On the instability of a spinning projectile due to a nonlinear Magnus moment

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MS received 9 August 1980; revised 11 February 1981

Abstract. A nonlinear differential equation representing the initial nutational vibrations of a spinning projectile in the presence of a nonlinear Magnus moment has been approximated by a perturbed Duffing's equation and the quantitative and qualitative agreement of the approximation brought out. Then the combined motion in nutation and precession has been examined and sufficient conditions for the Magnus instability of the normal motion established via Lyapunov's second method.

Keywords. Nonlinear vibrations; nonlinear ballistics; Magnus instability.

1. Introduction

The initial angular motion of a spinning projectile due to a nonlinear overturning moment of the Lock-Fowler type (Fowler & Lock 1922) and a nonlinear Magnus moment of the type considered by Rath & Sharma (1981) has been examined. The nonlinear equation in yaw has been approximated by a perturbed Duffing's equation which has been solved by the method of equivalent linearisation (Bogoliubov & Mitropolsky 1961). This Duffing's equation has been shown to possess a good quantitative and qualitative agreement with the original equation in yaw.

In § 6, the combined motion of nutation and precession has been considered and it has been shown that if the nonlinear moment parameters $\lambda$ and $q$ satisfy certain conditions, then the dynamic stability of the projectile can be lost in the presence of a negative Magnus moment even when the projectile is gyroscopically stable provided that the precessional angular velocity of the projectile exceeds a certain value dependent upon the stability factor of the projectile. The results derived in this paper include those obtained by Rath & Sharma (1976) as particular cases when $\lambda$ and $q$ are both zero.

2. Equations of motion

If $\delta$ be the angle of nutation, and $\phi$ the angle of precession of a spinning projectile due to a nonlinear overturning moment of the Lock-Fowler type and a nonlinear Magnus moment, then the initial angular motion of the projectile can be represented by the following pair of equations (Rath & Sharma, 1981):

\[
(\cos \delta + \phi' \sin^2 \delta)' + \epsilon \sin^2 \delta (1 - \lambda \sin^2 \delta) = 0, \tag{1}
\]
(\delta'^2 + \phi'^2 \sin^2 \delta) + \frac{1}{2s} \{1 - 4qs (1 - \cos \delta)\} (\cos \delta)' \\
+ 2e\phi' \sin^2 \delta (1 - \lambda \sin^2 \delta) = 0. \tag{2}

In these equations \(q\) and \(\lambda\) characterize the nonlinearities in the overturning moment and the Magnus moment respectively; \(s\) is the stability factor of the projectile and \(\epsilon\), a measure of the strength of the Magnus moment. The primes indicate differentiation with respect to the nondimensional time \(\tau\).

3. Nutational vibrations due to a weak Magnus moment

For a weak Magnus moment, \(|\epsilon| \ll 1\); so neglecting the relatively small term in equation (1) containing \(\epsilon\), and then integrating, we obtain

\[
\cos \delta + \phi' \sin^2 \delta = 1, \tag{3}
\]

using the initial condition

\[
\delta(0) = 0. \tag{4}
\]

Eliminating now the precessional velocity \(\phi'\) from equation (2) by means of equation (3), we have

\[
\left\{ \delta'^2 + \left(\frac{1}{1 + \cos \delta}\right)^2 \sin^2 \delta \right\} + \frac{1}{2s} \{1 - 4qs (1 - \cos \delta)\} (\cos \delta)' \\
+ 2e \cdot \frac{1}{1 + \cos \delta} \cdot \sin^2 \delta (1 - \lambda \sin^2 \delta) = 0, \tag{5}
\]

which, by substituting

\[
Z = \sin \left(\frac{\delta}{2}\right), \tag{6}
\]

can be transformed into the equation

\[
Z'' + \omega^2 Z + \frac{ZZ'^2}{1 - Z^2} + \frac{Z^3}{4(1 - Z^2)} + \frac{Z^3}{4s} + 2qZ^3 (1 - Z^2) \\
+ \frac{\epsilon}{2Z} \cdot Z^3 (1 - Z^2) \left\{1 - 4\lambda Z^2 (1 - Z^2)\right\} = 0, \tag{7}
\]

with \(\omega^2 = \frac{1}{4} \left(1 - \frac{1}{s}\right). \tag{8}\)

It may be noted that for a gyroscopically stable projectile, \(s > 1\).
4. Approximation to equation (7) by means of Duffing’s equation

