows for three different types of transition to a turbulent boundary layer. Natural transition is predicted by Granville's (1953) correlation. If the laminar boundary layer separates before natural transition has been predicted then the development of the laminar separation bubble is calculated by Horton's (1967) semi-empirical method. Transition can also be fixed at a specified point as long as neither of the other criteria has been satisfied upstream.

In the theoretical calculation the transition from laminar to turbulent boundary layer is fixed at $x/c = 0.014$ and 0.110 on the upper and lower surfaces respectively, which corresponds to the downstream ends of the transition trips used in the experiment. The theoretical method does not implicitly contain an estimate of the change in the state of the boundary layer as it passes over the trip, but this can be simulated approximately by increasing the momentum thickness at the trip on the upper surface. At an angle of 12.49° , which corresponds nearly to the maximum lift for the aerofoil in the experiment, an increase in the momentum thickness of the boundary layer of 0.00018 m at $x/c = 0.014$ on the upper surface leads to reasonable agreement with the measured momentum thickness at $x/c = 0.2$ and 0.4 as indicated in figure 17. On the lower surface there is no increase in momentum thickness at the trip in the calculation but the calculated value of 0.00076 m for the momentum thickness compares well with the measured value of 0.00064 m. The calculation of the turbulent boundary layer now includes the effect of longitudinal curvature on turbulence structure. However, the second order effects of Reynolds normal stress for the boundary layer and the pressure difference across the wake in the EIF (given by (14)) have not been included. Calculations with the same increase in momentum thickness at the upper surface trip are repeated for angles of incidence between 0° and 14° ; the lift coefficients are compared with the corrected experimental and inviscid values in figure 18. Up to an incidence of 12.49° , there is good agreement with the experimental values. However, the theoretical results extend beyond the range of the experimental values and indicate a decrease in the lift coefficient as the separation

Figure 17. Upper surface momentum thickness for NACA 4412, $\alpha = 12.49^\circ$, $Re = 4.17 \times 10^6$, $\Delta\theta = 0.00018$.

point moves further towards the leading edge. The calculated and measured pressure distributions at an angle of incidence of 12.49° are compared in figure 19. The two pressure distributions are remarkably similar with the theoretical methods developing slightly less suction over the rear of the aerofoil. If we use the experimentally supported criterion that a shape parameter of 4 identifies the point of separation, then from figure 20 separation in the experiment occurs at $x/c = 0.8$ whilst in the calculation it

Figure 18. Variation of C_L with α for NACA 4412.

Figure 20. Development of H on upper surface for NACA 4412.

Figure 19. Aerofoil pressure distribution of NACA 4412 at $\alpha = 12.49^\circ$.

Figure 21. Development of displacement thickness on upper surface of NACA 4412.

occurs at $x/c = 0.82$. In figure 20 there is good agreement between the predicted and measured values of shape parameter up to separation. However after separation the measured values of H rise to a value of 8.7 at the trailing edge whilst the predicted values rise very rapidly to a value of 14.0 at the trailing edge. There are several possible reasons for the incorrect prediction of the development of the momentum thickness and shape parameter for separated flow: the lack of representation of Reynolds normal stresses which are significant in separating and separated boundary layers; and an incorrect correlation between H and H_1 for high values of H . However, more experimental data will have to be collected before an empirical formula for Reynolds normal stress or an improvement of the (H, H_1) relationship can be established and either hypothesis checked. The good prediction of the pressure distribution shown in figure 19 indicates that the displacement effect of the separated boundary layer is well predicted and this is demonstrated by the comparison of the predicted and measured displacement thicknesses in figure 21.

6.2 A method for multiple element aerofoils

Although the flow about high lift aerofoils is usually at low freestream Mach numbers (typically 0.2), the high lift developed by the aerofoil leads to a small region of supersonic flow at the leading edge. Reliable calculations of the flow development can only be obtained if effects of compressibility are included. Cebeci *et al* (1986) describe the semi-inverse and quasi-simultaneous coupling of a finite element method for full potential inviscid flow, FELMA developed by BAe, with the viscous method outlined in §6.1. Results are presented for the NLR 7301 wing-flap configuration (Van den Berg 1979), for which a two-dimensional model was tested at a Mach number of 0.185 and a Reynolds number of 2.51 million. There were measurements of the pressure distribution on the main aerofoil and flap and the development of the viscous layers was measured at two gap settings for the flap of 2.6 and 1.3% of the aerofoil chord. For the smaller gap there is an interaction between the wake from the wing and the boundary layer on the upper surface of the flap.

