One-dimensional propagation of discontinuities in non-homogeneous linear viscoelastic semi-infinite media

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Abstract. One-dimensional wave-propagation is analysed in a non-homogeneous, isotropic, linearly viscoelastic semi-infinite medium, by the theory of singular surfaces. The characteristics of the medium at any point are assumed to be dependent upon the position of the point. The solution for the stress field, which is valid even after the wave-front has passed, is obtained in the form of Taylor's series by prescribing the time dependent stress boundary condition in the form of a Maclaurin's series.

Furthermore, it is shown that the higher order discontinuities satisfy the same propagation conditions as the stress waves and all the discontinuities decay as they traverse the material.

An application of inhomogeneity varying exponentially with position has been analysed. Diagrams for the above example with arbitrarily chosen parameters are presented.

Keywords. One-dimensional propagation; discontinuities; singular surfaces; inhomogeneity; stress boundary conditions; Maclaurin's series; Taylor's series.

1. Introduction

The knowledge of the distribution of density and elasticity in the earth, plays a central role in our understanding the chemical composition, phase changes and evolution of the earth. In a similar way, knowledge of the distribution of anelasticity provides additional information regarding the internal structure of the earth. Unfortunately, our knowledge of anelasticity is much poorer than our knowledge of elasticity and density. This is caused by the difficulty of isolating and accurately measuring attenuation effects in seismic records.

Studies of wave-propagation in the earth stratum under loads have usually assumed that the earth behaves to a first approximation as an ideal elastic or viscoelastic material. The stratum may be of finite depth or it may be so deep compared to the size of the loaded area that it can be regarded as a half-space. In either case the complete solutions to elastic or viscoelastic problems are known when material parameters are treated independent of position.

The formation of earth strata in nature tends, however, to result in depth variations of these parameters and this may be due principally, either to stratification of different materials or to the effect of superincumbent pressure. The case of homogeneous viscoelastic half-space has been treated extensively in the literature. By comparison little has been done on the non-homogeneous viscoelastic half-space when the material characteristics vary continuously with depth. This fact is likely to be true when the effects of over-burden pressure predominate. Furthermore these studies have been restricted to particular depth variations which have allowed the resulting simplification to be exploited.

We are not aware of solutions having been obtained for non-trivial problems of this type, when the non-homogeneity is of a general form. In this paper we obtain expressions for the higher order fields due to a surface loading of an isotropic half space, the density and creep function of which are arbitrary functions of depth, by the method of propagating discontinuities.

Discontinuities of most kinds which happen to occur in the mathematical analysis of physical quantities are, in fact, the idealisations of quantities which vary very rapidly within infinitesimally small intervals of time and space. Therefore a discontinuity is the mathematical representation of a physical change across an infinitesimally thin layer(film) for an infinitesimal length of time. The honour of the formulation of the theory of discontinuities, in the nineteenth century, goes to Hugoniot, who conceived of a wave as a disturbance limited rigorously to a surface, although its magnitude may be arbitrary. The theory was put on a sound footing by Hadamard (1903), during the later part of the nineteenth century and the early part of this century.

Earlier, wave-propagation problems in many branches of physics and applied mathematics have been treated by the method of geometrical optics. Luneberg (1944, 1948) demonstrated that the leading term in the asymptotic solution of an electromagnetic wave equation is equivalent to its optical solution, while Friedlander (1946), Kaline (1951), Keller *et al* (1956), Keller (1958), showed that the subsequent terms in the asymptotic expansion of the solution give corrections to the optical wave-theory. Friedlander (1946) generalised the method of analysing the propagation of discontinuities in a non-homogeneous elastic medium, which is more exact than optical theory and avoids lingering approximations.

The theory has been formulated and developed upto the present stage by Erickson (1955), Truesdell & Toupin (1960, p. 504), Thomas (1961), Coleman *et al* (1965a), Coleman & Gurtin (1965b) and Chen & Wicke (1971). The theory has been successfully and satisfactorily applied to wave-propagation in homogeneous, non-homogeneous and composite elastic media by Karal & Keller (1959), Keller (1964), Lindholm & Doshi (1965), Payton (1966), Henry & Cooper (1967), Chen (1976), Dhaliwal & Singh (1980), McCarthy & Teirston (1982). Recently, Achenbach & Reddy (1967), Sun (1971), Chen (1976), Robotonova (1977) also covered the homogeneous viscoelastic materials.

In our case we apply this method to one-dimensional wave propagation in the earth, treating it as a continuous, isotropic, non-homogeneous and linearly viscoelastic half-space material without sliding. Relations in general form for discontinuities of stress and its derivatives along with displacement derivatives across the wave-front, as functions of time with spatial co-ordinates as parameters, have been derived under the time-dependent boundary loading condition (Maclaurin's series). Solutions for the field variables at an arbitrary location are given, in the form of Taylor's expansion about the time of arrival of the singular surface, which are valid in the neighbourhood of the

singular surface. Also the existence and the strength of the higher order waves have been discussed.

To be more clear about the general results, one specific application of density and creep function with exponential variation, under time dependent stress boundary conditions, has been analysed. The results for the stress field, velocity, and time of arrival of the wave-front, along with higher order waves, are formulated and discussed.

Graphs for σ_H/σ_0 and σ/σ_0 , the non-dimensional stresses for the homogeneous and non-homogeneous cases respectively, have been plotted for three wave-front locations for three different values of non-homogeneity parameter.

2. Formulation of the problem

The earth is treated as a half-space $0 \leq z < \infty$, $-\infty < x, y < \infty$, which is non-homogeneous, isotropic and linearly viscoelastic material, assumed to be continuous without sliding and initially dead (undisturbed and unstressed). Coordinate axes are so chosen that the z -coordinate measures the depth into the ground, with $z = 0$, the ground surface. The theory assumes that with respect to these axes all components of stress, strain and displacement are zero, except the normal components $\sigma(z, t)$, $e(z, t)$, $w(z, t)$, parallel to the z -axis. It is also assumed that subsequent to time $t = 0$, a load $p(t)$, a function of time t is applied at the boundary $z = 0$, in the direction of the z -axis. The load $p(t)$ is assumed to be integrable and expandable in terms of Maclaurin's series.

