Making $\pi$ Accessible* 

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The expression $C = \pi D$, which gives the relationship between the circumference $C$ and the diameter $D$ of a circle, is one of the few formulas known to almost all children and adults, regardless of how long they have been out of school. School-going children are introduced to this relation in their 7th grade. How this is done in India is illustrated by a snapshot from a textbook (NCERT, Mathematics, Grade 7, Chapter 11, [1].)

There is no reference as to who proved it first, leave aside any proof of this fact. More so at no stage in the school curriculum, there is a reference to a proof of this for self-study. The aim of this article is to provide arguments that are rigorous and accessible to upper middle-grade students to illustrate the fact that $C/D$ is constant for every circle.

1 Introduction

Historically, the fact that the ratio of the circumference $C$ to the diameter $D$ of every circle is the same constant, was known to many ancient civilizations, and thus one must expect it was known to the Greeks. Aristotle (384–322 BC) asserted that it was impossible to compare the lengths of curves and line segments, and this belief was widely held until the seventeenth century. Thus, one cannot discuss the ratio of the circumference to the diameter. In Elements [2], Euclid (300 BC) proved, essentially, the existence of the area constant for circles ($A = \pi r^2$, $r$ being the radius). But he did not mention the invariance of $C/D = \pi$, or anything equivalent to it. Archimedes (287–212 BC) proved that $A = \frac{1}{2} C \times r$

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**Figure 1.** \( \pi \) in a school textbook.

What do you infer from the above table? Is this ratio approximately the same? Yes.

Can you say that the circumference of a circle is always more than three times its diameter? Yes.

This ratio is a constant and is denoted by \( \pi \) (pi). Its approximate value is \( \frac{22}{7} \) or 3.14.

So, we can say that \( \frac{C}{d} = \pi \), where \( C \) represents circumference of the circle and \( d \) its diameter.

or \( C = \pi d \)

We know that diameter \( (d) \) of a circle is twice the radius \( (r) \) i.e., \( d = 2r \)

So, \( C = \pi d = \pi \times 2r \) or \( C = 2\pi r \).

and showed that

\[
\frac{223}{71} < \frac{C}{D} < \frac{22}{7}.
\]

But he never explicitly stated that \( \frac{C}{D} \) is the same constant for every circle.

The aim of this note is to provide arguments that are accessible to upper middle-grade students for the facts:

1. The area of a circle is \( \pi r^2 \), where \( r \) is the radius of the circle and \( \pi \) denotes the area of a circle with unit radius (Corollary 3.2).

2. The area \( A_r \) and circumference \( C_r \) are related by \( 2A_r = C_r \times r \) (Theorem 4.1).

3. Definition of \( \pi \) (see 5.4).

4. The ratio \( \frac{C}{D} \) is the same constant for every circle (Theorem 4.2).

Throughout this article, we will use modern terminology, notations, and mathematical concepts unless it is important to understand the original approach.
Sections 2 to Section 5 need only concepts taught in middle grades at schools. Section 6 needs concepts taught at secondary/senior secondary level.

Most of the historical notes and some arguments are curated from [3–6].

2 Preliminaries

Definition 2.1. Two polygons are said to be similar if there is a correspondence between their vertices so that the corresponding angles are equal and corresponding sides are proportional.

Example 2.2. For pentagons ABCDE and FGHKL to be similar, the following is required:

1. Corresponding angles are equal: \( \angle A = \angle F, \angle B = \angle G, \angle C = \angle H, \angle D = \angle K, \) and \( \angle E = \angle L. \)


Lemma 2.3. (Proposition No.1, Book XII of Elements) Similar polygons inscribed in circles are to one another as the square of their diameters.
The diameter of a polygon is the largest distance between any pair of vertices. In other words, it is the length of the longest polygon diagonal (e.g., straight line segment joining two vertices).

Proof. Refer [2] \(\square\)

**Lemma 2.4.**

(i) The area of a regular polygon of \(n\) sides is half its perimeter times the length of the apothem.

