Is There Still Room for Paradoxes in Special Relativity*

In this article, we want to analyze a relativistic paradox in a very simple way, which, in our opinion, could be useful for a pedagogical purpose. Indeed we want to solve it using only Einstein’s relativity principle and the Einstein equivalence principle. We think it could be useful from a pedagogical point of view.

1. Introduction

In the special theory of relativity (SR), the relativity principle is visible, from the mathematical point of view, in the symmetry of transformations. Indeed, if for the observer $A$, in the reference frame of $B$, the lengths are contracted and time passes more slowly; for the observer $B$, it is exactly the opposite. It is well known that both are right. A well-known consequence of relativistic time dilation is the so-called transverse Doppler effect. If there is a source $A$ that emits an electromagnetic impulse with frequency $\nu_e$, and that is moving transverse to the line of sight at a speed $\nu$, the receiver $B$ measures a frequency:

$$\nu_r = \nu_e \sqrt{1 - \frac{\nu^2}{c^2}}.$$  \hspace{1cm} (1)

By remembering the redshift parameter

$$z = \frac{\nu_e - \nu_r}{\nu_r},$$ \hspace{1cm} (2)

the receiver measures

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\[ z = \frac{v_L - v_r}{v_r} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \approx 1 + \frac{1}{2} \frac{v^2}{c^2} - 1 = \frac{1}{2} \frac{v^2}{c^2}. \] (3)

Obviously, from the principle of relativity, if \( B \) is the source, it is observer \( A \) who measures a decrease in frequency and, therefore, a redshift. It is not possible to say that one or the other clock is absolutely running slower. For accelerated motion, instead, the spacetime transformations are not symmetric, and the two observers agree, for example, on which of the two clocks is slower. In other words, the slowdown of time is absolute, and the accelerated clock is absolutely slowed compared to the non-accelerated clock. Also, in this case, from the inertial observer point of view, time dilation is a consequence of SR. Indeed, from the so-called ‘Einstein’s clock hypothesis’, if a frame is accelerating, the time rate of the clock is equal to that of a co-moving unaccelerated frame. Instead, for the observer who is accelerating, the slowing of time is caused, thanks to the Einstein equivalence principle, by the ‘gravitational’ potential.

2. Selleri Paradox

The starting point of Selleri reflections is the so-called Sagnac effect [1–21]. There is a platform that rotates with an angular velocity \( \omega \), and there is an observer \( O \) at rest on the platform at distance \( r \) from the center. If \( O \) sends two electromagnetic signals in the opposite directions around a circumference, the counter-rotating ray will return to the starting point \( O \) earlier than the co-rotating one. This is an extremely simplified version of the experiment. Indeed the light beam is bent around in a loop by mirrors or an optical fiber. Let us consider an inertial reference frame \( O_i \). For example, the laboratory where the platform is rotating. For \( O_i \), the co-rotating ray must travel a distance greater than the ray traveling in the opposite direction, thus:

\[ c \Delta t_1 = 2 \pi r + \omega r \Delta t_1, \] (4)
\[ c \Delta t_2 = 2\pi r - \omega r \Delta t_2. \] (5)

Therefore,

\[ \Delta t_1 = \frac{2\pi r}{c - \omega r}, \] (6)

\[ \Delta t_2 = \frac{2\pi r}{c + \omega r}, \] (7)

going

\[ \Delta \tau = \Delta t_1 - \Delta t_2 = \frac{4\pi r^2 \omega}{c^2 - \omega^2 r^2} = \frac{4\pi r^2 \omega}{c^2 \left(1 - \frac{\omega^2 r^2}{c^2}\right)}. \] (8)

For the observer O, at rest with respect to the platform, the time is slower and he measures

\[ \Delta t_O = \Delta \tau \sqrt{1 - \frac{\omega^2 r^2}{c^2}} = \frac{4\pi r^2 \omega}{c^2 \sqrt{1 - \frac{\omega^2 r^2}{c^2}}}, \] (9)

despite, from his point of view, the length of the journey is the same in both directions. It is also possible to show that, from his perspective, the speed of light in the two directions is:

\[ \left\{ \begin{array}{l}
|v_1| = \frac{c}{c + \omega r} \\
|v_2| = \frac{c}{c - \omega r}.
\end{array} \right. \] (10)

Italian physicist Selleri considered the phase shift unexplainable within the context of SR and developed a paradox called the Selleri paradox. The previous relations (10) are typically shown in general relativity (GR) textbooks. We want to underline that the anisotropy of the speed of light is not a big surprise in GR as in many of Einstein’s works it is explained [22]. The Italian physicist Selleri conceived a paradox starting from this experiment. He considered the phase shift unexplainable within the context of SR and developed the aforementioned paradox, which appears, in his opinion,
to undermine the theory [23]. If we consider the ratio between the speeds of light given by (10), we obtain

\[
\frac{1 - \beta}{1 + \beta} \neq 1,
\]