In view of the above assumptions the following stress boundary value problem may be formulated:

$$\sigma(z, t) = e(z, t) = w(z, t) = \sigma^{(n)}(z, t) = e^{(n)}(z, t) = w^{(n)}(z, t) = 0, \quad \text{for } 0 \leq z < \infty, t < 0, n = 1, 2, \dots \quad (1)$$

$$\sigma(z, t) = p(t) = \sum_{n=0}^{\infty} \left(\frac{A_n}{n!} \right) t^n, \quad z = 0, 0 < t < \infty, \quad (2)$$

$$\sigma(z, t) < N, \quad z \rightarrow \infty, 0 < t < \infty,$$

where N is a finite constant, and the symbol $f^{(n)}(z, t)$ denotes the n th order partial derivative of f with respect to time.

Balance of momentum asserts that for each part of the body bounded by the pair of points z_a, z_b and for all times $t > 0$,

$$\frac{d}{dt} \int_{z_a}^{z_b} \rho(z) w^{(1)}(z, t) dz = \sigma(z_b, t) - \sigma(z_a, t), \quad (3)$$

in the absence of body forces.

The momentum balance law together with appropriate smoothness hypothesis, implies that for $z \neq Z(t)$,

$$\sigma_{,z}(z, t) = \rho(z) w^{(2)}(z, t), \quad (4)$$

where $z = Z(t)$, is the equation of the wave-front (§3). The constitutive equation is written as

$$w_{,z}(z, t) = J(z, 0) \sigma(z, t) + \int_{0^+}^t J^{(1)}(z, t-s) \sigma(z, s) ds. \quad (5)$$

In the above relations $\rho(z)$ is the continuous and differentiable function representing the mass density and $J(z, t)$ is the continuous and differentiable creep function. The comma denotes a partial derivative with respect to the indicated subscript variable. Due to continuity of the material everywhere, $\rho(z)$, $J(z, t)$ and $w(z, t)$ are assumed to be continuous across the wave-front at $z = Z(t)$.

3. Discontinuity analysis

3.1 Results from the theory of singular surfaces

The theory of singular surfaces and the conditions that must be satisfied across a singular surface for geometrical and for kinematical reasons have been established and presented by Truesdell & Toupin (1960, p. 504), Thomas (1961) and Chen (1976). Here we summarise only the main features of singular surface analysis, for self sufficiency of the problem at hand.

3.1a Surface of discontinuity: A surface of discontinuity (singular surface, wave-front) is a surface moving with time across which some field quantity, represented by some dependent variable as a function of position and time, suffers a jump. In the present case, the surface of discontinuity is assumed to separate the points uniformly at rest from those points which are disturbed. It is denoted by $\Sigma(t)$ and moves with a finite velocity in the direction of its normal i.e. z -axis.

3.1b Velocity of singular surface: It is given by (Singh and Gupta 1984b)

$$V^2(z) = (\rho(z) J(z, 0))^{-1}. \quad (6)$$

Since $J(z, 0)$ is associated with initial elastic response of the material one may then make the qualitative statement that wave-fronts in viscoelastic (non-homogeneous) materials propagate with elastic velocity in a similar non-homogeneous medium (Keller 1964).

3.1c Equation of surface of discontinuity: For any given time t , at the position z from the ground surface, the wave-front which is the locus of points of discontinuity is given by (Singh and Gupta 1984a)

$$t = h(z) = \int_0^z V^{-1}(x) dx, \quad (7)$$

with $h(0) = 0$, $\lim_{z \rightarrow \infty} h(z) \rightarrow \infty$, which confirms that the stress field is bounded.

Also the equation of the wave-front for the given position z , has been obtained (appendix A) as

$$z = Z(t) = C_0 t + g^{\frac{1}{2}}(C_0 t) \int_0^{C_0 t} [1 - g^{-\frac{1}{2}}(s)] ds, \quad (8)$$

where $g(z) = C_0^{-2} [J(z, 0)\rho(z)]^{-1}$, C_0 is the velocity of the wave-front in a homogeneous case.

3.1d Jump notation: The notation $|f|$ for the jump or the incremental value of the dependent variable $f(z, t)$, which is continuous everywhere in the medium except across the wave-front at the position, $z = Z(t)$, is presented as

$$|f| = \lim_{\substack{z^+ \rightarrow Z(t) \\ t^+ \rightarrow t}} f(z^+, t^+) - \lim_{\substack{z^- \rightarrow Z(t) \\ t^- \rightarrow t}} f(z^-, t^-) \neq 0, \quad (9)$$

where (z^+, t^+) and (z^-, t^-) are the points on the front and rear sides, respectively, of the surface of discontinuity.

3.1e Kinematic condition of compatibility: It is the condition which relates the jump of the derivatives of the function $f(z, t)$, to the jump of the normal derivative, and the time rate of change of the jump of the function and the speed of the singular surface (Thomas 1957; Truesdell & Toupin 1960, p. 504). It is given by

$$\frac{d|f|}{dt} = |f^{(1)}| + V|f_{,z}|, \quad (10)$$

where $V = V(z)|_{z=Z(t)}$.

3.1f Hadamard or dynamical compatibility condition: The compatibility condition for the function $f(z, t)$, which itself is continuous across the singular surface, when its normal and time derivatives are discontinuous is given by

$$|f^{(1)}| + V|f_{,z}| = 0. \quad (11)$$

3.1g Order of singular surface: Order zero – A singular surface is said to be of order zero with respect to a field variable $f(z, t)$, if the field variable is discontinuous across $\Sigma(t)$ i.e. $|f| \neq 0$.

Order one – The singular surface $\Sigma(t)$ will be of order one relative to $f(z, t)$ when $f(z, t)$ is continuous across $\Sigma(t)$ and its first order derivatives are discontinuous.

Order n – The singular surface will be of order n for $f(z, t)$ if (i) all the derivatives of $f(z, t)$ upto the $(n-1)$ th order along with $f(z, t)$ are continuous and (ii) the derivatives of the n th order are discontinuous across the wave front.

For our problem, we assume that the stress $\sigma(z, t)$ and its derivatives may be continuous everywhere in the medium, except across the wave-front. Due to the integrity and the non-sliding nature of the material, the displacement variable $w(z, t)$ is continuous everywhere, even across the wave-front. Hence, in the case of the displacement field, order zero discontinuity does not exist, whereas in the case of stress, velocity and strain fields, it may exist.