(ii) Given a circle of radius \(r\), let \(P_n\) denote a regular polygon of \(n\) sides inscribed in the circle and \(P_{cir}\) denote the regular polygon of \(n\) sides circumscribing the circle. Then

\[
\text{area}(P_{cir}) = \frac{1}{2} \times r \times \text{perimeter}(P_{cir}).
\]

Proof. (i) Let \(F\) be the center (a point inside the regular polygon at an equal distance from each vertex) of the regular \(n\)-gon \(P_n\). Then \(P_n\) can be thought of as \(n\) congruent triangles, each congruent to \(\triangle ABF\). Let \(FG\) be the right bisector of \(AB\). Then \(FG\) is called the apothem of the regular polygon. We denote by \(p_n\) the apothem of the regular \(n\)-sided polygon. Now (refer to Figure 3 and Figure 4),

\[
\begin{align*}
\text{area}(P_n) &= n \times \text{area}(\triangle FAB) \\
&= n \left(\frac{1}{2} (AB \times FG)\right) = \frac{1}{2} \times \text{perimeter}(P_n) \times p_n(1)
\end{align*}
\]
Lemma 2.5. Suppose $C$ is a circle. Let $\mathcal{P}_n$ denote a $2^n$-sided regular polygon inscribed in $C$, and let $Q_n$ denote a $2^n$-sided regular polygon circumscribing $C$. Then, for every $n \geq 1$, there exist $\mathcal{P}_n$ and $Q_n$ such that the following holds:

(i) \( \text{perimeter}(Q_{n+1}) > \text{perimeter}(Q_n) > \text{perimeter}(\mathcal{P}_{n+1}) < \text{perimeter}(\mathcal{P}_n) \).

(ii) \( \text{area}(Q_{n+1}) > \text{area}(Q_n) > \text{area}(\mathcal{P}_n) < \text{area}(\mathcal{P}_{n+1}) \).

Proof. We illustrate the proofs for $n = 2$ geometrically. For more detailed proofs, one may refer [5].

Let a circle of radius $r$ with center at $O$ be given. We construct $\mathcal{P}_n$ and $Q_n$ inductively as follows: Construct perpendicular diameters $AC$ and $BD$. Then $ABCD$ is our regular inscribed polygon. Also, construct square $KLMN$ by drawing lines $KBN$ and $LDM \parallel AC$ and lines $KAL, NCM \parallel DB$, see Figure 5.

This gives us, for $n = 2$, inscribed regular polygon $\mathcal{P}_n$ and circumscribed regular polygon $Q_n$ of sides $2^n$. Clearly,

\[
\text{perimeter}(Q_2) > \text{perimeter}(\mathcal{P}_2) \quad \text{and} \quad \text{area}(Q_2) > \text{area}(\mathcal{P}_2).
\]
**Figure 5.** Inscribed and circumscribed for \( n = 2 \).

**Figure 6.** Doubling the number of sides.

Next, we construct \( \mathcal{P}_3 \) and \( Q_3 \) as follows (see Figure 6). Locate the midpoints \( A_1, B_1, C_1, D_1 \) of arcs \( AB, BC, CD, \) and \( DA \), respectively, and construct the inscribed octagon \( AA_1BB_1CC_1DD_1 \). This is our regular polygon \( \mathcal{P}_3 \). Clearly,

\[
\text{perimeter}(\mathcal{P}_3) > \text{perimeter}(\mathcal{P}_2) \quad \text{and} \quad \text{area}(\mathcal{P}_3) > \text{area}(\mathcal{P}_2).
\]

Next, draw tangents at the vertices of octagon \( AA_1BB_1CC_1DD_1 \) to get the circumscribed octagon, \( AE_1A_1F_1B_1G_1B_1I_1C_1J_1C_1L_1DN_1D_1M_1A \). This is our regular circumscribed polygon \( Q_3 \).

By comparing the sectorial triangles (see Figure 7) and using tri-
angle inequality, we have

\[ \text{perimeter}(Q_2) > \text{perimeter}(Q_3) \] and \[ \text{area}(Q_2) > \text{area}(Q_3). \]

Having constructed \( \mathcal{P}_n \) and \( Q_n \), we construct \( \mathcal{P}_{n+1} \) and \( Q_{n+1} \) as above. Clearly, \( \text{Figure 8} \) illustrates the inequalities:

\[ \text{perimeter}(Q_n) > \text{perimeter}(\mathcal{P}_n) \] and \[ \text{area}(Q_n) > \text{area}(\mathcal{P}_n). \]

\( \text{Figure 9} \) and \( \text{Figure 10} \) illustrate the inequalities

\[ \text{perimeter}(\mathcal{P}_{n+1}) < \text{perimeter}(\mathcal{P}_n) \] and \[ \text{perimeter}(Q_{n+1}) > \text{perimeter}(Q_n). \]

\[ \square \]
Figure 9. Doubling the sides of inscribed polygon.

