Counting Candy Sequences*  
An Enumeration Problem

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A bottle contains 7 Type A and 7 Type B candies. Each morning, you select two candies at random: If they are of opposite types, eat them both; otherwise, eat one and return the other to the bottle. Continue until all candies are eaten. If ever the bottle contains only one candy, select and eat it. How many distinct candy sequences are possible?

A Problem is Born

Do you have a sweet tooth? If you do, you will sympathize with Johnny, the ten-year-old son of a mathematician—an optimization theorist, to be precise. To optimize his trips to the candy store, the father bought 7 candies of Type A (apple flavored, wrapped in aqua wrapper) and 7 candies of Type B (butter flavored, wrapped in brown wrapper) and placed them in a bottle.

Concerned that Johnny might eat too many candies and ruin his health, and to teach him responsible behavior, Johnny’s mother imposed a rule: “Every morning, draw two candies at random. If they are of different types, eat them both. Otherwise, eat one and return the other to the bottle. What you eat each day, write on the wall calendar. If the bottle has only one candy left, then select and eat it. When the bottle is empty, ask your father for a refill.”

The mathematician could not pass by the opportunity to pose to Johnny (and anyone else who would listen) a math problem: “How many distinct candy sequences are there?” Two sequences are distinct if they differ with respect to what Johnny eats on at least one day.

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1. Brute Force Enumeration

Let us begin with very few candies of each type in the bottle. If the bottle contains one Type A candy and one Type B candy, then Johnny will eat them both on Day 1, because his only choice is to draw $AB$. Let us denote a pair of candies of opposite types $AB$ by the single letter $Z$. If the bottle contains two Type A candies and one Type B candy, then Johnny may eat either $AZ$ or $ZA$ on two successive days. Thus, there are two candy sequences.

If the bottle contains two Type A candies and two Type B candies, then on Day 1, three things can happen:

1. Johnny chooses $Z = AB$, eats both candies, and the bottle has one candy of each type, which he will eat on Day 2.

2. Johnny chooses $BB$, eats only one Type B candy, and the bottle has two Type A candies and one Type B candy, which he will eat in two more days as $AZ$ or $ZA$, as we already argued above.

3. Johnny chooses $AA$, eats only one Type A candy, and the bottle has one Type A candy and two Type B candies, which he will eat in two more days as $BZ$ or $ZB$.

Hence, beginning with two candies of each type, there are five possible sequences of candies: $ZZ, BAZ, BZA, ABZ, AYZ$. Note that there is at least one $Z$ in each sequence. Note also that these sequences happen with probabilities $(2/3, 1/18, 1/9, 1/18, 1/9)$, respectively. Moreover, note that the sequences differ in length (which denotes the number of days needed to eat all candies): All four candies are eaten in either two or three days (with probabilities $2/3$ and $1/3$, respectively). Stochastic aspects of the problem are studied in [1] by Anh Do et al. In this article, we focus on counting the candy sequences, which was not addressed in [1].

If the bottle initially contains 3 Type A and 3 Type B candies, then the 25 possible candy sequences are shown in Figure 1 as paths that start at the lower-left corner, move horizontally if a Type A candy is eaten (because $AA$ is drawn), vertically if a Type B candy
is eaten (because BB is drawn), and diagonally north-east if two candies \( AB = Z \) are eaten, and end at the upper-right corner. These same sequences may be written as words (sorted in lexicographic order), with the number of letters in each word indicating the number of days needed to eat all 6 candies.

\[
\begin{align*}
AABBZ & \quad AABZB & \quad AAZBB & \quad ABABZ & \quad ABAZB \\
ABBAZ & \quad ABBZA & \quad ABZZ & \quad AZBZ & \quad AZZB \\
BAABZ & \quad BAAZB & \quad BABAZ & \quad BABZA & \quad BAZZ \\
BBAAZ & \quad BBAZA & \quad BBZA & \quad BZAZ & \quad BZZA \\
ZABZ & \quad ZAZB & \quad ZBAZ & \quad ZBZA & \quad ZZZ
\end{align*}
\]

Ten-year-old Johnny stopped counting candy sequences further. But he did make a discovery: “For all \( a, b \geq 1 \), every sequence has a \( Z \), and the last \( Z \) is never followed by both A and B.”