In the present analysis, we shall concentrate on the problem of formulation of discontinuities of stress and its derivatives rather than on the problem of discontinuities of displacement derivatives, since the former is of more practical interest for physical situations, where the time of application of the load on the body is comparable with the characteristic wave-propagation time.

3.2 Derivation of discontinuities in stress and stress derivatives

Partially differentiating the constitutive equation (5) $(n+1)$ times with respect to t we

get (appendix B)

$$w_{,z}^{(n+1)}(z, t) = J(z, 0)\sigma^{(n+1)}(z, t) + \sum_{i=1}^{n+1} J^{(i)}(z, 0)\sigma^{(n+1-i)} + \int_{0^+}^t J^{(n+2)}(z, t-s)\sigma(z, s)ds, \quad z \neq Z(t), \quad (12)$$

$$\text{where } J^{(i)}(z, 0) = \left. \frac{\partial^i J(z, t)}{\partial t^i} \right|_{t=0}.$$

Since the stress $\sigma(z, t)$ along with its derivatives $\sigma^{(n)}(z, t)$ is bounded and continuous everywhere, except on the wave-front at $z = Z(t)$, and since $J(z, t)$ along with its derivatives is also bounded and continuous everywhere, the integral term in (12) across the wave-front is infinitesimally small. Thus neglecting the increment of the integral across the wave-front, we get at $z = Z(t)$,

$$|w_{,z}^{(n+1)}| = J|\sigma^{(n+1)}| + \sum_{i=1}^{n+1} J^{(i)}|\sigma^{(n+1-i)}|, \quad (13)$$

where $J^{(i)} = J^{(i)}(z, 0)|_{z=Z(t)}$.

Again partially differentiating the equation of motion (4) with respect to t , we get at $z = Z(t)$,

$$|\sigma_{,z}^{(n)}| = \rho|w^{(n+2)}|, \quad \text{with } \rho = \rho(z)|_{z=Z(t)}. \quad (14)$$

Using (10), we get from (13) and (14),

$$\frac{d}{dt}|w^{(n+1)}| - |w^{(n+2)}| = JV|\sigma^{(n+1)}| + V \sum_{i=1}^{n+1} J^{(i)}|\sigma^{(n+1-i)}|, \quad (15)$$

and

$$\frac{d}{dt}|\sigma^{(n)}| - |\sigma^{(n+1)}| = \rho V|w^{(n+2)}|. \quad (16)$$

Then from (15) and (16), using (6), we get

$$\rho V \frac{d}{dt}|w^{(n+1)}| = \frac{d}{dt}|\sigma^{(n)}| + 2 \sum_{i=1}^{n+1} \alpha_i(t)|\sigma^{(n+1-i)}|, \quad (17)$$

$$\text{where } \alpha_i(t) = \left. \frac{J^{(i)}(z, 0)}{2J(z, 0)} \right|_{z=Z(t)}, \quad \text{with } J(z, 0)|_{z=Z(t)} \neq 0. \quad (18)$$

Thus α_i are functions of time, provided $J(z, t)$ cannot be expressed in the factored form separately in terms of z and t .

From relation (16), substituting the value of $d/dt|w^{(n+1)}|$ in (17), we obtain the first order non-homogeneous differential equation for $|\sigma^{(n)}|$ in the form

$$P(D)|\sigma^{(n)}| = F_n(t), \quad n \geq 1, \quad (19)$$

where the differential operation, $P(D) = D + \alpha_1(t) + \beta(t)$,

$$\beta(t) = -\frac{1}{2}D(\log \rho V), \quad \text{and}$$

$$F_n(t) = \frac{1}{2}[D^2 + 2\beta(t)D]|\sigma^{(n-1)}| - \sum_{i=1}^n \alpha_{i+1}(t)|\sigma^{(n-i)}|, \quad D = \frac{d}{dt}. \quad (20)$$

The kinematic compatibility condition from the balance of momentum is obtained as

$$|\sigma| = -V\rho|w^{(1)}|. \quad (21)$$

Using (21) in (17) for $n = 0$, we get

$$P(D)|\sigma| = 0. \quad (22)$$

Then the solutions to (19) and (22) are given by

$$\begin{aligned} |\sigma^{(n)}| &= B_n(\rho V)^{\frac{1}{2}}(\rho V)_0^{-\frac{1}{2}} \exp[-Q(t)] \\ &+ (\rho V)^{\frac{1}{2}} \exp[-Q(t)] \int_0^t (\rho V)^{-\frac{1}{2}} F_n(s) \exp[Q(s)] ds, \end{aligned} \quad (23)$$

and

$$|\sigma| = B_0(\rho V)^{\frac{1}{2}}(\rho V)_0^{-\frac{1}{2}} \exp[-Q(t)], \quad (24)$$

where $Q(t) = \int_0^t \alpha_1(s) ds$, $(\rho V)_0 = (\rho V)|_{t=0}$, B_n are arbitrary constants to be determined from the boundary condition.

Case 1: When the creep function $J(z, t)$ is expressible in separable form in terms of z and t , i.e.

$$J(z, t) = J_0 J_1(z) f(t), \quad (25)$$

and also when the density function is written as

$$\rho(z) = \rho_0 \rho_1(z), \quad (26)$$

we have

$$\alpha_i = \frac{1}{2} f_0^{(i)} / f_0, f_0^{(i)} = \frac{\partial^i f}{\partial t^i} \Big|_{t=0}, f_0 = f(0) \neq 0, \text{ as constants}$$

$$\text{and } Q(t) = \alpha_1 t, h(z) = h_1(z)/C_0 = C_0^{-1} \int_0^z g_1^{-\frac{1}{2}}(s) ds,$$

$$\begin{aligned} g_1(z) &= [\rho_1(z) J_1(z)]^{-1}, \\ C_0 &= (J_0 \rho_0 J(0))^{-\frac{1}{2}}. \end{aligned} \quad (27)$$