Figure 22. Variation of lift and drag for NLR aerofoil: large flap gap, NLR configuration 1: gap = 2.6°; Re = 2,510,000; $M = 0.185$. (Δ experiment; + FELMA viscous – SI; $\square$ FELMA viscous – QS; ∇ FELMA inviscid.)

In figures 22 and 23 the experimental curves for lift and drag coefficient are compared with the predicted inviscid and viscous results. For the viscous solution results from the SI and QS couplings are coincident, but there is some discrepancy over the precise level of C_L between theory and experiment. This discrepancy is larger for the smaller flap gap which indicates that the origin lies in viscous modelling. The theory is able to predict the stall of the aerofoil at an incidence that is approximately one degree too low. The accuracy of the drag prediction is encouraging but it must be noted that this is an idealised test case with the aerofoil designed to avoid separation on the lower surfaces.

Figure 23. Variation of lift and drag for NLR aerofoil: small flap gap, NLR configuration 2: gap = 1.3°; Re = 2,510,000; $M = 0.185$. (Δ experiment; + FELMA viscous – SI; $\square$ FELMA viscous – QS; ∇ FELMA inviscid.)

Figure 24. Variation of wing and flap lift coefficients for NLR aerofoil: large flap gap, NLR configuration 1: gap = 2.6%; $Re = 2,510,000$; $M = 0.185$. (Δ experiment; + FELMA viscous - SI; $\square$ FELMA viscous - QS; ∇ FELMA inviscid.)

In figure 24 the variation of lift coefficient with incidence for the wing and flap are presented independently for the large gap configuration. Since the major portion of the lift is generated on the wing, the trend in the variation of lift coefficient with incidence for the wing follows closely the trend for the complete aerofoil given in figure 22. The lift coefficient of the flap, on the other hand, reduces with incidence until the stall, at which point it begins to rise. Figures 25a, b show how the boundary layer develops on the wing upper surface at the trailing edge with the variations of shape parameter H in figure 25a and momentum thickness θ in figure 25b. For this case the agreement with experiment is very good except close to the stall where the theory predicts higher values of H and a stall at a slightly lower angle of incidence. In figure 26 the viscous and inviscid pressure distributions at an angle of incidence

Figure 25. Variation of shape parameter H (a) and momentum thickness (b) at wing trailing-edge with incidence. NLR configuration 1: gap = 2.6%; $Re = 2,510,000$; $M = 0.185$. (Δ experiment; + FELMA viscous - SI; $\square$ FELMA viscous - QS; ∇ FELMA inviscid.)

Figure 26. Pressure distribution for NLR aerofoil: large flap gap, NLR configuration 1; $\alpha = 13.1^\circ$; $Re = 2,510,000$; $M = 0.185$. (Δ experiment; — FELMA viscous — SI; — FELMA viscous QS; — FELMA inviscid.)

13.1° (which is just prior to the stall) are compared with experimental values: the pressure coefficients from the SI and QS procedures cannot be distinguished. Generally the agreement between the experiment and viscous theory is very good with the marked change in the distribution of pressure over the flap being well predicted.

The issue of computational efficiency is addressed by comparing the rates of convergence for the SI and QS schemes with and without multi-grid on the finest grid employed within FELMA: multi-grid is a technique used in the inviscid method to accelerate convergence. The case chosen for this exercise is the NLR configuration with a large gap at an incidence of 11.6° , which is a relatively simple case with fully attached flow over the two components. The conclusions drawn here on the usefulness of the strong coupling procedure are also applicable to more difficult cases with separated flow. The convergence histories for circulation around the wing and flap are given in figures 27a & b. For each case a total of 500 viscous calculations are performed. When multi-grid is used only three grid levels are employed since the

Figure 27. (a) Comparison of convergence properties of SI and QS schemes including multi-grid acceleration. NLR configuration 1 – multi-grid: $\alpha = 11.6^\circ$; $Re = 2,510,000$; $M = 0.185$.

Figure 27. (b) Comparison of convergence properties of SI and QS scheme without multi-grid acceleration. NLR configuration 1 – no multi-grid: $\alpha = 11.6^\circ$; $Re = 2,510,000$; $M = 0.185$.

overall grid dimensions would not permit further global subdivision. For the multi-grid case three inviscid cycles are performed per viscous calculation, whilst for the non-multi-grid cases the ratio is five, which ensures approximately the same inviscid work-load per viscous calculation. The run times indicated are in CRAY XMP CPU seconds. The difference between the total run times in figures 27a & b is due to the overheads associated with multi-grid i.e. residual calculation and interpolation, which have not been fully optimised on the CRAY. The figures also confirm that the difference in prediction of circulation that exists between the two coupling methods is small. A closer examination of these differences points to minor difficulties for the si coupling in matching the velocities precisely in the near wake regions. However, it is clearly evident from figure 27a that with multi-grid the qs procedure converges significantly faster than si. Full convergence for qs has been obtained within 50 s while it takes perhaps 200 s for si. Without employing multi-grid, there appears to be little to choose between the two viscous coupling methods. Therefore rapid overall convergence is only obtained with both a fast viscous coupling procedure which provides a good estimate of the changes in the boundary conditions in the EIF and a fast inviscid solution which propagates these modifications in boundary condition throughout the field.