Making $\epsilon = 0$ in equation (7), we have

$$Z'' + \omega^2 Z + \frac{ZZ''^2}{1 - Z^2} + \frac{Z^3}{4(1 - Z^2)} + \frac{Z^3}{4s^2} 2qZ^3 (1 - Z^2) = 0,$$  \hspace{1cm} (9)

representing the nutational vibrations in the absence of the Magnus moment. If these vibrations are small \( i.e. |Z| \ll 1 \), a good approximation (Sharma 1975) to equation (9) is provided by Duffing's equation

$$Z'' + \omega^2 Z + \tilde{\omega}^2 Z^3 = 0,$$  \hspace{1cm} (10)

in which $\tilde{\omega}^2 = \frac{1}{4} + \frac{1}{4s^2} + 2q$. \hspace{1cm} (11)

The initial conditions considered for this purpose are

$$Z(0) = 0,$$  \hspace{1cm} (12)

which is an equivalent form of (4) and

$$Z'(0) = b/2, \quad (0 < b < 1),$$  \hspace{1cm} (13)

where $b$ is a measure of the initial disturbance.

As a natural extension of the approximation mentioned above, the perturbed Duffing’s equation

$$Z'' + \omega^2 Z + \tilde{\omega}^2 Z^3 = - \epsilon \frac{Z^3}{2Z'} (1 - Z^2) \{1 - 4 \lambda \ Z^2 (1 - Z^2)\},$$  \hspace{1cm} (14)

may be considered to represent the nutational oscillations of the projectile governed by equation (7). The equivalent linear equation of (14) by the method of equivalent linearization (Bogoliubov & Mitropolsky 1961) is

$$Z'' + \lambda_\epsilon (a) \ Z' + K_\epsilon (a) \ Z = 0,$$  \hspace{1cm} (15)

where $\lambda_\epsilon (a) = \frac{\epsilon}{32 \omega^2} \{16 - 12 (4\lambda + 1) \ a^2 + 80 \ \lambda \ a^4 - 35 \ \lambda \ a^6\}$, \hspace{1cm} (16)

$$K_\epsilon (a) = \omega^2 + \frac{3}{2} \tilde{\omega}^2 \ a^2,$$  \hspace{1cm} (17)

$$a = b/2\omega. \hspace{1cm} (18)$$

Solutions of equations (14) and (15) provide good quantitative estimates for the solution of equation (7). A comparison of these solutions for a positive and negative $\epsilon$ is given in table I for some data.
Table 1. Comparison of solutions

<table>
<thead>
<tr>
<th>Case I</th>
<th>( b = 0.018 )</th>
<th>( s = 3.25 )</th>
<th>( q = 2.0 )</th>
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<tr>
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<td>( \epsilon = 0.0001161 )</td>
<td>( \tau )</td>
<td>( Z ) from (7)</td>
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<td>0.00169</td>
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</tr>
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5. Stability of nutational motion

Setting \( Y = Z' \), equation (14) may be rewritten as the system

\[
Z' = Y
\]

\[
Y' = - \omega^2 Z - \tilde{\omega}^2 Z^3 - \frac{e Z^2}{2Y} (1 - Z^3) \left\{ 1 - 4\lambda Z^2 (1 - Z^3) \right\}. \tag{19}
\]

With \( Z(0) / Y(0) \neq 0 \), it is seen that \( Y = 0, Z = 0 \) is an isolated singular point of this system. Now it is interesting to note that the stability properties of this trivial solution are fully preserved by the perturbed Duffing's equation. This is established by Lyapunov's second method as follows.

Consider the Lyapunov function

\[
V(Y, Z) = Y^2 + \omega^2 Z^2 + \frac{1}{2} \tilde{\omega}^2 Z^4, \tag{20}
\]
which is positive definite. Its derivative formed by means of (19) turns out to be

\[ V' = -\epsilon Z^3 (1 - Z^2) \psi (\lambda, Z), \]  

in which

\[ \psi (\lambda, Z) = 1 - 4\lambda Z^2 + 4\lambda Z^4. \]  

For all positive \( h < 1/\sqrt{2} \), it is not difficult to see that (see Appendix)

\[ \psi (\lambda, Z) > 0, \text{ whenever } |Z| \leq h, \]  

and \( \lambda \leq 1. \)
In view of the above facts, we conclude (La Salle & Lefschetz 1961) that the trivial solution of the system (19) is

\[
\begin{align*}
(a) & \text{ asymptotically stable, if } \epsilon > 0, \\
(b) & \text{ stable, if } \epsilon = 0; \text{ and} \\
(c) & \text{ unstable, if } \epsilon < 0.
\end{align*}
\]