Then the solutions, given by (23) and (24) reduce to

$$\begin{aligned} |\sigma^{(n)}| &= B_n(\rho_1 V_1)^{\frac{1}{2}}(\rho_1 V_1)_0^{-\frac{1}{2}} \exp(-\alpha_1 t) \\ &+ (\rho_1 V_1)^{\frac{1}{2}} \exp(-\alpha_1 t) \int_0^t (\rho_1 V_1)^{-\frac{1}{2}} F_n(s) \exp(\alpha_1 s) ds, \end{aligned} \quad (28)$$

with

$$F_n(t) = \frac{1}{2} [D^2 + 2\beta(t)D] |\sigma^{(n-1)}| - \sum_{i=1}^n \alpha_{i+1} |\sigma^{(n-i)}|,$$

and

$$|\sigma| = B_0(\rho_1 V_1)^{\frac{1}{2}}(\rho_1 V_1)_0^{-\frac{1}{2}} \exp(-\alpha_1 t). \quad (29)$$

Case 2: When the medium is homogeneous (Achenbach & Reddy 1967), we take

$$J = J_0 f(t), f = f_0, \quad (30)$$

so that α_i are again given by (27) and are constants, and $Q(t) = \alpha_1 t$, $h(z) = z/C_0$, $\beta(t) = 0$.

Then the solutions (23) and (24) reduce to

$$|\sigma_H(n)| = B_n \exp(-\alpha_1 t) + \exp(-\alpha_1 t) \int_0^t F_n(s) \exp(\alpha_1 s) ds,$$

with

$$F_n(t) = \frac{1}{2} D^2 |\sigma^{(n-1)}| - \sum_{i=1}^n \alpha_{i+1} |\sigma^{(n-i)}|, \quad (31)$$

and

$$|\sigma_H| = B_0 \exp(-\alpha_1 t), \quad (32)$$

where H stands for the field variable for the homogeneous case.

4. Stress field

In the present work we also propose to seek a solution for the stress field at the arbitrary position z , in the form of Taylor's expansion about the time of arrival of the wave-front, $t = h(z)$. This solution is valid for the time, after the wave-front has passed the position z and in the neighbourhood of the wave-front. Therefore the stress field for the given position z and time $t \geq h(z)$, can be expressed as

$$\sigma(z, t) = \sum_{n=0}^{\infty} \frac{(t-h(z))^n}{n!} \frac{\partial^n \sigma}{\partial t^n} \Big|_{t=h(z)}. \quad (33)$$

Since at the fixed position $z = Z(t)$ in the medium, the material particles are at rest until the disturbance arrives at the time $t = h(z)$, i.e. for initially dead medium, $\sigma(z, t)$ and its derivatives $\sigma^{(n)}(z, t)$ are identically zero, and they have finite values even after the wave-front has passed. These derivatives or at least some of them are the propagating discontinuities.

As the material on the front side of the wave-front is uniformly at rest, we have

$$|\sigma^{(n)}| = \frac{\partial^n \sigma}{\partial t^n} \Big|_{t=h(z)}. \quad (34)$$

Therefore the stress field in terms of discontinuities can be written as

$$\sigma(z, t) = \sum_{n=0}^{\infty} \frac{[t-h(z)]^n}{n!} |\sigma^{(n)}|. \quad (35)$$

The coefficients $|\sigma^{(n)}|$ in the RHS can be obtained from (23) and (24) by replacing t by $h(z)$. After doing this and using the stress boundary condition (2), we get

$$B_n = A_n. \quad (36)$$

5. Higher order waves

Here we discuss the existence and strength of higher order waves with respect to the stress and displacement fields across the wave-front $\Sigma(t)$ at any arbitrary position $z = Z(t)$. A time dependent boundary load in the form of a Maclaurin's series (2) is prescribed.

5.1 Stress waves

(i) *Order zero* – The strength of the stress waves of order ‘zero’ is given by

$$|\sigma_H| = A_0 \exp(-\alpha_1 t), |\sigma| = A_0 (\rho V)_0^{\frac{1}{2}} (\rho V)^{-\frac{1}{2}} \exp(-\alpha_1 t). \quad (37)$$

(ii) *Order one* – The stress waves of order ‘one’ exist if and only if,

$$|\sigma_H| = 0, \text{ i.e., } A_0 \exp(-\alpha_1 t) = 0$$

$$\text{and } |\sigma| = 0, \text{ i.e., } A_0 (\rho V)^{\frac{1}{2}} \exp(-\alpha_1 t) = 0. \quad (38)$$

From (38), for the homogeneous or the non-homogeneous cases, when $A_0 = 0$ and $A_1 \neq 0$, the stress waves of order ‘one’ exist for all times $t > 0$. But when $A_0 \neq 0$, $A_1 \neq 0$, these waves exist only when $t \rightarrow \infty$ and are continuous. In the case of non-homogeneity, when $A_0 \neq 0$, $(\rho V) = 0$, $A_1 \neq 0$, these waves can exist for finite times and are continuous across the wave-front, but $(\rho V) \neq 0$ for the propagation of waves in real materials. Thus the existence of waves of order one depends upon the boundary condition.

The strength of these waves, when $A_0 = 0$ at an arbitrary position $z = Z(t)$ across the wave-front is given by

$$|\sigma_H^{(1)}| = A_1 \exp(-\alpha_1 z/C_0),$$

$$|\sigma^{(1)}| = A_1 (\rho V)^{\frac{1}{2}} \Big|_{t=h(z)} \cdot [(\rho V)_0^{-\frac{1}{2}} \cdot \exp(-\alpha_1 h(z))], \quad (39)$$

and the waves are given by

$$\sigma_H^{(1)}(z, t) = \sum_{m=1}^{\infty} \frac{(t-z/C_0)^{m-1}}{(m-1)!} |\sigma_H^{(m)}|,$$

$$\sigma^{(1)}(z, t) = \sum_{m=1}^{\infty} \frac{(t-h(z))^{m-1}}{(m-1)!} |\sigma^{(m)}|. \quad (40)$$

(iii) *Waves of order ‘n’* – The stress waves of order n exist iff,

$$|\sigma_H^{(i)}| = \sigma_0 |\sigma_{H, zzz \dots}^{(i)}| = 0, \text{ i.e., } A_i \exp(-\alpha_i t) = 0,$$

$$|\sigma^{(i)}| = \sigma_0 |\sigma_{, zzz \dots}^{(i)}| = 0, \quad (41)$$

i.e., $A_i (\rho V)^{\frac{1}{2}} \exp(-\alpha_i t) = 0$, $i = 0, 1, \dots, (n-1)$.

