Traditionally, students are taught to find the equivalent resistance between two arbitrary nodes of an electrical circuit using Kirchhoff’s laws. In this article, we introduce a procedure for finding the equivalent resistance based on a variational principle that is consistent with the Kirchhoff’s laws. This method is easy to track for a nontrivial arrangement of resistors with large numbers of nodes and conceptually straightforward to implement for a circuit whose components may not be in series or parallel with each other, such as the familiar Wheatstone bridge. Our pedagogical method, with lower computational costs than the Kirchhoff’s laws, is accessible to undergraduates with some background in matrix algebra and calculus.

1 Introduction

The problem of finding the equivalent resistance between a pair of nodes for any generic circuit is more than a century old. This problem is analogous to finding the equivalent capacitance of a circuit and the equivalent thermal conductivity of a material from Fourier’s law. This seemingly trivial problem has been investigated extensively with applications in different disciplines. In physics, the knowledge of equivalent resistances is used in electrical circuits, and in transport and percolation theory. In mathematics, the equivalent resistance is, in fact, a measure of distance known as ‘resistance distance’ [1]. Equivalent resistances are also applied to study random walks on finite networks that are
regarded as Markov chains, and in graph theory [2]. These examples highlight a few of the applications of finding the equivalent resistance of an electrical circuit.

The traditional approach to finding this equivalent resistance that undergraduates learn is through Kirchhoff’s laws [3]. However, it is a formidable task to find the equivalent resistance for a non-trivial arrangement of resistors using Kirchhoff’s laws. The complexity of the problem increases rapidly as the number of nodes in the circuit increases. Students are also susceptible to conceptual difficulties to find the equivalent resistance in circuits, which can be neither separated into series or parallel or a combination of the two. The usual equivalent resistance formulas for series and parallel resistors may not be directly applicable in such circuits, and they may require non-intuitive solutions and special tricks.

In this article, we present a simple recipe to find the equivalent resistance between arbitrary nodes of a circuit from the variational principle [4, 5], which is consistent with the nodal analysis of Kirchhoff’s laws [6]. The method to find the equivalent resistance from the variational approach in electrical circuits is pedagogical in nature, universally applicable to all resistor circuits, computationally inexpensive, and easy to implement conceptually. Undergraduates will benefit immensely by seeing a new strategy with lower computational complexity since the equivalent resistance between all possible pairs of nodes in the circuit is computed simultaneously with the eigendecomposition of a single matrix.

Unlike previous work on finding the equivalent resistance in a circuit, no assumptions are made such as the resistances are equal to each other [12] or the resistances are placed on a lattice [13] or the circuit is symmetric in structure [14].

This article is organized as follows: In Sec. 2, we state the variational principle for an electrical circuit with an example that lays the foundation of the article. In Sec. 3, we demonstrate a strategy to find the equivalent resistance in circuits (such as the Wheatstone bridge) with the variational principle. For interested readers, we have included some problems in Sec. 4 that enhance the understanding of the material presented here. We conclude with
2 Statement of the Variational Principle for an Electrical Circuit

The variational principle states that given an electrical circuit with resistances assigned between the nodes in the circuit, the potentials (voltages) on the nodes are distributed such that the total electric power in the circuit is minimized [4, 5]. The constraint for this variational problem is that a constant voltage difference is maintained between two nodes in the circuit that is fixed by the external battery. The equivalent resistance will be calculated between the two nodes that have a fixed potential difference set by the e.m.f. $\mathcal{E}$ of the battery. The potentials obtained for the remaining nodes of the circuit through this minimization method are consistent with Kirchhoff’s circuit laws as shown in Sec. 3.

It is well-known that for any resistor circuit, the electric power $P$ dissipated across a resistance $R$ is proportional to the potential drop across the resistance. In fact, the total power dissipated in a circuit is proportional to the square of the e.m.f. $\mathcal{E}$ of the circuit [15]. The proportionality constant for this relation is the equivalent resistance $R_{\text{equ}}$ between the two nodes across which the potential difference is fixed by the battery. The mathematical relationship between the total power dissipated in a circuit and the e.m.f. of the circuit is

$$P_{\text{tot}} = \frac{\mathcal{E}^2}{R_{\text{equ}}} \quad (1)$$

We will now demonstrate how the equivalent resistance for a cir-
circuit with two resistors in series can be found with the variational principle. For the series circuit in Figure 1, consider the potentials on nodes (1) and (3) to be denoted by $E$ and 0 respectively, with the potential difference $E$ between them held fixed. This choice of nodal potentials is consistent with Kirchhoff’s laws since the voltage drops by $E$ across a battery of fixed e.m.f. $E$ and there are no resistors between the positive (and negative) ends of the battery and nodes (1) and (3). The variable potential on node (2) is denoted by $y$. The total electric power dissipated by this circuit is

$$P(y) = \frac{(E - y)^2}{R_1} + \frac{(y - 0)^2}{R_2}. \quad (2)$$

Before we find the equivalent resistance, we list two limiting cases to draw physical insights into the variational principle.