(25)

That the original equation (7) also has the stability properties (25) can be seen by first rewriting it in the equivalent form

\[
\left(\frac{4Z^2 + Z^3}{1 - Z^2}\right)' - \left(\frac{Z^2}{s}\right)' + (4qZ^4)' + 4\epsilon Z^2 \psi (\lambda, Z) = 0, \tag{26}
\]

and then constructing its Lyapunov function \( V(Z, Z') \) as follows

\[
V(Z, Z') = \frac{4Z^2 + Z^3}{1 - Z^2} - \frac{Z^2}{s} + 4qZ^4 = \frac{4Z^2 + Z^4}{1 - Z^2} + 4\epsilon \omega Z^2 + 4qZ^4, \tag{27}
\]

due to (8). The derivative of \( V \) by means of (26) turns out to be

\[
V' = -4\epsilon Z^2 \psi (\lambda, Z). \tag{28}
\]

From (27) and (28), the stability properties (25) are easily seen to follow, when \( \lambda \leq 1 \) and \(|Z| \leq h\).

6. Magnus instability

The normal motion of the projectile is characterized by

\[
\begin{align*}
l & = 1, \quad m = n = 0, \\
p_1 & = N, \quad q_1 = r_1 = 0,
\end{align*}
\]

(29)

where \( l, m, n \) are the components of the unit vector \( \vec{\Lambda} \) defining the axis of the projectile; \( p_1, q_1, r_1 \) are the components of the angular velocity vector of the body axes defined by the vectors \( \vec{\Lambda}, \vec{\Lambda}l - \vec{X}, \vec{X} \times \vec{\Lambda}; \vec{X} \) being the unit vector defining the direction of motion of the centre of mass of the projectile. The perturbations considered about this normal motion are

\[
\begin{align*}
l & = \cos \delta = 1 + \xi, \\
m & = \sin \delta \cos \phi = \overline{m}, \\
n & = \sin \delta \sin \phi = \overline{n},
\end{align*}
\]

(30)
and \[ p_1 = N + \alpha, \quad q_1 = \phi' \sin \delta = \tilde{q}_1, \quad r_1 = \delta' = \tilde{r}_1. \] (31)

Since it has been assumed that there is no decay in the axial spin of the projectile, we note that \( \alpha = 0 \). In view of equations (30) and (31), equations (1) and (2) may be rewritten as

\[
(\xi + \phi' \sin^2 \delta)' + \epsilon \sin^2 \delta (1 - \lambda \sin^2 \delta) = 0, \tag{32}
\]

\[
(\delta'' + \phi'' \sin^2 \delta)' + \left(\frac{\xi}{2s}\right)' + (q'q')' + 2\epsilon\phi' \sin^2 \delta (1 - \lambda \sin^2 \delta) = 0. \tag{33}
\]

Also, the fundamental relation

\[
(1 + \xi)^2 + m^2 + \tilde{n}^2 = 1,
\]

gives \( \xi^2 + 2\xi + \sin^2 \delta = 0. \) (34)

Consider now the functions

\[
V_1 = \xi + \phi' \sin^2 \delta, \quad \quad \quad \quad V_2 = \delta'' + \phi'' \sin^2 \delta + \left(\frac{\xi}{2s}\right) + q'q', \quad \quad \quad \quad V_3 = \xi^2 + 2\xi + \sin^2 \delta. \tag{35}
\]

A Lyapunov function using an arbitrary parameter \( \eta \) can now be constructed as follows:

\[
V = 2\eta V_1 + V_2 - \left(\eta + \frac{1}{4s}\right) V_3
\]

\[
= \delta'' + \phi'' \sin^2 \delta + 2\eta \phi' \sin^2 \delta -
\]

\[
- \left(\eta + \frac{1}{4s}\right) \sin^2 \delta - \left(\eta + \frac{1}{4s} - q\right) \xi^2. \tag{36}
\]