From (41) for the homogeneous or the non-homogeneous case, when $A_i = 0$, $i = 0, 1, \dots, (n-1)$, $A_n \neq 0$, these waves exist for all times $t > 0$, but when $A_i \neq 0$, $A_n \neq 0$, these waves exist only when $t \rightarrow \infty$ and are continuous. In the case of non-homogeneous media, when $A_i \neq 0$, $A_n \neq 0$, $\rho V = 0$, these waves can also exist for finite times, but the condition $\rho V = 0$ does not hold for wave-propagation. Thus the existence of the waves of higher order depends upon the coefficients in the Maclaurin’s series representing the boundary stress.

The strength of these waves, when $A_i = 0$, $i = 0, 1, \dots, (n-1)$, $A_n \neq 0$, across the wave-front, is given by

$$|\sigma_H^{(n)}| = A_n \exp(-\alpha_1 z/C_0),$$

$$|\sigma^{(n)}| = A_n (\rho V)^{\frac{1}{2}} \Big|_{t=h(z)} \cdot (\rho V)_0^{-\frac{1}{2}} \cdot \exp[-\alpha_1 h(z)], \quad (42)$$

and the waves are given by

$$\begin{aligned}\sigma_H^{(n)}(z, t) &= \sum_{m=n}^{\infty} \{ (t - z/C_0)^{m-n} / (m-n)! \} |\sigma_H^{(m)}|, \\ \sigma^{(n)}(z, t) &= \sum_{m=n}^{\infty} \{ [t - h(z)]^{m-n} / (m-n)! \} |\sigma^{(m)}|. \end{aligned} \quad (43)$$

5.2 Displacement waves

(i) Due to continuity of the displacement field across the wave-front everywhere, the strength of these waves is zero i.e. $|w_H| = 0$, $|w| = 0$, and the waves are given by

$$\begin{aligned}w_H(z, t) &= \sum_{m=0}^{\infty} (t - z/C_0)^m |w_H^{(m)}| / m!, \\ w(z, t) &= \sum_{m=0}^{\infty} [t - h(z)]^m |w^{(m)}| / m!. \end{aligned} \quad (44)$$

(ii) *Order 'one'* (Velocity waves) – On account of the continuity of the displacement field across the wave-front, the velocity waves exist and their strength is given by

$$\begin{aligned}|w_H^{(1)}| &= -(A_0/C_0\rho_0)\exp(-\alpha_1 t), \\ |w^{(1)}| &= -A_0(\rho V)^{-\frac{1}{2}} \cdot (\rho V)_0^{-\frac{1}{2}} \cdot \exp(-\alpha_1 t). \end{aligned} \quad (45)$$

At an arbitrary position these are given by

$$\begin{aligned}|w_H^{(1)}| &= -A_0(C_0\rho_0)^{-1} \exp(-\alpha_1 z/C_0), \\ |w^{(1)}| &= -A_0(\rho V)^{-\frac{1}{2}} \Big|_{t=h(z)} \cdot (\rho V)_0^{-\frac{1}{2}} \cdot \exp[-\alpha_1 h(z)]. \end{aligned} \quad (46)$$

The waves are given by

$$\begin{aligned}w_H^{(1)}(z, t) &= \sum_{m=1}^{\infty} (t - z/C_0)^{m-1} |w_H^{(m)}| / (m-1)!, \\ w^{(1)}(z, t) &= \sum_{m=1}^{\infty} [t - h(z)]^{m-1} |w^{(m)}| / (m-1)!. \end{aligned} \quad (47)$$

(iii) *Order 'two'* (Acceleration waves) – The displacement waves of order 'two' exist if,

$$\begin{aligned}|w_H^{(1)}| &= |w_{H,z}| = 0, \text{ i.e. } A_0 \exp(-\alpha_1 t) = 0, \\ |w^{(1)}| &= |w_{,z}| = 0 \text{ i.e. } A_0(\rho V)^{-\frac{1}{2}} \exp(-\alpha_1 t) = 0. \end{aligned} \quad (48)$$

The explanation for the existence of these waves is similar to the one given above in the case of stress waves of order 'one'. The strength of these waves when $A_0 = 0$, $A_1 \neq 0$ is given by

$$\begin{aligned}|w_H^{(2)}| &= -A_1(C_0\rho_0)^{-1} \exp(-\alpha_1 t), \\ |w^{(2)}| &= -A_1(\rho V)^{-\frac{1}{2}} \cdot (\rho V)_0^{-\frac{1}{2}} \cdot \exp(-\alpha_1 t), \end{aligned} \quad (49)$$

and at an arbitrary position by

$$\begin{aligned}|w_H^{(2)}| &= -A_1(C_0\rho_0)^{-1} \exp(-\alpha_1 z/C_0), \\ |w^{(2)}| &= -A_1(\rho V)^{-\frac{1}{2}} \Big|_{t=h(z)} \cdot (\rho V)_0^{-\frac{1}{2}} \cdot \exp[-\alpha_1 h(z)]. \end{aligned} \quad (50)$$

These waves are given by

$$\begin{aligned} w_H^{(2)}(z, t) &= \sum_{m=2}^{\infty} (t - z/C_0)^{m-2} |w_H^{(m)}|/(m-2)!, \\ w^{(2)}(z, t) &= \sum_{m=2}^{\infty} [t - h(z)]^{m-2} |w^{(m)}|/(m-2)!. \end{aligned} \quad (51)$$

Therefore, we conclude that in general the existence of the higher order waves for finite times in homogeneous or non-homogeneous media depends upon the vanishing of the first coefficients upto the order less by one and with non-zero coefficient of required order for stress waves in the Maclaurin's series of the time-dependent boundary stress. Whereas in the case of displacement waves of higher order, existence depends upon the vanishing of the first coefficients upto the order less by two and with non-zero coefficient of order less by one of the required order of the wave. It is also observed that the decay or growth of these waves with distance from the boundary depends upon the sign of α_1 . The strength of the waves is affected by the non-uniformity of the material characteristics.