**Figure 1.** If initially there are 3 Type A and 3 Type B candies in the bottle, then there are 25 possible candy sequences.

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**2. Recursive Computation**

The brute force enumerations need not be carried out further. However, it informs us that depending on which types of two candies are drawn on Day 1, the problem simplifies to either one

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Can you prove Johnny’s theorem?

Although tedious, brute force enumerations show us how to proceed more efficiently!
fewer candy of one type, or one fewer candy of both types. Hence, we can count the candy sequences more efficiently.

Suppose that initially, the container has $a$ type A candies and $b$ type B candies. Without loss of generality, assume $a \leq b$. Clearly, if $a = 0, b \geq 1$, then Johnny will eat one Type B candy every day. Hence, there is only one candy sequence. If $a = 1, b \geq 1$, then Johnny will eat one Type B candy each day, and the only Type A candy he will eat on Day $i = 1, 2, \ldots, b$. Thus, there are $b$ distinct candy sequences. Recall when two candy sequences are distinct.

When $a, b \geq 2$, we develop a recursive relation as follows: Let us denote the state of the candy selection process by $(i, j)$, where $i$ denotes the cumulative number of candies of Type A, and $j$ of Type B, already eaten. Let $N_{a,b}$ denote the number of distinct sequences of candy selections starting from state $(0, 0)$ and ending at state $(a, b)$. Clearly, for all $a, b \geq 1$, $N_{a,b} = N_{b,a}$, $N_{0,b} = 1, N_{1,b} = b$. Thereafter, by conditioning on the candies selected on Day 1, and letting the path go from state $(0, 0)$ to one of states $(1, 0), (1, 1), (0, 1)$, we obtain the recursive relation:

$$N_{a,b} = N_{a-1,b} + N_{a-1,b-1} + N_{a,b-1}. \quad (1)$$

For consistency, we choose to define $N_{0,0} = 1$. Table 1 lists the number of distinct sequences of candy selections or paths going from state $(0, 0)$ to state $(a, b)$ for $0 \leq a, b \leq 7$. It is computed by evaluating (1) using the following codes in the freeware R.

```r
### Computing candy sequence numbers
n=7  # candies of each type
A = matrix(0, nrow=n, ncol=n)
for (j in 1:n){A[1,j]=j}  # Row 1 of A
for (i in 2:n){
    for (j in 1:n){
        if (i>j) {A[i,j]=A[j,i]}
    }  # end j
}  # end i
```
A # print candy sequence numbers

<table>
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<th>a</th>
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<tbody>
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<tr>
<td>7</td>
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</tbody>
</table>

\[ N_{2,2} = 5 \quad \text{and} \quad N_{3,3} = 25 \]

In section 1, we had documented all \( N_{2,2} = 5 \) and \( N_{3,3} = 25 \) candy sequences. From the computations in Table 1, we learn that \( N_{7,7} = 19825 \). Thus, if the bottle contains 7 candies of Type A and 7 candies of Type B, then there are 19,825 distinct candy sequences until all candies are eaten. Can we generalize?

### Table 1. Number of distinct candy sequences, or paths going from state \((0,0)\) to state \((a,b)\), for \(0 \leq a, b \leq 7\).

When a particular problem is solved, try to generalize it.

3. **Analytic Formula**

Using the same R codes with larger values of \( a = b = n \), we computed some more central candy sequence numbers:

\[ N_{10,10} = 3,317,445, \quad N_{14,14} = 3.256957 \times 10^9, \]
\[ N_{30,30} \approx 3.980085 \cdots \times 10^{21}, \quad N_{100,100} \approx 8.497919 \cdots \times 10^{74}. \]

Of course, an analytic formula is desirable. For that purpose, we first study Delannoy numbers, named after the French army officer and amateur mathematician Henri–Auguste Delannoy (1833–1915) and published in 1895. See [2]. Thereafter, we connect the Delannoy numbers to the candy sequence numbers.

An analytic formula is desirable.
Figure 2. The number of paths on $G_D$ from $(0, 0)$ to $(a, b)$ is called the Delannoy number $D(a, b)$.