![Diagram of inscribed polygon](image)

Figure 10. Doubling the sides of circumscribed polygon.

![Diagram of circumscribed polygon](image)

3 Area of the Unit Circle

3.1 Euclid (300 BC), Archimedes (287–212 BC)

We continue with the intuitive definition of area. The area of a geometric figure is the amount of space inside it [1]. With that understanding the space inside a circle has an area. The main aim of this section is to understand this. The tool used is:

**Principle of Exhaustion:** Given two unequal magnitudes, from the greater of which is subtracted a magnitude larger than its half, and from the remainder a magnitude greater than its half removed, then after a finite number of such operations a quantity is reached which has a magnitude less than that of the smaller of
the two original magnitudes.

**Lemma 3.1.** Given any circle $C$ and any real number $\epsilon > 0$,

(i) There exists a polygon $P$ inside $C$ such that

$$a(C) - a(P) < \epsilon.$$  

(ii) There exists a polygon $P$ inside $C$ such that

$$a(P) - a(C) < \epsilon.$$  

**Proof.** Let $a(S)$ denote the area of a region $S$. On Circle $C$, inscribe the square $ABCD$ and let $PQRS$ be the circumscribing circle, as shown in Figure 11.

Then

$$a(ACBD) = \frac{1}{2}a(PQRS) > \frac{1}{2}a(C).$$

Thus,

$$a(C) - a(ACBD) = a(2) < \frac{1}{2}a(C).$$

Next, construct the regular octagon $AEDFBGCH$. Then

$$a(\Delta AHC) = \frac{1}{2}a(ACVT) > \frac{1}{2}a(\text{sector} AHC).$$

By similar arguments, we see that the octagon $AEDFBGCH$ includes not only the square $ABCD$ but also includes an area which
Figure 12. Construction of circle (C).

is more than half the area between circle (C) and the square \( ACBD \). Thus

\[
\alpha(C) - \alpha(AEDFBGCH) < \frac{1}{2} \alpha(2) < \frac{1}{4} \alpha(C).
\]

If we continue this process, by the Principle of Exhaustion, after a finite number of steps, we will get a regular polygon \( P \) inside circle (C) such that

\[
\alpha(C) - \alpha(P) < \epsilon.
\]

\[ \square \]

**Theorem 3.1.** Proposition (No.2, Book XII of Elements)

*Areas of circles are in the ratio of squares of their diameter.*

**Proof.** Let circles (1) and (2) have diameters \( XY \) and \( AB \), respectively, as shown in Figure 12. Let \( \alpha(F) \) denote the area of a region \( F \).

Suppose \( XY^2 : AB^2 \neq \alpha(1) : \alpha(2) \). Let there exists a circle (3) such that

\[
XY^2 : AB^2 = \alpha(1) : \alpha(3).
\] (2)

(The existence of the fourth proportional as an area is assumed by Euclid). Then either \( \alpha(3) \) is larger than \( \alpha(2) \) or is smaller than \( \alpha(2) \). Without loss of generality, let

\[
\alpha(3) < \alpha(2).
\]
Now by Lemma 3.1, we can find a regular polygon $P$ inside circle (2) such that $a(2) - a(P)$ is smaller than any given magnitude, say $a(2) - a(3)$. Thus,

$$a(2) - a(P) < a(2) - a(3),$$

i.e.,

$$a(P) > a(3).$$  \(3\)

Now inscribe inside circle (1) a polygon $P'$ similar to $P$. Then, by Lemma 3.1 and (2),

$$XY^2 : AB^2 = a(P') : a(P) = a(1) : a(3).$$

Thus,

$$a(P') : a(1) = a(P) : a(3).$$

Since $a(P') < a(1)$, as $P'$ is inscribed in circle (1), we get $a(P) < a(3)$, a contradiction to (3). Hence the theorem.