6. Example

We consider a specific case when the characteristics of the earth i.e. the creep compliance (in the separable form) and density, which vary slowly with depth are represented as under –
creep compliance:

$$\begin{aligned} J(z, t) &= J(z)f(t) = J_0 J_1(z)f(t) = J_0(1 + \varepsilon e^{-az})^{-2}(1 - \varepsilon_1 e^{-m}), \\ \text{density: } \rho(z) &= \rho_0(1 - \varepsilon e^{-az})^2, \end{aligned} \quad (52)$$

where

$$\varepsilon_1, a, \eta \geq 0, \rho_0, J_1 > 0, 0 \leq \varepsilon \ll 1.$$

Also $J_0(1 + \varepsilon)^{-2}(1 - \varepsilon_1) < J(z, t) \leq J_0$, for $z, t \in [0, \infty)$,
and $\rho_0(1 - \varepsilon)^2 < \rho(z) \leq \rho_0$ for $z \in [0, \infty)$.

We also specify the boundary condition as:

$$\sigma(0, t) = \sigma_0 \exp(-t) = \sigma_0(1 - t + t^2/2), \text{ where } t \text{ is small.} \quad (53)$$

Then the coefficients A_n , the Maclaurin's series become,

$$A_0 = \sigma_0, A_1 = -\sigma_0, A_2 = \sigma_0, A_n = 0, \text{ for } n \geq 3. \quad (54)$$

Then we calculate the results derived in the previous sections,

$$\begin{aligned} V(z) &= C_0[1 + \varepsilon e^{-az}]/[1 - \varepsilon e^{-az}], C_0 = [J_0 \rho_0(1 - \varepsilon_1)]^{-\frac{1}{2}}, \\ V(z) &= C_0[1 + 2\varepsilon e^{-az} + 2\varepsilon^2 e^{-2az}] + O(\varepsilon^3). \end{aligned} \quad (55)$$

Since $\rho V = \rho_0 C_0[1 - \varepsilon^2 e^{-2az}]$, we make all calculations by retaining the 2nd order and neglecting the higher order terms. Further we have,

$$\begin{aligned} \alpha_n &= \frac{1}{2}(-1)^{n+1} \eta^n \varepsilon_1 / (1 - \varepsilon_1), \text{ which in the special case } \varepsilon_1 = 0.5 \text{ becomes} \\ \alpha_n &= \frac{1}{2}(-1)^{n+1} \eta^n. \end{aligned} \quad (56)$$

The equation of the wave front, for the given time t , is obtained as

$$\begin{aligned} t = h(z) &= (2/aC_0) \log [(e^{az/2} + \varepsilon e^{-az/2})/(1 + \varepsilon)], \\ &= (z/C_0) - (2\varepsilon/aC_0)(1 - e^{-az}) + (\varepsilon^2/aC_0)(1 - e^{-2az}) + 0(\varepsilon^3). \end{aligned} \quad (57)$$

The equation of the wavefront for the given position z , is obtained as

$$z = Z(t) = C_0 t + (2\varepsilon/a)(1 - e^{-aC_0 t}) - (\varepsilon^2/a)(1 - e^{-aC_0 t} + 3e^{-2aC_0 t}). \quad (58)$$

The jumps in the stress and stress derivatives across the wave-front are obtained as

$$\begin{aligned} |\sigma| &= |\sigma_H| + P_0(t)|\sigma_H|, \\ |\sigma_H| &= \sigma_0 \exp(-\eta t/2), \text{ where } P_0(t) = \frac{1}{2}\varepsilon(1 - e^{-2aC_0 t}), \\ |\sigma^{(1)}| &= |\sigma_H^{(1)}| + P_0(t)|\sigma_H^{(1)}| + P_1(t)|\sigma_H|, \\ |\sigma_H^{(1)}| &= \sigma_0(1 + \eta^2 t) \exp(-\eta t/2), \\ \text{where } P_1(t) &= -\frac{1}{2}\varepsilon^{(5/8)} aC_0(1 - e^{-2aC_0 t}), \\ |\sigma^{(2)}| &= |\sigma_H^{(2)}| + P_0(t)|\sigma_H^{(2)}| + P_1(t)|\sigma_H^{(1)}| + P_2(t)|\sigma_H|, \\ |\sigma_H^{(2)}| &= \sigma_0 \{1 - [(13/16)\eta^3 + (5/8)\eta^2]t + (25/128)\eta^4 t^2\} \exp(-\eta t/2), \\ P_2(t) &= \frac{1}{2}\varepsilon^2 aC_0(aC_0 + \eta/2)(1 - e^{-2aC_0 t}). \end{aligned} \quad (59)$$

Substituting $t = h(z)$ from (57) in (59), we obtain the required first three co-efficients for the Taylor's expansion of the stress field $\sigma(z, t)$ about the time $t = h(z)$ as:

$$\begin{aligned} |\sigma| &= |\sigma_H| + Q_0(z)|\sigma_H|, \quad k = \frac{1}{2}\eta/aC_0, \\ |\sigma_H| &= \sigma_0 \exp(-\eta z/2C_0), \\ \text{where } Q_0(z) &= 2\varepsilon k(1 - e^{-az}) + \varepsilon^2[2k^2(1 - e^{-az})^2 + \frac{1}{2}(1 - 2k)(1 - e^{-2az})], \\ |\sigma^{(1)}| &= |\sigma_H^{(1)}| + Q_0(z)|\sigma_H^{(1)}| + Q_1(z)|\sigma_H|, \\ |\sigma_H^{(1)}| &= \sigma_0[-1 + (5/8)\eta^2 z/C_0] \exp(-\eta z/2C_0), \\ Q_1(z) &= -(5/2)K\eta\varepsilon(1 - e^{-az}) + \varepsilon^2[-5k^2\eta(1 - e^{-az})^2 \\ &\quad + \eta(5k/4 - 1/4k)(1 - e^{-2az})], \\ |\sigma^{(2)}| &= |\sigma_H^{(2)}| + Q_0(z)|\sigma_H^{(2)}| + Q_1(z)|\sigma_H^{(1)}| + Q_2(z)|\sigma_H|, \\ |\sigma_H^{(2)}| &= \sigma_0[1 - (\eta^2/16)(13\eta + 10)(z/C_0) + (25/128)\eta^4 z^2/C_0^2] \exp(-\frac{1}{2}\eta z/C_0), \\ Q_2(z) &= \eta^2/8 \{26\varepsilon k(1 - e^{-az}) + \varepsilon^2[77k^2(1 - e^{-az})^2 \\ &\quad + (-13k + 1 + k/k^2)(1 - e^{-2az})]\}. \end{aligned} \quad (60)$$