- $R_1 \gg R_2$: Intuitively, there should be a big potential drop across the nodes of a circuit with a large resistance between them. From the variational principle point of view, to minimize the total power in Eq. (2), the numerator of the second term must be as small as possible to compensate for the large contribution from $1/R_2$. Thus we expect $y \sim 0$, such that there is a big potential drop across the large resistance $R_1$ and $E - y \sim E$.

- $R_1 \ll R_2$: Contrary to the previous case, the potential drop across a small resistance in a circuit should be negligible. Thus from the variational principle’s viewpoint, the term $(E - y)^2$ must be small ($\sim 0$) to minimize the large contribution of $1/R_1$ compared to $1/R_2$.

According to the variational principle, to obtain the equivalent resistance, the scalar function $P(y)$ in Eq. (2) needs to be minimized with respect to $y$. The same potential would be obtained, had Kirchhoff’s laws been used instead.
The potential \( y^* \) that minimizes the function \( P(y) \) is the solution to the equation \( P'(y) = 0 \):

\[
y^* = \frac{R_2}{R_1 + R_2} \mathcal{E}.
\]  

(3)

This result in Eq. (3) confirms the intuitive predictions for the two limiting cases.

For the rest of the article, we will focus on finding the equivalent resistances rather than the nodal potentials. Using Eqs. (2) and (3), the power dissipated is

\[
P(y^*) = \frac{\mathcal{E}^2}{R_1 + R_2}.
\]  

(4)

On comparing with Eq. (1), \( R_{\text{eq}} = R_1 + R_2 \) as expected.

3 Equivalent Resistance from the Variational Principle

We now extend the strategy to compute the equivalent resistance for generic resistor circuits from the variational principle by minimizing the total power. To do this, we start by denoting the nodal potentials of an electrical circuit with \( n \) nodes by \( V_1, V_2, \ldots, V_n \). Without any loss of generality, we will find the equivalent resistance between nodes \( \{1\} \) and \( \{n\} \), subject to the following constraint. The potential difference across this pair of nodes is fixed by the e.m.f. \( \mathcal{E} \) of the battery in the circuit, i.e. \( V_1 - V_n = \mathcal{E} \). By denoting the resistance between nodes \( \{i\} \) and \( \{j\} \) as \( R_{ij} \), the total electric power can be expressed as a quadratic function of the nodal potentials,

\[
P = \frac{1}{2} \sum_{i,j=1}^{n} \frac{(V_i - V_j)^2}{R_{ij}}.
\]  

(5)

Finally, the scalar function \( P \) of Eq. (5) is minimized to find the equivalent resistance by the Lagrange multiplier method. The method of the Lagrange multiplier is computationally cheap. With the eigendecomposition of a matrix, the equivalent resistance between all the pairs of nodes in the circuit can be computed. On
the other hand, the regular Kirchhoff method requires a matrix inversion each time the equivalent resistance between two nodes is computed. Moreover, in the Lagrange multiplier method, it is not necessary to find the voltages on all the nodes explicitly to find the equivalent resistances.

As mentioned earlier, the technique to find nodal potentials in a circuit through power minimization is equivalent to finding them by using Kirchhoff’s laws. Kirchhoff’s (second) law states that the sum of the nodal currents in a circuit is equal to 0. Mathematically, it can be represented as

$$\sum_{j=1}^{n} I_{ij} = \sum_{j=1}^{n} \frac{(V_i - V_j)}{R_{ij}} = 0, \quad \text{for } i = 2, \ldots, n - 1. \quad (6)$$

Here $I_{ij}$, $(R_{ij})$ denotes the current (resistance) between nodes $(i)$ and $(j)$, and $V_i$ denotes the potential on node $(i)$. The nodes $(1)$ and $(n)$ do not satisfy this equation since they are connected with the battery and current flows in and out of them. For more details, refer to Eq. (9) in [6].

On the other hand, the nodal potentials obtained by minimizing the power function in Eq. (5),

$$P = \frac{1}{2} \sum_{j=1}^{n} \frac{(V_i - V_j)^2}{R_{ij}}, \quad (7)$$

are the solutions to the $(n - 2)$ simultaneous equations

$$\frac{\partial P}{\partial V_i} = \sum_{j=1}^{n} \frac{(V_i - V_j)}{R_{ij}} = 0, \quad \text{for } i = 2, \ldots, n - 1 \quad (8)$$

The nodes $(1)$ and $(n)$ are not part of this set of equations since they are connected with the battery and the potential difference across them is held constant. This is the constraint for the power minimization problem. Both the Eqs. (6) and (8) lead to the same solutions and thus the methods are thus equivalent.

