Linear Maps Between the Sequence Spaces $c$ and $c_o$

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This article aims to provide an elementary proof for the non-existence of a linear isometry from the Banach space $c$ to the Banach space $c_o$. This elementary proof was communicated to the first author by Prof. William B. Johnson. It is emphasized that an isometry need not be onto. In order to place this result in an appropriate perspective, an elementary proof for the existence of a linear homeomorphism between the Banach spaces $c$ and $c_o$ is also given. Also, the readers’ attention is drawn to the fact that $c$ and $c_o$ are not nearly isometric to each other.

In [1], Banach called two Banach spaces $X$ and $Y$ to be nearly isometric if the Banach–Mazur distance $(X, Y)$ between them is 0. It should be noted that this is essentially a short expository article with a touch of history.

Introduction

If $X$ is a non-empty set, then a sequence in $X$ is nothing but a function $f : \mathbb{N} \to X$, where $\mathbb{N}$ is the set of natural numbers. Usually, we denote $f(n)$ by $x_n$ and the sequence $f$ by $(x_n)$. Further, $X^\mathbb{N}$ denotes the set of all sequences in $X$, that is, $X^\mathbb{N} = \{x = (x_n) : (x_n) \text{ is a sequence in } X\}$. When $X = \mathbb{R}$ or $\mathbb{C}$, we call a member of $\mathbb{R}^\mathbb{N}$ or $\mathbb{C}^\mathbb{N}$ real or complex sequence respectively. Note that both $\mathbb{R}^\mathbb{N}$ and $\mathbb{C}^\mathbb{N}$ are linear spaces over $\mathbb{R}$ and $\mathbb{C}$ respectively—vector addition and scalar multiplication are defined co-ordinate (or component) wise, where $x_n$ is called the $n$-th co-ordinate of the sequence $x = (x_n)$. For the scalar field of these linear spaces, we use a notation $\mathbb{K}$, that is, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Keywords
Sequence spaces, linear homeomorphism, linear isometry.

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Among the simplest and most important normed linear and Banach spaces are those of sequences—linear subspaces of the linear space $\mathbb{K}^\mathbb{N}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Now we explain the following subspaces of $\mathbb{K}^\mathbb{N}$:

(a) $\ell_\infty = \{ x = (x_n) : x \text{ is a bounded sequence in } \mathbb{K}, \text{ that is, } \exists M_x > 0 \text{ (depending on } x) \text{ such that } |x_n| < M_x \forall n \}.$

(b) $c = \{ x = (x_n) \in \ell_\infty : (x_n) \text{ converges in } \mathbb{K}, \text{ that is, } x_n \to k \text{ in } \mathbb{K} \}.$

(c) $c_0 = \{ x = (x_n) \in \ell_\infty : x_n \to 0 \text{ in } \mathbb{K} \}.$

Note that $c_0 \subseteq c \subseteq \ell_\infty$. On $\ell_\infty$, we can assign the following supremum norm: $||x||_\infty = \sup_n |x_n|.$

Both $c$ and $c_0$ are Banach spaces and the dual of each of them can be identified with the space $l_1$. It is known that each of $\ell_\infty$, $c$ and $c_0$ is a Banach space. Since $c_0 \subsetneq c$ and $c_0 \neq c$, the inclusion map from $c_0$ to $c$ is a linear isometry which is not onto. Note that an isometry is always one-to-one. In this short note, our main query is: Does there exist a linear isometry from $c$ to $c_0$? This query assumes significance due to the fact that both $c$ and $c_0$ are Banach spaces, and the dual of each of them can be identified with the space $l_1$, where $l_1 = \{ x = (x_n) \in \mathbb{K}^\mathbb{N} : \sum_{n=1}^{\infty} |x_n| < \infty \}$ and the norm $\| \cdot \|_1$ on $l_1$ is defined by $\|x\|_1 = \|(x_n)\|_1 = \sum_{n=1}^{\infty} |x_n|$. Here, we would like to emphasize that on various discussion forums (available online, for example, [2]), one may find proofs for the non-existence of an onto linear isometry, that is, the non-existence of a surjective linear isometry from $c$ to $c_0$ or they provide proofs for the non-existence of a linear isometry from $c$ to $c_0$ that require a deeper knowledge of the subject. In this article, our main focus is to provide an elementary proof for the stronger assertion.

In order to answer the aforesaid query, first we go for a weaker linear map. We now show that $c$ and $c_0$ are linearly homeomorphic, that is, there exists a linear bijection $T : c \to c_0$ which is a homeomorphism as well, that is, both $T$ and $T^{-1}$ are continuous.
Theorem 1. There exists a linear homeomorphism \( T : (c, \| \cdot \|_\infty) \to (c_0, \| \cdot \|_\infty) \).