Then the stress-field for $t \geq h(z)$ in the neighbourhood of the wave-front is obtained as

$$\sigma(z, t) = \sigma_H(z, t) + \delta(\sigma(z, t)), \quad (61)$$

where $\delta(\sigma(z, t)) = Q_0(z)\sigma_H(z, t) + [Q_1(z)(t - z/C_0)$
 $+ \frac{1}{2}(t - z/C_0)^2 Q_2(z)]|\sigma_H^{(1)}| + \{(t - z/C_0)(2\varepsilon/aC_0)(1 - e^{-az})$
 $+ \frac{1}{2}(t - z/C_0)^2 Q_0(z) + \varepsilon^2[(2/a^2 C_0^2)(1 - e^{-az})^2 - (1/aC_0)(t - z/C_0)$
 $\times (1 - e^{-2az})]\}|\sigma_H^{(2)}|,$

and

$$\begin{aligned}\sigma_H(z, t) = & \sigma_0 \{1 + (t - z/C_0)[-1 + (5/8)\eta^2]z/C_0 \\ & + \frac{1}{2}(t - z/C_0)^2[1 - (\eta^2/16)(10 + 13\eta)(z/C_0) + (25/128)\eta^4 z^2/C_0^2]\} \\ & \times \exp(-\frac{1}{2}\eta z/C_0).\end{aligned}$$

7. Discussion of results

Here we discuss the results regarding the stress field, the velocity and the time of arrival of the wave-front, and the strength of higher order waves across the wave-front at the given position z from the surface of the earth, based upon the analyses of specific applications for homogeneous and non-homogeneous cases.

7.1 Stress field

The stress field at the position z and time $t = z/C_0$, in the homogeneous case, is given by

$$\sigma_H(z, z/C_0) = \sigma_0 \exp(-\eta z/2C_0),$$

whereas in the non-homogeneous case at the same position z and time $t = h(z)$, we have

$$\begin{aligned}\sigma(z, h(z)) = & \sigma_0 \{1 + 2\varepsilon k(1 - e^{-az}) + \varepsilon^2[2k^2(1 - e^{-az})^2 \\ & + \frac{1}{2}(1 - 2k)(1 - e^{-2az})]\} \exp(-\eta z/2C_0).\end{aligned}$$

Hence the stress field at any depth z from the ground surface of the earth at the time of arrival of the maiden disturbance depends upon the non-homogeneity parameter ε and the relaxation parameter η . It seems to be greater than that for the homogeneous case.

The stress in the non-homogeneous case at time $t = z/C_0$ is given by

$$\begin{aligned}\sigma(z, z/C_0) = & \sigma_0 \exp(-\eta z/2C_0) + Q_0(z)\sigma_0 \exp(-\eta z/2C_0) \\ & + [(2\varepsilon/aC_0)(1 - e^{-az}) - (\varepsilon^2/aC_0)(1 - e^{-2az})]|\sigma_H^{(1)}| \\ & + (2\varepsilon^2/a^2 C_0^2)(1 - e^{-az})^2 |\sigma_H^{(2)}|.\end{aligned}$$

7.2 Velocity and time of arrival of the wave-front

The velocity is given as follows.

Homogeneous case: $C_0^2 = [J_0 \rho_0 (1 - \varepsilon_1)^{-1}]$,

this is non-zero and constant everywhere for $t > 0$, and $\varepsilon_1 \neq 1$.

Non-homogeneous case: $V(z) = C_0(1 + 2e^{-az} + 2\varepsilon^2 e^{-2az})$,

$$V(t) = C_0[1 + 2\varepsilon e^{-aC_0 t} - 2\varepsilon^2(1 - 3e^{-aC_0 t})e^{-aC_0 t}],$$

which is non-zero and variable for all z , $0 < z < \infty$, and $t > 0$. Time of arrival is given by

homogeneous case: $t_H = z/C_0$,

non-homogeneous case: $t = z/C_0 - (2\varepsilon/aC_0)(1 - e^{-az}) + (\varepsilon^2/aC_0)(1 - e^{-2az})$.

Since $a > 0$, we conclude that $V(t)$ or $V(z) \geq C_0$ and $t \leq t_H$, as $\varepsilon \geq 0$. Therefore the disturbances in the non-homogeneous case move faster or slower and arrive at any given position z , earlier or later than those in the homogeneous case, as $\varepsilon > 0$ or $\varepsilon < 0$, respectively. Thus from the measure of time of arrival of a disturbance at any depth z , we can measure the non-homogeneity of the earth at that position.

7.3 Higher order waves

(i) *Stress waves*: The strength of the stress field or waves of order 'zero' across the wave-front at any depth z , is given by

$$|\sigma_H| = \sigma_0 \exp(-\eta z/2C_0), |\sigma| = \sigma_0[1 + Q_0(z)] \exp(-\eta z/2C_0),$$

which is non-zero for $0 \leq z < \infty$ and $\sigma_0 \neq 0$.

Since $|\sigma_H|$ or $|\sigma|$ is not zero for $t > 0$ and $\sigma_0 \neq 0$, the waves of order 'one' do not exist for the prescribed surface loading. But if the surface loading be prescribed, for example, in the form of a sine function of time i.e.

$$\sigma(0, t) = \sigma_0 \sin t = \sigma_0[t + (t^3/3!) + \dots],$$

then we get,

$$|\sigma_H| = 0, |\sigma| = 0.$$

Thus the stress waves of order 'one' exist for this prescribed surface loading. The strength of waves of order 'one' is given by $|\sigma_H^{(1)}| = \sigma_0 \exp(-\eta z/2C_0)$, $|\sigma^{(1)}| = \sigma_0(1 + Q_0(z)) \exp(-\eta z/2C_0)$ which is also non-zero, for $0 \leq z < \infty$ and $\sigma_0 \neq 0$. For the higher order waves, if we prescribe the loading condition as

$$\sigma(0, t) = \sigma_0 f(t), \sigma_0 \neq 0,$$

with $f(0) = f^{(1)}(0) = \dots = f^{(n-1)}(0) = 0$, $f^{(n)}(0) \neq 0$, then the waves of order n exist and their strength is given by

$$|\sigma_H^{(n)}| = \sigma_0 \exp(-\eta z/2C),$$

$$|\sigma^{(n)}| = \sigma_0[1 + Q_0(z)] \exp(-\eta z/2C_0),$$

and the stress field of order n is given by

$$\sigma_H^{(n)}(z, t) = \sum_{m=n}^{\infty} (t - z/C_0)^{m-n} |\sigma_H^{(m)}| / (m-n)!,$$

$$\sigma^{(n)}(z, t) = \sum_{m=n}^{\infty} [t - h(z)]^{m-n} |\sigma^{(m)}| / (m-n)!.$$