It is worth pointing out that since $P$ in Eq. (5) is a quadratic function, the minimization algorithms presented in this section in the context of electrical circuits are applicable to other quadratic functions of interest\(^2\).
**Lagrange Multiplier**

We will now demonstrate the method of the Lagrange multipliers to find the minimum of the power function that is subject to a constraint [19]. The power function in Eq. (5) is expressed in terms of a matrix equation as:

$$P = \frac{1}{2} \sum_{i,j=1}^{n} \frac{(V_i - V_j)^2}{R_{ij}} = \sum_{i,j=1}^{n} L_{ij} V_i V_j,$$

(9)

where

$$L = \begin{pmatrix} s_1 & -\frac{1}{R_{12}} & \cdots & -\frac{1}{R_{1n}} \\ -\frac{1}{R_{21}} & s_2 & \cdots & -\frac{1}{R_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{R_{n1}} & -\frac{1}{R_{n2}} & \cdots & s_n \end{pmatrix},$$

(10)

$$s_i = \sum_{k=1}^{n} \frac{1}{R_{ik}}, \text{ for } i = 1, \ldots, n.$$

It is subject to the constraint that

$$V_\eta - V_\sigma = E,$$

(11)

where $\eta$ and $\sigma$ can be any pair of nodes from the set of nodes \{1, \ldots, n\}. The equivalent resistance will be computed between these nodes ($\eta$) and ($\sigma$).

To use the Lagrange multiplier method, we start by writing an expression that is a sum of the function we want to minimize and the constraint equation multiplied by an arbitrary constant $\lambda$. The $\lambda$ is called the ‘Lagrange multiplier’, and its value will be determined through an optimization method.

To familiarize oneself with this technique, a simple example to minimize a mathematical function is shown in Appendix A.

For the problem at hand, the Lagrange multiplier method finds the optimum value of the function.
\[ f(V_1, \ldots, V_n, \lambda) = P - \lambda \left( V_\gamma - V_\sigma - \mathcal{E} \right) \]
\[ = \sum_{i,j=1}^{n} L_{ij} V_i V_j - \lambda (V_\gamma - V_\sigma - \mathcal{E}), \quad (12) \]

by solving a set of \((n + 1)\) simultaneous equations for the \((n + 1)\) unknowns \((V_1, \ldots, V_n, \lambda)\). The simultaneous equations are obtained by differentiating \(f(V_1, \ldots, V_n, \lambda)\) with respect to its variables, i.e.,

\[ \nabla_x f(V_1, \ldots, V_n, \lambda) = 0, \quad (13) \]

where \( x = V_1, \ldots, V_n, \lambda \). The simultaneous set of equations are

\[ \nabla_{V_i} f = \sum_{i,j=1}^{n} L_{ij} \left( \delta_{kj} V_j + V_j \delta_{jk} \right) - \lambda (\delta_{jk} - \delta_{ck}) \]
\[ = 2 \sum_{j=1}^{n} L_{kj} V_j - \lambda (\delta_{jk} - \delta_{ck}) = 0, \quad \text{for } k = 1, \ldots, n, \quad (14) \]
\[ \nabla_{\lambda} f = -(V_\gamma - V_\sigma - \mathcal{E}) = 0. \]

Notice that the matrix \( L \) has a zero eigenvalue that is characteristic of a Laplacian matrix; therefore, it is not invertible. However, to find a solution for the set of the nodal potentials, we need to invert the matrix \( L \). We circumvent this problem of inversion by adding the identity matrix times a small constant \( \epsilon \) to define the modified Laplacian matrix \( L(\epsilon) = L + \epsilon I \). Finally, we will take the limit of \( \epsilon \to 0 \), and the results will be independent of \( \epsilon \). A detailed analysis of this method is shown in Appendix B.