3.1 Delannoy Numbers

**Definition** (Delannoy numbers). Consider an integer lattice endowed with horizontal edges, vertical edges, and slope-one diagonals joining the (near) lattice points. See Figure 2. The number of paths on this graph $G_D$ going from $(0, 0)$ to $(a, b)$, always moving north, east, or north-east (that is, always moving closer to the destination), is called the Delannoy number $D(a, b)$.

Clearly, $D(0, a) = 1 = D(a, 0), D(1, a) = 2a + 1 = D(a, 1)$ for all $a = 1, 2, \ldots$, and for $a, b \geq 1$, $D(a, b) = D(b, a)$ and by conditioning on the initial segment going from $(0, 0)$ to one of points $(1, 0), (1, 1), (0, 1)$, the following recursive relation holds:

$$D(a, b) = D(a - 1, b) + D(a - 1, b - 1) + D(a, b - 1).$$  \hspace{1cm} (2)

Thus, Delannoy numbers satisfy the same recursive relation (1) as the candy sequence numbers $N_{a,b}$ do; but they begin with larger boundary values (given by the sequence of odd numbers instead of the sequence of natural numbers).

Again, for consistency, we choose to define $D(0, 0) = 1$. **Table 2** documents the Delannoy numbers $D(a, b)$ for $0 \leq a, b \leq 7$. The table is computed by evaluating (2) with a slight change in the R codes: On line 3, replace `A[1,j]=j` by `A[1,j]=2*j+1`. 

Each segment of a path must get you closer to the destination.

The same recursive relation, but different boundary values!
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<td>2241</td>
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<td>19825</td>
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</tr>
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</table>

**Remark 1:** Note that $D(b - 1, b) = N_{b,b} = D(b, b - 1)$ for all $b \geq 1$. Is this a coincidence? Do you see any other patterns between Tables 2 and 1? Wait until we derive analytic expressions.

Delannoy numbers have closed-form expressions, two of which are given in Theorem 1. The proof of the first formula is readily available in [2]. However, failing to find a documented proof of the second formula (the statement of which we did find in [2]), we constructed our own proof.

**Theorem 1:** For any $0 \leq a \leq b$, we have

$$D(a, b) = \sum_{k=0}^{a} \binom{a}{k}\binom{a + b - k}{a}.$$  \hspace{1cm} (3)

$$D(a, b) = \sum_{k=0}^{a} \binom{a}{k}\binom{b}{k}2^k.$$  \hspace{1cm} (4)

**Proof.** Formula (3) holds for $a = 0, b \geq 0$, since $1 = \binom{0}{0}\binom{b}{0}$. For $0 < a \leq b$, we evaluate $D(a, b)$ via direct counting. Decompose the paths from $(0, 0)$ to $(a, b)$ into subgroups according to the number of diagonal segments contained within each path. If there are $k$ diagonal segments, then there are $(a - k)$ horizontal segments and $(b - k)$ vertical segments. Conversely, any linear ordering of such number of segments constitutes a path. Using

Are there other patterns between Tables 2 and 1?

“Something old, something new, something borrowed, something blue.”

A smart decomposition aids us in counting.
Trinomial coefficients are products of two binomial coefficients.

The truth of formula (4) does not depend on where it come from.

Such definitions are born of deep insights.

Recursive relation (2) holds true.

Thus, recursive relation (2) holds, and the proof is complete. □
The infinite-dimensional lower triangular matrix $L$ with rows and columns indexed by $i = 0, 1, 2, \ldots$, introduced in the proof of Theorem 1, will be needed again. Therefore, let us explicitly write down the $i$-th row of $L$ as follows:

$$L_{i,*} = \left( \binom{i}{0}, \binom{i}{1}, \sqrt{2}, \binom{i}{2}, \binom{i}{1} \sqrt{3}, \ldots, \binom{i}{1} \sqrt{i}, 0, 0, \ldots \right). \quad (6)$$

In particular, using (4) and (3), the central Delannoy numbers (when $a = b$) are given by

$$D(a, a) = \sum_{k=0}^{a} \binom{a}{k}^{2} = \sum_{k=0}^{a} \binom{a}{k} \binom{2a-k}{a} = \sum_{l=0}^{a} \binom{a}{l} \binom{a+l}{l}. \quad (7)$$

The central Delannoy numbers are listed in the Online Encyclopedia of Integer Sequences (OEIS) as A001850. See [3]. Moreover, by partitioning these paths according as the farthest point from $(0, 0)$ just before either $i = a$ or $j = a$ or both, we have

$$D(a, a) = 2 \sum_{h=0}^{a-2} D(h, a-1) + 2 \sum_{k=0}^{a-2} D(a-1, k) + 3D(a-1, a-1)$$

$$= 4 \sum_{k=0}^{a-1} D(a-1, k) - D(a-1, a-1). \quad (8)$$

We are now ready to derive an analytic formula for the candy sequence numbers $N_{a,b}$.