Regarding \( V \) as a quadratic form of the four variables \( \delta', \phi' \sin \delta, \sin \delta \) and \( \xi \), we assert that \( V \) is positive definite (Efimov & Rozendorn 1975) if the principal minors of the determinant \( \Delta_4 \) are all positive, where
These conditions turn out to be equivalent to

\[-\eta - \frac{1}{4s} + q > 0,\]

\[-\eta^2 - \eta - \frac{1}{4s} > 0;\]

i.e.,

\[\eta < \omega^2 + q - \frac{1}{4},\]  \hfill (37)

\[-\frac{1}{2} - \omega < \eta < -\frac{1}{2} + \omega;\]  \hfill (38)

on account of (8). Since for a gyroscopically stable projectile \(\omega < \frac{1}{2}\), inequality (38) requires that \(\eta\) be negative. Moreover, if

\[q \geq - (\frac{1}{2} - \omega)^2,\]  \hfill (39)

then (37) is always implied by (38). The derivatives of \(V\) formed by means of (32) and (33) turn out to be

\[V' = -2\epsilon \sin^2 \delta (1 - \lambda \sin^2 \delta) (\phi'' + \eta),\]  \hfill (40)

due to (34) and (35). Hence, it follows that the normal motion (29) of the projectile is

(a) asymptotically stable, if \(\epsilon > 0\), (b) stable if \(\epsilon = 0\); and (c) unstable, if \(\epsilon < 0\); provided that

\[\lambda \leq 1,\]  \hfill (41)

\[\phi'' + \eta > 0;\]  \hfill (42)

and conditions (38) and (39) hold. On account of (38), condition (42) can be realised, by requiring that

\[\phi' > \omega + \frac{1}{2}.\]  \hfill (43)

Thus we conclude that the gyroscopic stability of the projectile is lost in the presence of a negative Magnus moment (i.e. \(\epsilon < 0\)) whenever its precessional velocity \(\phi'\) exceeds \(\omega + \frac{1}{2}\) under the conditions (38), (39) and (41).
7. Conclusions

The nonlinear equation (7) representing the nutational oscillations of the projectile has been approximated by the perturbed Duffing's equation (14) and the equivalent linear equation (15). These approximations have been obtained assuming the Magnus moment to be weak, i.e. \( |\epsilon| < 1 \). While both the approximate equations preserve the stability properties (25) of the original equation (7) for all \( \epsilon \), their quantitative estimates depend significantly on the magnitude of \( \epsilon \). This has been brought out by comparing the initial amplitudes of oscillations as shown in figure 1 and table 2. While equation (14) provides an excellent approximation for the complete range 0.0001 to 0.1 of \( \epsilon \), considered for comparison, the approximation provided by equation (15) becomes poorer as \( \epsilon \) increases from 0.0001 to 0.1. The fluctuations in the \% errors shown in table 2 are expected to be due to the parabolic interpolations used for estimating the amplitudes of oscillations from the numerical solutions.

The author is thankful to Prof. P C Rath, Institute of Armament Technology, Pune for encouragement and to Dr L K Wadhwa, Director, CASSA, Bangalore for permission to publish this work. Thanks are also due to the referees for suggestions which led to the improvement of this work. Finally, the author thanks...
Appendix

The extrema of the function $\psi(\lambda, Z)$ occur at points $Z = 0, \pm 1/\sqrt{2}$. Their nature, however, depends upon the sign of the parameter $\lambda$.

When $\lambda < 0$, the function $\psi(\lambda, Z)$ assumes a minimum at the point $Z = 0$ and, a maximum at either of the points $Z = \pm 1/\sqrt{2}$.

When $\lambda > 0$, the function $\psi(\lambda, Z)$ assumes a maximum at the point $Z = 0$ and a minimum at either of the points $Z = \pm 1/\sqrt{2}$.

Further we note that $\psi(\lambda, 0) = 1$, $\psi(\lambda, \pm 1/\sqrt{2}) = 1 - \lambda$ and $\psi(\lambda, \pm 1) = 1$. Since the absolute minimum of the function in the interval $[-1, 1]$ will occur either at an end point of this interval or at an extremum point in this interval, we have

$$\psi(\lambda, Z) \geq \min [1, 1 - \lambda].$$

Here, we note that $1 - \lambda$, occurring as an argument in $\min [1, 1 - \lambda]$, is the value of the function $\psi(\lambda, Z)$ at the points $Z = \pm 1/\sqrt{2}$. Hence, it follows that

$$\psi(\lambda, Z) > 0 \text{ for } |Z| < 1/\sqrt{2}$$

whenever $\lambda \leq 1$. This fact is also brought out in the graphical representations of $\psi(\lambda, Z)$ as shown in figures 2(a–d).

**Figure 2.** Graphs of $\psi(\lambda, Z)$ when a. $\lambda = 1$, b. $\lambda > 1$, c. $\lambda < 0$, d. $0 < \lambda < 1$. 
Magnus instability

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