6.3 *Methods for two-dimensional compressible flow*

The selection of experimental results to assess the accuracy of theoretical methods in the transonic speed range is particularly difficult because the problems associated with the wind-tunnel technique are most pronounced in this speed range. In particular, tests are usually conducted in slotted wall tunnels which certainly reduce the overall interference but make it extremely difficult to estimate the remaining effects accurately. Also the spurious effects produced by the roughness bands which are used to promote transition are marked if the band is placed in a region which is nearly sonic.

These problems have been alleviated during the tests on the RAE 5225 aerofoil (Ashill *et al* 1987) in a $2.44 \times 2.44 \text{ m}^2$ ($8 \times 8 \text{ ft}^2$) pressurised wind tunnel at RAE, Bedford. The aerofoil is of 'advanced' design with considerable rear camber and a model of 0.635 m was positioned across the working section of the tunnel and tested at Reynolds numbers of 6 and 20 million. The reduction in the adverse effects produced by the wind-tunnel technique is achieved by two features. The transition was fixed (at 5% chord) by air injection through small holes normal to the surface. This technique produces minimal disturbance to the boundary layer and thus the overall forces. The pressures were measured on the solid tunnel walls which enables the calculation of the interference effects (Ashill & Weeks 1982) giving the blockage, incidence correction and induced curvature.

The theoretical method is a coupling of the lag-entrainment method (Green *et al* 1973) to the full potential method of Garabedian & Korn (1971) with the partially conservative correction introduced by Lock (1980). In the 'standard' method (Collyer & Lock 1979), called VGK, there is a direct coupling between the inviscid method and the first-order boundary-layer equations. In the latest method (Ashill *et al* 1987), called BVGK, the coupling has been changed to semi-inverse and the inverse boundary layer method now can include the second-order terms.

The comparison between the experiment and theory is given for two cases: flow A with $M_\infty = 0.598$, $\alpha = 1.38^\circ$ and a Reynolds number of 20 million and flow B with $M_\infty = 0.735$, $\alpha = 1.16^\circ$ and a Reynolds number of 6 million, taken from Ashill *et al* (1987). The value of the freestream Mach number, M_∞ , and the incidence have been

Figure 28. Pressure distributions, comparison between theory and experiment. Flow A, RAE5225, $M_\infty = 0.598$, $C_L = 0.433$, $Re = 20 \times 10^6$.

corrected for the effect of tunnel-wall interference. However the variation of wall-induced upwash along the chord of the model is sufficient for a camber correction to be made to the aerofoil geometry which is subsequently used in the calculations. The corrected value of the lift coefficient is specified in the calculations and the incidence and pressure distributions are predicted.

For A the flow is wholly subcritical and is attached on both surfaces. The pressure distribution predicted by both the standard method and BVGK are shown to compare well with experimental values in figure 28. However the values of the incidence and drag predicted by BVGK at the measured lift are in closer agreement with the measured values than those by the standard method. An analysis of the effect on overall forces and pitching moment of successively adding the various second-order corrections to the standard method is presented in the histograms of figure 29. The figure only gives an indication of the relative importance of the various terms since for a non-linear system the effects are not additive. The second-order effects are identified by the following labels.

IEIF – The pressure in the EIF is used to calculate the development of the boundary layer (see § 3.2).

Figure 29. Effect of various correction terms on lift, drag and pitching moment. Flow A, RAE5225, $M_x = 0.598$, $C_L = 0.433$, $Re = 20 \times 10^6$.

INPG and INST – Allowance is made in the streamwise momentum equation (8) for the effects of normal pressure gradient (term ③ in (8)) and Reynolds normal stress (term ④ in (8)).

IHH1 – The revised form of the shape parameter relationship, (29), is used.

ILRN – Corrections are made to allow for low Reynolds number effects (see Ashill *et al* 1987 for details).

ICURVB – Curvature effects are included in the lag equation (27) (see Green *et al* 1973 for details).