### 3.2 Candy Sequence Numbers

The candy sequence number, $N_{a,b}$, counts the number of paths going from $(0, 0)$ to $(a, b)$, on a reduced graph $G_D \setminus R$, where the subset $R$ consists of $(a + b + 2)$ edges whose starting points are unreachable: $a$ north-leaning edges $[(h, b-1), (h, b)]$ for $h = 0, 1, \ldots, a-1$; $b$ east-leaning edges $[(a-1, k), (a, k)]$ for $k = 0, 1, \ldots, b-1$; and two more edges $[(a, 0), (a, 1)]$ and $[(0, b), (1, b)]$. See Figure 3, where a path goes east if a Type A candy is eaten (because AA is drawn), etc. The paths must avoid edges in $R$ because candies are drawn in pairs.

For easy access later, we explicitly write down the $i$-th row of $L$.

Another smart partitioning aids in efficient counting.

“Ready or not, here it comes!”

Since candies are drawn in pairs, permissible paths must avoid some edges on $G_D$. 

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**Figure 3.** The number of paths on $G_D \setminus R$ from $(0, 0)$ to $(a, b)$ is called the candy sequence number $N_{a,b}$.

What we observed in Remark 1, is indeed true! Theorem 2 proves it.

Can you anticipate Theorem 3?

Since candies are drawn in pairs, we must remove some inadmissible paths from all Delannoy paths.

Theorem 2 proves that $N_{a,a} = D(a, a-1) = D(a-1, a)$ for $a \geq 1$, a feature we had noted in Remark 1 by comparing Tables 1 and 2. These numbers are listed in the OEIS as A002002.

Theorem 3 expresses analytically the general candy sequence number $N_{a,b}$, for any $a, b \geq 1$, as a function of two Delannoy numbers: Can you guess which two Delannoy numbers? Can you guess which function of the two Delannoy numbers? There are two correct answers to these two questions.

**Theorem 2:** For any $a \geq 1$,

$$N_{a,a} = D(a-1, a) = \sum_{k=0}^{a-1} \binom{a}{k} \binom{a-1}{k} 2^k = \sum_{j=0}^{a-1} \binom{a}{j+1} \binom{a+j}{j}.$$

**Proof.** Starting from all paths joining $(0, 0)$ to $(a, b)$ on $G_D$, we remove those paths that do use exactly one of the edges in $R$. Note that no path can use more than one edge in $R$. Therefore,

$$N_{a,a} = D(a, a) - \sum_{h=0}^{a-1} D(h, a-1) - \sum_{k=0}^{a-1} D(a - 1, k)$$

$$= D(a, a) - 2 \sum_{k=0}^{a-1} D(a-1, k).$$

Doubling (9) and substituting (8), we have

$$2N(a, a) = D(a, a) - D(a - 1, a - 1) = 2D(a - 1, a).$$
Hence, \( N_{a,a} = D(a - 1, a) = \langle L_{a-1}, L_{a} \rangle \). Thereafter, substituting (4) and (3), we complete the proof.

As an illustration, we compute \( N_{4,4} = D(3, 4) = \langle L_{3}, L_{4} \rangle = 1 \cdot 1 \cdot 2^0 + 3 \cdot 4 \cdot 2 + 3 \cdot 6 \cdot 2^2 + 1 \cdot 4 \cdot 2^3 = 129 \).