Proof. Consider the mapping \( T : c \to c_0 \) defined by:

\[
T(x) = T((x_n)) = T(x_1, x_2, \ldots, x_n, \ldots) = (x_{\infty}, x_1 - x_{\infty}, x_2 - x_{\infty}, \ldots),
\]

where \( x_{\infty} = \lim_{n \to \infty} x_n \), that is, the sequence \( x = (x_n) \) converges to \( x_{\infty} \) in \( c_0 \). It is easy to verify that \( T \) is linear and one-to-one. Now if \( y = (y_1, y_2, \ldots) \in c_0 \), then \( y_n \to 0 \) and hence \( y_1 + y_n \to y_1 \), that is, the sequence \( (y_1 + y_n) \) belongs to \( c \). But \( y = (y_1, y_2, \ldots) = T(y_1 + y_2, y_1 + y_3, \ldots) \). Hence \( T \) is also onto, and consequently, \( T \) is a linear bijection.

Now, if \( x = (x_n) \in c \), then \( |x_n - x_{\infty}| \leq |x_n| + |x_{\infty}| \quad \forall \ n \in \mathbb{N} \). But \( \|x\|_{\infty} = \sup_n |x_n| \Rightarrow |x_n| \leq \|x\|_{\infty} \quad \forall \ n \in \mathbb{N} \). Thus, \( \lim_{n \to \infty} x_n \leq \|x\|_{\infty} \Rightarrow |x_{\infty}| \leq \|x\|_{\infty} \Rightarrow |x_{\infty}| \leq \|x\|_{\infty} \). Hence \( |x_n - x_{\infty}| \leq 2\|x\|_{\infty} \quad \forall \ n \in \mathbb{N} \) and so \( \|Tx\|_{\infty} \leq 2\|x\|_{\infty} \quad \forall \ x \in c \). Hence \( T \) is continuous at 0. Since \( T \) is linear, \( T \) is continuous on \( c \). Also, for \( n \in \mathbb{N} \) \( |x_n| = |x_n - x_{\infty} + x_{\infty}| \leq |x_n - x_{\infty}| + |x_{\infty}| \leq 2\|Tx\|_{\infty} \) and hence \( \|x\|_{\infty} = \sup_n |x_n| \leq 2\|Tx\|_{\infty} \). By taking \( y = Tx \), we see that \( \|T^{-1}y\|_{\infty} \leq 2\|y\|_{\infty} \quad \forall \ y \in c_0 \). Hence \( T^{-1} \) is continuous at 0. But \( T^{-1} \) also being linear, \( T^{-1} \) is continuous on \( c_0 \). Hence \( T \) is a linear homeomorphism.

Remark. Note that both \( c \) and \( c_0 \) are Banach spaces. Hence if \( T : c \to c_0 \) is a linear isomorphism as well as bounded (that is, continuous), then \( T^{-1} : c_0 \to c \), by the open mapping theorem, is also bounded. That is, \( T \) becomes a linear homeomorphism. Hence a linear isomorphism \( T : c \to c_0 \) is a linear homeomorphism if and only if \( T \) is continuous, and so we can talk of continuous isomorphisms of \( c \) onto \( c_0 \).

In [1, p.242], Banach–Mazur distance \( (X, Y) \) between two Banach spaces \( X \) and \( Y \) was defined to be \( \inf \{ \log \|T\| \|T^{-1}\| \} \), where \( T \) runs through all linear homeomorphisms of \( X \) onto \( Y \). If \( (X, Y) = 0 \), then \( X \) and \( Y \) are called nearly isometric to each other. Note that the Banach–Mazur distance is a metric on the space of Banach spaces.
that if there exists an onto isometry $T$ from $X$ to $Y$, then $\|T\| = 1 = \|T^{-1}\|$ and consequently, $(X, Y) = 0$.

In particular, in [1], it has been queried if $c$ and $c_o$ are nearly isometric to each other -- (★). Apparently, it was already known that $c$ and $c_o$ are not isometric to each other.

In [3], it was shown that $\|T\| \cdot \|T^{-1}\| \geq 2$ for each linear homeomorphism $T : c \rightarrow c_o$. While in [4], it was established that any linear homeomorphism $T$ of $c$ onto $c_o$ must satisfy $\|T\| \cdot \|T^{-1}\| \geq 3$. Moreover, the existence of a linear homeomorphism $T$ of $c$ onto $c_o$ with $\|T\| \cdot \|T^{-1}\| = 3$ was also proved in [4]. Therefore, this result of Cambern gives a complete solution to Banach’s query stated in (★). In particular, $c$ and $c_o$ are not even nearly isometric to each other, let alone being isometric.