In each case the changes are small but significant. For the second flow B the comparison between the predicted pressure distributions and the experimental values is given in figure 30. There is a region of supercritical flow on the upper surface but flow is essentially shock free. However there is a severe adverse pressure gradient towards the trailing edge on both surfaces and BVGK predicts separation just upstream of the trailing edge on the upper surface. Overall the prediction of the pressure distribution by BVGK is in better agreement with experiment than the standard method for this case. It should also be noted that the values of the incidence and drag predicted with BVGK are in much closer agreement with experiment than the values of the standard method. The relative importance of the various modifications to the standard method are illustrated in figure 31. Compared with flow A the corrections are in general larger with IEIF and ICURVB having the greatest effect on lift whilst INST and ILRN have the largest effect on drag.

From this brief summary of the results given in Ashill *et al* (1987) it can be seen that the inclusion of the higher-order effects is essential if reliable predictions of the forces (especially drag) are to be produced.

Figure 30. Pressure distributions, comparison between theory and experiment. Flow B, RAE5225, $M_\infty = 0.735$, $C_L = 0.403$, $Re = 6 \times 10^6$.

Figure 31. Effect of various correction terms on lift, drag and pitching moment. Flow B, RAE5225, $M_\infty = 0.735$, $C_L = 0.403$, $Re = 6 \times 10^6$.

7. Conclusions

The general principles of interactive methods for calculating viscous flows are now well established:

- (1) Generalisations of the concept of the displacement effect of the viscous layers, as modelled through a 'transpiration velocity through the wall in the equivalent inviscid flow, have been derived which enable VII methods to be applied in many cases for which the usual assumptions of first order boundary-layer theory are certainly violated.
- (2) Similar generalisation of the momentum integral equations allow boundary-layer methods of 'integral' type to incorporate several 'higher-order' features, in particular the effects of normal pressure gradients induced by the curvature of the flow and of Reynolds normal stresses, which would otherwise require a full solution of the Navier-Stokes (NS) equations to simulate; and this work also suggests that similar improvements are needed to methods of 'differential' type to bring them up to an even higher standard.
- (3) The importance of properly modelling the near wake region (even though the primary objective remains the prediction of the pressures and forces on the wing itself) is now fully appreciated; and our understanding of how to do this in the framework of a VII method – involving jumps in the components of velocity both normal and tangential to the wake surface in the EIF, to allow respectively for the wake thickness and curvature effects – has been greatly improved.
- (4) A similar improvement in our understanding of the mechanism which governs the coupling between the inviscid and viscous components of a VII method has been achieved. In particular, this work has led (at least in two dimensions) to
 - (a) improvements in the choice of relaxation factors in 'direct' iterative procedures.
 - (b) rationalisation of the 'semi-inverse' method and extensions to its range of validity in calculating separated flows.
 - (c) suggestions for the development of new 'quasi-simultaneous' schemes, which show signs of proving the most efficient and robust of all the techniques for treating 'difficult' interactive problems.

As regards the development and assessment of practical prediction methods, it is fair to say:

- (1) In two dimensions – for single aerofoils – a number of existing methods give impressive results even up to high lift coefficients and stalling conditions.
- (2) For multiple-element aerofoils at low speeds, good results have also been obtained for lift but drag predictions are still not satisfactory. The chief obstacle to producing adequate drag predictions remains the modelling of the interaction between the wakes from one element and the boundary layer developing on the next; and for this problem local solutions of the NS equations may be needed.
- (3) Up to the present, no attempt has been made to model the flow in junction regions or near the wing tips. Some progress in this direction is surely possible with interactive methods, even though eventually no doubt full NS solutions will be needed, at least locally.
- (4) Similarly, it is doubtful how far VII techniques can be advanced in the solution of flow problems involving massive three-dimensional separations, particularly those involving the formation of discrete vortex sheets. There is some possibility that these vortices could be at least partially simulated through the inviscid Euler solution, and

their interaction with the boundary layer taken care of by a VII scheme; but it remains to be seen how effective this might be.

This brings us finally to the main question mark regarding the future of VII methods, as opposed to full numerical solutions of the NS equations for the complete configuration. At present, most NS methods are severely limited by (a) excessively long computing times as the grid is refined close to the surface and (b) inadequacies in turbulence modelling. Although recent advances in numerical techniques for solving the NS equations have been most impressive (see for example Flores *et al* 1986), it does seem likely that similar techniques could also be used to speed up the treatment of the inviscid (Euler) component of a VII method; so that, provided it is also possible to improve the efficiency of the coupling technique and extend the range of boundary-layer solutions in three dimensions, the advantages in terms of computing time that the best VII methods certainly possess now should still be retained. Whether this will continue to outweigh their undoubted limitations as regards generality of application, is much more doubtful.

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