**Theorem 3:** For any \( 1 \leq a \leq b \),

\[
N_{a,b} = \frac{[D(a - 1, b) + D(a, b - 1)])/2}{[D(a, b) - D(a - 1, b - 1)])/2}.
\]

\[
(10)
\]

\[
(11)
\]

**Proof 1.** As reasoned in (9), we also have

\[
N_{a,b} = D(a, b) - \sum_{h=0}^{a-1} D(h, b - 1) - \sum_{k=0}^{b-1} D(a - 1, k)
\]

\[
= \langle L_{a}, L_{b} \rangle - \sum_{h=0}^{a-1} \langle L_{h}, L_{b-1} \rangle - \sum_{k=0}^{b-1} \langle L_{a-1}, L_{k} \rangle
\]

\[
= \langle L_{a}, L_{b} \rangle - \langle L_{a}^{(1)} \sqrt{2}, L_{b-1} \rangle - \langle L_{a-1}, \frac{L_{b}}{\sqrt{2}} \rangle
\]

\[
(12)
\]

using (5), where \( L_{a}^{(1)} \) is the vector obtained by dropping the first element from the vector \( L_{a} \).

Again using (5), note that \( L_{b} = L_{b-1} + (0, \sqrt{2}L_{b-1}) \). Hence,

\[
\langle L_{a}, L_{b} \rangle = \langle L_{a}, L_{b-1} \rangle + 2 \langle L_{a}^{(1)} \sqrt{2}, L_{b-1} \rangle
\]

\[
(13)
\]

and likewise, since \( L_{a} = L_{a-1} + (0, \sqrt{2}L_{a-1}) \), we have

\[
\langle L_{a}, L_{b} \rangle = \langle L_{a-1}, L_{b} \rangle + 2 \langle L_{a-1}, \frac{L_{b}}{\sqrt{2}} \rangle
\]

\[
(14)
\]

Taking averages on the two sides of (13) and (14), we have

\[
\langle L_{a}, L_{b} \rangle = \frac{1}{2} [\langle L_{a-1}, L_{b} \rangle + \langle L_{a}, L_{b-1} \rangle]
\]

\[
+ \langle L_{a}^{(1)} \sqrt{2}, L_{b-1} \rangle + \langle L_{a-1}, \frac{L_{b}}{\sqrt{2}} \rangle
\]

which when substituted in (12), yields

\[
N_{a,b} = \frac{1}{2} [\langle L_{a-1}, L_{b} \rangle + \langle L_{a}, L_{b-1} \rangle]
\]

\[
= \frac{1}{2} [D(a - 1, b) + D(a, b - 1)],
\]
proving (10). Thereafter, (11) follows from (2). \hfill \Box

Thus, Theorem 3 tells us that $N_{a,b}$ is the average of two Delannoy numbers $D(a - 1, b)$ and $D(a, b - 1)$, as well as the half-difference between two Delannoy numbers $D(a, b)$ and $D(a - 1, b - 1)$. In keeping with the spirit of twos, let us provide a second (hopefully easier) proof of Theorem 3.

**Proof 2.** Clearly, when $a + b = 2$, we have $a = 1 = b$ for which $N_{1,1} = 1$, while $[D(0, 1) + D(1, 0)]/2 = [1 + 1]/2 = 1$ and $[D(1, 1) - D(0, 0)]/2 = [3 - 1]/2 = 1$. Thus, the theorem holds for $a + b = 2$.

If $a = 1, b \geq 2$, the theorem holds by mathematical induction on $b$ since $N_{1,b+1} = b + 1$, while $[D(0, b + 1) + D(1, b)]/2 = [1 + (2b + 1)]/2 = 1 + b$ as well as $[D(1, b + 1) - D(0, b)]/2 = [(2b + 2 + 1) - 1]/2 = b + 1$. Likewise, the theorem holds for $a \geq 2, b = 1$.

The rest of the proof is by complete (or strong) induction on $(a+b)$ for any $a, b \geq 1$. In the previous paragraph, we have checked that the theorem holds for all $a+b \leq 3$. Next, suppose that the theorem holds all $a+b \leq m$, for some $m \geq 3$. Then for any $a+b = m+1$, using (1), the induction hypothesis, algebraic rearrangement, (2) and (2) again, we have the following chain of equalities:

$$
N_{a,b} = N_{a-1,b} + N_{a-1,b-1} + N_{a,b-1} \\
= \frac{D(a - 1, b) - D(a - 2, b - 1)}{2} \\
+ \frac{D(a - 1, b - 1) - D(a - 2, b - 2)}{2} \\
+ \frac{D(a, b - 1) - D(a - 1, b - 2)}{2} \\
= \frac{D(a - 1, b) + D(a - 1, b - 1) + D(a, b - 1)}{2} \\
- \frac{D(a - 2, b - 1) + D(a - 2, b - 2) + D(a - 1, b - 2)}{2} \\
= \frac{D(a, b) - D(a - 1, b - 1)}{2} = \frac{D(a - 1, b) + D(a, b - 1)}{2}. \\
$$