\[\square\]

**Corollary 3.2.** Let $\pi$ denote the area of the unit circle. Then, the area of a circle of radius $r$ is $\pi r^2$.

**Proof.** Let $C_r$ denote a circle of radius $r$ units and $\pi := \text{area}(C_1)$. Then by Theorem 3.1,

$$\text{area}(C_r) : \text{area}(C_1) = r^2 : 1,$$

implying $\text{area}(C_r) = \pi r^2$.

\[\square\]

**Note**

Using similar arguments, Archimedes computed many other areas and volumes. He also found approximations for the number $\pi$ by inscribing and circumscribing regular polygons, e.g., he obtained the relation.

$$\frac{223}{71} < \pi < \frac{22}{7}.$$
4 Ratio of Circumference to Diameter

There is no place in Euclid’s work where the length of anything but a line segment is mentioned. The problem, as explained in [3] comes to this:

*How can one compare the lengths of two curves in the plane or two surfaces in three dimensions? With areas in the plane, the answer is relatively simple — one eventually arranges things so one area is contained in the other. But with curves, this is not possible. So there is no really obvious answer to the question, In what circumstances can we tell easily whether one curve is longer or shorter than another?*

How did Archimedes deal with the problem?

4.1 Archimedes (287–212 BC)

We begin with a naive definition of ‘circumference’ of a circle (as given in school textbooks, see page 218 of [1]):

**Definition 4.1.** The circumference of a circle is the (linear) distance around it. That is, the circumference would be the length of the circle if it were opened up and straightened out to a line segment.

**Assumption 4.2.** The perimeter of an inscribed regular n-gon is less than the circumference of the circle, which is less than the perimeter of a circumscribed regular n-gon.

Geometrically, this seems obvious. Next theorem gives the definition of ‘circumference’ without using π.

**Theorem 4.1.** The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle, i.e.,

\[
\text{Area of circle} = \frac{1}{2} \times (\text{Circumference of the circle}) \times (\text{Radius of the circle}).
\]
**Proof.** Suppose we begin with a circle with radius $r$, area $A$, and circumference $C$.

Let $T$ be a right triangle with side lengths $r$ and $C$. For the sake of contradiction, suppose area (T) $\neq A$. There are two cases, either area (T) $> A$ or area (T) $< A$.

Suppose area (T) $< A$, i.e., $\epsilon := A - \text{area (T)} > 0$. Now, using Lemma 3.1 (i), there is an inscribed regular polygon $P_{in}$ such that

$$A - \text{area}(P_{in}) < \epsilon = A - \text{area (T)}.$$

This implies

$$\text{area}(P_{in}) > \text{area}(T). \quad (4)$$

Let $r'$ be the length of the perpendicular, as shown in Figure 14. Then $r' < r$, and hence
Figure 15. Circumscribed polygon.

\[
\text{area}(P_{\text{in}}) = \frac{1}{2} \text{perimeter}(P_{\text{in}}) < \frac{1}{2} C = \text{area}(T). \tag{5}
\]

Note that in the last inequality, we are using assumption (4.2). Inequalities (5) and (4) contradict each other.

Now suppose area (T) > A, i.e., \( \epsilon := \text{area}(T) - A > 0 \). Once again using Lemma 3.1(ii), there is a circumscribed regular polygon \( P_{\text{circ}} \) such that

\[
\text{area}(P_{\text{in}}) - A < \epsilon = \text{area}(T) - A.
\]

So

\[
\text{area}(P_{\text{circ}}) < \text{area}(T). \tag{6}
\]

Thus

\[
\text{area}(P_{\text{circ}}) = \frac{1}{2} r \text{perimeter}(P_{\text{in}}) > \frac{1}{2} C = \text{area}(T). \tag{7}
\]

For every circle, the ratio of its circumference to its diameter is the constant \( \pi \).

Note that in the last inequality, we are using assumption, (4.2). Inequalities (7) and (6) contradict each other. Hence

Area of circle = \( \frac{1}{2} \) (Circumference of the circle)\( \times \) (Radius of the circle).
Theorem 4.2. For every circle, the ratio of its circumference to its diameter is the constant π.