(ii) *Displacement waves*: Since the displacement field is continuous across the wave-front everywhere $0 \leq z < \infty$, the waves of order 'one' (velocity waves) exist and their

 Figure 1. Stress σ/σ_0 and σ_H/σ_0 versus time for three different locations of the rod at $z/C_0 = 0.1, 1.0$ and 2.0

strength is given by

$$|w_H^{(1)}| = -\frac{\sigma_0}{C_0\rho_0} \exp(-\eta z/2C_0), |w^{(1)}| = [1 + Q_0(z)] |w_H^{(1)}|,$$

which is non-zero and finite for $0 \leq z < \infty$ and $\sigma_0 \neq 0$, $C_0\rho_0 \neq 0$.

Again since $|w_H^{(1)}|$ or $|w^{(1)}|$ is non-zero for $t > 0$ and $\sigma_0 \neq 0$, the waves of order 'two' i.e. acceleration waves do not exist for the prescribed surface condition. But if we modify our loading condition as given above in the case of stress-waves of order 'one' or higher order waves, the acceleration waves or higher order displacement waves will similarly exist.

7.4 Numerical results

Graphs for σ/σ_0 and σ_H/σ_0 versus time t , for the non-homogeneous and homogeneous cases for three depths $z/C_0 = 0.10, 1.00, 2.00$ where $C_0 = 1.2909944$, have been drawn for the following values of various parameters (figure 1):

$$J_0 = 1, \rho_0 = 1.2, \varepsilon_1 = 0.5, \eta = 0.5, a = 1,$$

and for three different values of ε , viz $\varepsilon = 0.1, 0.2, 0.3$.

It is observed from the graphs that the wave-front in the non-homogeneous case moves faster than in the homogeneous case. At the time of arrival of the wave-front, the non-dimensional stress σ/σ_0 across it, for a small non-dimensional depth such as $z/C_0 = 0.1$, decreases with increase in the value of ε i.e. with increase of non-homogeneity, whereas at comparatively deep depths $z/C_0 = 1.00, 2.00$, it increases with ε . But the stress σ/σ_0 in the non-homogeneous case remains less than σ_H/σ_0 in the homogeneous case for the same depths and times. However σ/σ_0 at the given time and depth decreases with increase in non-homogeneity.

Appendix A

The time t taken by the wave-front to traverse a distance z from the boundary $z = 0$, is given by

$$t = z/C_0, \quad (A1)$$

where C_0 is the speed of the wave in the homogeneous case.

Let the time t in the non-homogeneous case be given by

$$t = h(z) = \int_0^z V^{-1}(x)dx, \quad (A2)$$

where $V(z)$ is the corresponding speed in non-homogeneous case.

Let us express (A2) as

$$t = z/C_0 + f(z), \quad (A3)$$

$$\text{where} \quad f(z) = \int_0^z [V^{-1}(x) - C_0^{-1}]dx. \quad (A4)$$

Let us assume that when (A3) is solved for z , we get

$$z = C_0 t + F(t). \quad (A5)$$

From (A3) and (A5) we get

$$F(t) + C_0 f[C_0 t + F(t)] = 0. \quad (\text{A6})$$

Expanding f about $C_0 t$ (the distance in the homogeneous case) we get

$$F(t) = -[C_0 f(C_0 t)]/[1 + C_0 f'(C_0 t)], \quad (\text{A7})$$

after retaining terms upto the first approximation. Through (A4) this is equivalent to

$$F(t) = [C_0^{-1} V(C_0^{-1} t)] \int_0^{C_0 t} [1 - C_0 V^{-1}(x)] dx. \quad (\text{A8})$$

Thus the distance z traversed by the wave, in time t is given by

$$z = C_0 t + g^{\frac{1}{2}}(C_0 t) \int_0^{C_0 t} (1 - g^{-\frac{1}{2}}(x)) dx, \quad (\text{A9})$$

$$\text{where} \quad g(z) = C_0^{-2} V^2(z) = C_0^{-2} [J(z, 0)\rho(z)]^{-1}. \quad (\text{A10})$$

Appendix B

We have

$$w_{,z}(z, t) = J(z, 0)\sigma(z, t) + \int_{0^+}^t J^{(1)}(z, t-s)\sigma(z, s)ds. \quad (\text{B1})$$

Differentiating partially with respect to t we get

$$\begin{aligned} w_{,z}^{(1)}(z, t) &= J(z, 0)\sigma^{(1)}(z, t) + J^{(1)}(z, 0)\sigma(z, t) \\ &+ \int_{0^+}^t J^{(2)}(z, t-s)\sigma(z, s)ds. \end{aligned} \quad (\text{B2})$$

Again differentiating (B2) partially with respect to t we get

$$\begin{aligned} w_{,z}^{(2)}(z, t) &= J(z, 0)\sigma^{(2)}(z, t) + J^{(1)}(z, 0)\sigma^{(1)}(z, t) \\ &+ J^{(2)}(z, 0)\sigma(z, t) + \int_{0^+}^t J^{(3)}(z, t-s)\sigma(z, s)ds, \end{aligned} \quad (\text{B3})$$

and so on.

Thus differentiating partially (B1) n times, with respect to t we get

$$\begin{aligned} w_{,z}^{(n)}(z, t) &= J(z, 0)\sigma^{(n)}(z, t) + \sum_{i=1}^n J^{(i)}(z, 0)\sigma^{(n-i)} \\ &+ \int_{0^+}^t J^{(n+1)}(z, t-s)\sigma(z, s)ds. \end{aligned} \quad (\text{B4})$$

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