(11)

where \(\beta = \frac{vr}{c} \). He notes that the result (11) holds if \( r \to \infty \) and \( \omega \to 0 \) but the product \( \omega r \) remains constant. He writes: "Suppose that one builds a set of circular platforms with radii

\[ R_1, \ldots, R_i, \ldots (R_1 < \ldots R_i < \ldots), \]

and makes them spin with angular velocities \( \omega_1, \ldots, \omega_i, \ldots \) in such away that

\[
\omega_1 R_1 = \omega_2 R_2 = \ldots \omega_i R_i = v,
\]

(12)

where \( v \) is a constant velocity.

Obviously, (11) applies to all such platforms, but the respective centripetal accelerations \( \omega^2 R \) will tend to zero with growing \( R_i \). For this reason, a little piece of the rim of a platform with a very large radius for a short time will be equivalent to an inertial frame. Therefore he states that, for this inertial frame, the speeds of light in the forward and backward directions are different in violation of SR.

3. Relativity and Equivalence Principles

It is obvious that SR is a very successful theory, and its theoretical predictions are confirmed in every type of experiment, and no contradictions arise. So something must be wrong, and indeed, the 'paradox' has been resolved in the literature. For example, a well-written paper is [24]. The problem, in our opinion, is that these explanations require a thorough knowledge of GR and are not accessible to undergraduate students. For this reason, we want to follow a simple reasoning suitable for high school. First of all,
we can observe that the ‘gravitational potential’ \( \frac{1}{2} \omega^2 R^2 \) is constant as you go from one of Selleri’s platforms to the next. We want to underline that it is the gravitational potential that breaks the symmetry of SR. Thus a little piece of the rim of a platform will be Minkowskian when the ‘gravitational potential’ goes to zero, not when the acceleration \( (\omega^2 R) \) goes to zero, as assumed by Selleri. This is the essence of our reflections: Selleri’s limiting frame is not inertial, so there’s no paradox. To better clarify this concept, let us remember that Kündig was the first who tested, in a more precise way, the validity of the transverse Doppler shift we mentioned at the beginning of this article. Indeed, he performed an experiment sending gamma rays from the center of a rotating disk [25–27]. On the rim, there is a revolving absorber, and we expect the Lorentz time distortion

\[
    t_r = \frac{t_L}{\sqrt{1 - \frac{\omega^2 R^2}{c^2}}},
\]

(13)

where \( t_L \) is the laboratory time while \( t_r \) is the time measured by the revolving watch. Moreover, \( \omega \) is the angular velocity of the rotor, \( r \) is the distance between the center and the rim, and \( c \) is the speed of light. For the observer in the laboratory, time dilation is a consequence of SR, while thanks to the Einstein equivalence principle, for the revolving observer, it is a consequence of the centrifugal gravitational potential. In both cases, we should have a negative shift, and that is a blue-shift

\[
    z \approx -\frac{1}{2} \frac{\omega^2 R^2}{c^2},
\]

(14)

By introducing a constant \( k \) such that

\[
    z \approx -k \frac{1}{2} \frac{\omega^2 R^2}{c^2},
\]

(15)

Kündig found

\[
    k = 1.0065 \pm 0.011,
\]

(16)
in excellent agreement with relation (14). In summary, if the receiver is moving in a circle around the source, we have a blueshift, while if the source is moving in a circle around the absorber, we have a redshift. There is no symmetry. In the case of SR, instead, from relation (3), we see that the receiver always measures a positive shift and that is a redshift if the radiation is received along a direction with an angle of $\frac{\pi}{2}$ as always happens to Selleri observer. Indeed, we repeat that, with the inertial frame, the situation is perfectly symmetrical, and from the receiver point of view, it is the emitter that is moving, and it is the time of the emitter that is slowing down. In Selleri’s hypotheses, this symmetry is never recovered. Instead, in Selleri’s hypothesis, the receiver is subject to the constant centrifugal potential, and it is its time that is absolutely slowed down, and it will always measure a blueshift. To become an inertial system, it must measure a redshift recovering the perfect symmetry of SR. This occurs if the inertial potentials disappear, and this will never happen if $\omega R$ is constant. Therefore, for the Selleri observer, the speeds of light in the forward and backward directions are different as a standard consequence of GR.

4. Conclusions

In this pedagogical letter, starting from the well-known Sagnac effect, we have analyzed a relativistic paradox that an Italian physicist had proposed. At this late date, SR is not open to serious challenges; there must be something wrong with Selleri’s paradox. However, the published explanations, although rigorous, are very complex for undergraduates. We have proposed a very simple way to solve it, with little mathematics using the principle of relativity and the equivalence principle.

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Suggested Reading


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