Proof. Follows from Theorem 4.1 and Corollary (3.2).

Remarks About Greek Methods

(1) They were rigorous and were derived from well-stated axioms.

(2) They successfully avoided the concept of limits.

(3) Concepts like area and volume were not defined, but only methods of computing each individually were given.

5 Making Concepts Precise, Using Completeness Property

This section aims to show how intuitive arguments can be made precise. For this we need some concepts and properties of real numbers normally taught in secondary/higher secondary grades at schools, (see [7]).

5.1 Real Numbers

The set of real numbers includes the set of rational numbers, as a proper subset. Further, like rational numbers, there are two binary operations defined on \( \mathbb{R} \), called \textit{addition} and \textit{multiplication} with properties similar to that of \( \mathbb{Q} \), the rational numbers. Also, there is an \textit{order} between real numbers, with properties similar to that of rational numbers. This makes \( \mathbb{R} \) an ‘ordered field’, similar to that of \( \mathbb{Q} \), for more details, refer to [7]. However, what distinguishes real numbers from rational numbers\(^1\) is the \textit{completeness property}.

5.2 Completeness Property

\textit{Every monotonically increasing sequence of real numbers that is bounded above\(^2\) is convergent.}

\(^1\)A sequence of real numbers \( a_1, a_2, a_3, \ldots \) is said to be monotonically increasing if \( a_{n+1} \geq a_n \ \forall n \).

\(^2\)A sequence of real numbers \( a_1, a_2, a_3, \ldots \) is said to be bounded above if there exists a constant \( M \) such that \( |a_n| \leq M \ \forall n \).
This version of the completeness property is also known as the ‘fundamental axiom of analysis’, see [8].

5.3 Definition of π as Area of the Unit Circle

Recall our notations,

\[
\begin{align*}
\text{area}(C_1) &= \text{area of the unit circle (to be found)}, \\
\text{area}(P_n) &= \text{area of the inscribed regular } 2^n\text{-gon}, \\
\text{area}(Q_n) &= \text{area of the circumscribed regular } 2^n\text{-gon}
\end{align*}
\]

By Lemma 2.5 we get

\[
\text{area}(Q_{n+1}) > \text{area}(Q_n) > \text{area}(P_n) < \text{area}(P_{n+1}).
\]

Thus, the sequences \(\{\text{area}(P_n)\}_{n \geq 2}\) and \(\{\text{area}(Q_n)\}_{n \geq 2}\) of real numbers are such that

\[
\text{area}(P_n) < \text{area}(C_1) < \text{area}(Q_n) \text{ for every } n \geq 2.
\]

The sequence \(\{\text{area}(P_n)\}_{n \geq 2}\) is strictly increasing and \(\{\text{area}(Q_n)\}_{n \geq 2}\) is strictly decreasing. Both sequences are coming closer and closer to the required value. Does it really happen? By the completeness property, there exist real numbers \(L_1\) and \(L_2\) such that

\[
|\text{area}(P_n) - L_1| \to 0, \text{ and } |\text{area}(Q_n) - L_2| \to 0.
\]

Using Lemma 3.1, it follows that that \(L_1 = L_2\).

5.4 Definition of \(\pi\)

The common value of \(L_1\) and \(L_2\) is called the area of the unit circle and is denoted by the Greek letter \(\pi\).

6 Circumference of Circle

For a circle of radius \(r\), if \(Q_n\) is a \(2^n\)-sided regular polygon circumscribing the circle, then

\[
\text{perimeter}(Q_n) \to 2\pi r.
\]

Theorem 6.1. For a circle of radius \(r\), if \(Q_n\) is a \(2^n\)-sided regular polygon circumscribing the circle, then

\[
\text{perimeter}(Q_n) \to 2\pi r.
\]
Proof. From Lemma (2.4), we have

\[ \text{area}(Q_n) = \frac{1}{2} r \text{perimeter}(Q_n), \]

where \( Q_n \) is a \( 2^n \)-sided regular polygon circumscribing the circle of radius \( r \). Now, using Theorem 4.2, we have

\[ \text{perimeter}(Q_n) \rightarrow 2\pi r. \]

\[ \square \]

Definition 6.1. For a circle of radius \( r \), its circumference is defined as \( 2\pi r \).

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