Now in order to show that there is no linear isometry from $c$ to $c_o$, we need to observe that if $z_1$ and $z_2$ are complex numbers satisfying $|z_1 + z_2| = 2$ and $|z_1| = 1 = |z_2|$, then $z_1 = z_2$. Also, we thank Prof. William B. Johnson for communicating to us a simple proof of the following result. Here again, we would like to emphasize that an isometry need not be onto.

**Theorem 2.** There is no linear isometry from $c$ to $c_o$.

**Proof.** [5] Suppose that $T : c \rightarrow c_o$ is a linear isometry if possible. Note that the sequence $e = (1, 1, \ldots)$ belongs to $c$, but not to $c_o$. Here, $e = (\gamma_n)$ where $\gamma_n = 1 \ \forall \ n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $e_n = (\alpha_m)$ where $\alpha_m = 0$ if $m \neq n$ and $\alpha_m = 1$.

Since $\|e + e_n\|_\infty = 2 \ \forall \ n \in \mathbb{N}$, $\|T(e + e_n)\|_\infty = 2 \ \forall \ n \in \mathbb{N}$. Hence there exists $t_n \in \mathbb{N}$ such that $|T(e + e_n)(t_n)| = 2$, that is, $|T(e)(t_n) + T(e_n)(t_n)| = 2$. – (★★). Note that $|T(e)(t_n) + T(e_n)(t_n)| \leq |T(e)(t_n)| + |T(e_n)(t_n)|$. Since $\|e\|_\infty = 1$, $\|T(e)\|_\infty = 1$ and hence $|T(e)(t_n)| \leq 1 \ \forall \ n \in \mathbb{N}$. Similarly $|T(e_n)(t_n)| \leq 1 \ \forall \ n \in \mathbb{N}$. But since $|T(e)(t_n) + T(e_n)(t_n)| = 2$, we must have $|T(e)(t_n)| = 1 = |T(e_n)(t_n)|$. Hence by the observation made before this result, $T(e)(t_n) = T(e_n)(t_n) \ \forall \ n \in \mathbb{N}$.

Now for $n \neq m$, $\|e_n + e_m\|_\infty = 1$. So for $n \neq m$, $\|T(e_n + e_m)\|_\infty = 1$. Therefore, $|T(e_n + e_m)(t_n)| \leq 1$. Now if for some $n \neq m$, $\|T(e_n + e_m)\|_\infty = 1$. Then for $n \neq m$, $|T(e_n + e_m)(t_n)| = 1$. This implies that $T(e_n + e_m)(t_n)$ is a constant function. But since $T$ is a linear isometry, $T(e_n + e_m)$ is an isometry. Therefore, $T(e_n + e_m)$ must be the zero function. However, this contradicts the fact that $\|T(e_n + e_m)\|_\infty = 1$. Hence, there is no linear isometry from $c$ to $c_o$. \hfill $\square$
$t_n = t_m$, then $|T(e_n)(t_n) + T(e_m)(t_n)| = |T(e_n)(t_n) + T(e_m)(t_m)| = 2|T(e)(t_n)| = 2$. We arrive at a contradiction. Hence if $n \neq m$, then $t_n \neq t_m$. Therefore, $|T(e)(t_n)| = 1$ for infinitely many distinct $t_n$’s. But this is impossible since $T(e) \in c_o$. Therefore, there is no linear isometry from $c$ to $c_o$. \hfill \Box

**Corollary 1.** [6, p.34, Prob. 1.49(c)] There is no surjective linear isometry from $c$ to $c_o$.

Recall that a point $p$ in a non-empty convex subset $A$ of a vector space $V$ is an extreme point (also known as an extremal point) of $A$ if and only if

$$p = \frac{1}{2}(x + y) \text{ with } x, y \in A \Rightarrow p = x = y.$$ 

Those who are familiar with the concept of extreme points can also prove Theorem 2 using the following:

(a) the closed unit ball in $c_o$ has no extreme point, while the closed unit ball in $c$ does have.

(b) the closed unit ball of any infinite-dimensional subspace of $c_0$ has no extreme points.

At the beginning of this article, we have already mentioned that the scalar field can be either $\mathbb{R}$ or $\mathbb{C}$. Though in [3, 4], the scalar field was taken to be $\mathbb{C}$, the results of these papers are equally valid if we choose $\mathbb{R}$ as the scalar field. See the emphasis on the scalar field given at the beginning of the article [7].

**Suggested Reading**