The proof is now complete. \hfill \Box

**Remark 2:** Theorem 2 is recovered from Theorem 3 as a special case when we put $a = b$ in (10).
Remark 3: Substituting (3) and (4) in (10), for any \(0 \leq a \leq b\), we have

\[
N_{a,b} = \sum_{k=0}^{a} \binom{a}{k} \binom{a+b-k}{a} \frac{1}{2} \left( 1 - \frac{k}{a+b-k} \right) < \frac{1}{2} D(a,b)
\]

\[
N_{a,b} = \sum_{k=0}^{a} \binom{a}{k} \binom{b}{k} \left( 1 - \frac{k}{H} \right) 2^k > \left( 1 - \frac{a}{H} \right) D(a,b),
\]

where \(H = \frac{2ab}{a+b}\) is the harmonic mean of \(a\) and \(b\).

As illustration, we compute \(N_{4,6}\) using (11):

\[
D(4, 6) = 1 \cdot 1 \cdot 2^0 + 4 \cdot 6 \cdot 2 + 6 \cdot 15 \cdot 2^2 + 4 \cdot 20 \cdot 2^3 + 1 \cdot 15 \cdot 2^4 = 1289
\]

\[
D(3, 5) = 1 \cdot 1 \cdot 2^0 + 3 \cdot 5 \cdot 2 + 3 \cdot 10 \cdot 2^2 + 1 \cdot 10 \cdot 2^3 = 231.
\]

Hence, \(N_{4,6} = [1289 - 231]/2 = 529\). Alternatively, using (10) we compute \(N_{4,6} = [D(4, 5) + D(3, 6)]/2 = [681 + 377]/2 = 529\).

Delannoy numbers appear in many contexts. Interested readers may learn about them and their uses from [2, 4, 5, 6].

4. Variations

In this article, we have focused only on counting candy sequences. The stochastic aspects of the problem, such as \(X\), the number of days Johnnny needs to finish eating all candies, \(Y\), the number of days he eats two candies, and their mathematical expectations \(E[X]\) and \(E[Y]\) are studied in [1] by Ahn Do et al. (2022), who also study such random variables and their expectations under several alternative rules Mommy imposed on Johnny:

(1) “Eat both candies if they are different, but if they are the same, then eat none and return both candies to the bottle.”

(2) “Eat both candies if they are different, but if they are the same, then draw a third candy: If it is different, then eat one of each kind and return one duplicate kind; if it is the same, then return all three candies to the bottle.”
Mommy teaches Johnny a life lesson of sharing.

(3) “Eat both candies if they are different, but if they are the same, then eat one and give the other to your sister to eat.”

Under each of Rules (1) and (2), there are infinitely-many candy sequences because the number of days Johnny will eat no candy at all can be arbitrarily large. In fact, the number of days until all $a$ Type A candies are eaten is the sum of independent geometric random variables with success probabilities $(a - j)(b - j)/(a + b - 2j)$ for Rule (1), and $1 - \left(\frac{a - j}{3}\right) + \left(\frac{b - j}{3}\right) / (a + b - 2j)$ for Rule (2), for $j = 0, 1, \ldots, a - 1$. When all $a$ Type A candies are eaten, if only one Type B candy remains (that is, if $b = a + 1$), then on the next day, Johnny will choose and eat it. But if two or more Type B candies remain (that is, if $b \geq a + 2$), then those candies will never be eaten: They will be chosen in twos, augmented by a third under Rule (2), if possible, and returned to the bottle. Poor Johnny will never get a refill!

Under Rule (3), every day, two candies are eaten either by Johnny alone or by Johnny and his sister, except maybe on the last day. Hence, all candies are eaten in exactly $[(a + b)/2]$ days, after which Johnny gets a refill. Suppose that, under Rule (3), altogether, there are $M_{a,b} = M_{b,a}$ candy sequences until all candies are eaten. If $a = 0$, then $M_{0,b} = 1$, for all $b \geq 1$; and if $a = 1$, then accounting for the day when the single Type A candy is eaten, $M_{1,b} = [(1 + b)/2]$, for all $b \geq 0$. Finally, for $2 \leq a \leq b$, we have

$$M_{a,b} = M_{a-1,b-1} + M_{a-2,b} + M_{a,b-2}.$$  \hspace{1cm} (15)

The recursive relation (15) is different from (1) and (2); of course, the initial conditions also differ. Below we give the R codes to compute $M_{a,b}$ (by modifying the R codes to compute $N_{a,b}$). We document $M_{a,b}$ (for $0 \leq a, b \leq 7$) in Table 3. Again, for consistency, we define $M_{0,0} = 1$.

### Candy sequence numbers under Rule (3)

# If AB, eat both; else, eat 1, give 1 to sister
n=7+1 # add 1 since starting from Row 0
B = matrix(0, nrow=n, ncol=n)
for (j in 1:n){B[1,j]=1}  # Row 0
for (j in 1:n){B[2,j]=ceiling(j/2)}  # Row 1
for (i in 3:n){
  for (j in 1:n){
    if (i>j) {B[i,j]=B[j,i]}
  }  # end j
}  # end i
M  # print candy sequence numbers

<table>
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<th>3</th>
<th>4</th>
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<td>126</td>
<td>267</td>
<td>393</td>
</tr>
</tbody>
</table>

Table 3. Number of distinct candy sequences, or paths going from state (0, 0) to state (a, b), under Rule (3) for 0 ≤ a, b ≤ 7.

Executing the R codes for larger a = b = n, we note that

\[ M_{7,7} = 393; \quad M_{10,10} = 8953; \quad M_{14,14} = 616227 \]
\[ M_{30,30} \approx 1.825203 \cdots \times 10^{13}; \quad M_{100,100} = 2.513427 \cdots \times 10^{46}. \]

Indeed, \{M_{a,a}, a \geq 0\} = \{1, 1, 3, 7, 19, 51, 141, 393, 1107, \ldots\} is listed as A002426 in the OEIS; they are called central trinomial coefficients [the coefficient of \(X^a\) in \((1 + X + X^2)^n\)]; they were discovered by Euler in 1763 to have an analytic expression

\[ M_{a,a} = \sum_{k=0}^{\lfloor a/2 \rfloor} \binom{a}{k, a-2k, k} = \sum_{k=0}^{\lfloor a/2 \rfloor} \binom{a}{2k} \binom{2k}{k}. \]

Unlike humans, the computer does not complain about such repetitive computations.

Having no computer available, how did Euler discover it? No doubt, he possessed impressive mathematical ingenuity.
Similarly, the reader can derive $M_{a,b}$ for $1 \leq a < b$ as

$$M_{a,b} = \begin{cases} \sum_{k=0}^{\lfloor a/2 \rfloor} \binom{c}{c-a+k,a-2k,k} = \sum_{k=0}^{\lfloor a/2 \rfloor} \binom{c}{a-k} \binom{a-k}{k} & \text{if } a + b = 2c \\ \sum_{k=0}^{\lfloor a/2 \rfloor} \left[ \binom{c}{c-a+k,a-2k,k} + \binom{c}{c-a+1+k,a-1-2k,k} \right] & \text{if } a + b = 2c + 1 \end{cases}$$

Let us end the article by posing a new counting problem.

**New Problem:** Suppose that a bottle has 7 Type A, 7 Type B and 7 Type C candies. Each morning, you select three candies at random, eat only the distinct types of candies selected, and return any duplicates to the bottle. Continue until the bottle is empty. If ever the bottle has fewer than three candies, you will choose them all, etc. How many distinct candy sequences are there? Hint: Use a 3-D extension of a Delannoy graph, reduced appropriately. Please write to me after you solve this problem.

**Acknowledgement**

I thank the 2022 IUPUI High School Mathematics Committee for choosing the computation of $N_{10,10}$ as a contest problem and then demanding an analytic solution for the general case. I also thank Debolina Chatterjee for giving me feedback on an earlier draft.

**Suggested Reading**