From Eq. (14), the solution to the nodal potentials are obtained by multiplying the inverse of \( L(\epsilon) \), i.e., \( G(\epsilon) \) on both sides of the equation:

\[ V_i = \sum_{k=1}^{n} \frac{1}{2} G_{ik}(\epsilon)(\delta_{ik} - \delta_{ck}) \]
\[ = \frac{1}{2} \left[ G_{i\gamma}(\epsilon) - G_{i\sigma}(\epsilon) \right]. \quad (15) \]
Using the constraint in Eq. (11) and the solution to the nodal potentials \( \{V_i\} \), we find that

\[
V_\eta - V_\sigma = \frac{\lambda}{2} \left[ G_{\eta\eta}(\epsilon) - G_{\eta\sigma}(\epsilon) - G_{\sigma\eta}(\epsilon) + G_{\sigma\sigma}(\epsilon) \right]
\]

\[
= \frac{\lambda}{2} \left[ G_{\eta\eta}(\epsilon) - 2G_{\eta\sigma}(\epsilon) + G_{\sigma\sigma}(\epsilon) \right]
\]

(16)

since \( G \) is symmetric. Thus,

\[
\lambda = \frac{2(V_\eta - V_\sigma)}{G_{\eta\eta}(\epsilon) - 2G_{\eta\sigma}(\epsilon) + G_{\sigma\sigma}(\epsilon)}.
\]

(17)

By substituting the value of \( \sum_{j=1}^{n} L_{ij} V_j \) from Eq. (14) and \( \lambda \) from Eq. (17), the minimum power is

\[
P = \sum_{i,j=1}^{n} L_{ij} V_i V_j
\]

\[
= \sum_{i=1}^{n} V_i \sum_{j=1}^{n} L_{ij} V_j
\]

\[
= \sum_{i=1}^{n} V_i \frac{\lambda}{2} (\delta_{ii} - \delta_{ij})
\]

\[
= \frac{\lambda}{2} (V_\eta - V_\sigma)
\]

\[
= \frac{\epsilon^2}{G_{\eta\eta}(\epsilon) - 2G_{\eta\sigma}(\epsilon) + G_{\sigma\sigma}(\epsilon)}.
\]

(18)

On comparing the power to the form \( \frac{\epsilon^2}{R_{\text{equ}}^\eta} \) in Eq. (1), we find the equivalent resistance between nodes \( \{\eta\} \) and \( \{\sigma\} \) to be

\[
R_{\text{equ}}^{\eta\sigma}(\epsilon) = G_{\eta\eta}(\epsilon) - 2G_{\eta\sigma}(\epsilon) + G_{\sigma\sigma}(\epsilon).
\]

(19)

By taking the limit of \( \epsilon \to 0 \) and substituting the value of \( G(\epsilon) \) from Eq. (B.3) derived in Appendix B, the equivalent resistance is

\[
R_{\text{equ}}^{\eta\sigma} = \lim_{\epsilon \to 0} \left( G_{\eta\eta}(\epsilon) - 2G_{\eta\sigma}(\epsilon) + G_{\sigma\sigma}(\epsilon) \right)
\]

\[
= \lim_{\epsilon \to 0} \left( g_{\eta\eta}(\epsilon) - 2g_{\eta\sigma}(\epsilon) + g_{\sigma\sigma}(\epsilon) \right)
\]

\[
= g_{\eta\eta}(0) - 2g_{\eta\sigma}(0) + g_{\sigma\sigma}(0).
\]

(20)
Since
\[ g_{ij}(\epsilon) = \sum_{k=1}^{n} \psi_{ki} \frac{1}{\lambda_k + \epsilon} \psi_{kj}^* \]  \hspace{1cm} (21)

the eigenvalues \((\lambda_i)\) and the eigenvectors \((\Psi_i)\) of the matrix \(L\) are sufficient to determine all the possible pairs of equivalent resistance.\(^3\)

**Computational complexity:** To implement this method, the primary computational complexity comes from determining the eigenvectors and eigenvalues of the matrix \(L\). The complexity of finding the dominant eigenvalue and eigenvector of a matrix of size \(k\) with the power iteration algorithm is \(O(k^3)\) [21]. Since \(k\) such values need to be obtained, the complexity is \(O(k^3)\). For a circuit with \(k\) nodes, all the pairwise equivalent resistances can be determined from the eigenvectors and eigenvalues of \(L\). Thus, for a circuit with \(k\) nodes, the complexity of the problem is still \(O(k^3)\). This is significantly cheaper than Kirchhoff’s laws by an order of magnitude.

Note that the smallest eigenvalue \(\lambda_1 = 0\) does not appear in the sum. The \(\epsilon\)-dependent term \(\frac{1}{\lambda_k + \epsilon}\) in \(g_{ij}(\epsilon)\) will simply be \(\frac{1}{\lambda_1}\) in the limit of \(\epsilon \to 0\).

Kirchhoff’s laws require a matrix inversion each time the equivalent resistance is computed between a pair of nodes in the circuit. The matrix inversion problem is reducible to matrix multiplication and thus it can be performed as fast as matrix multiplication. The trivial algorithm [22] for multiplying matrices of size \(k\) is \(O(k^3)\) while the current fastest algorithm [23] has a complexity of \(O(k^{2.373})\). For an electrical circuit of \(k\) nodes, if we wanted to find all the pairwise equivalent resistances (i.e., \(k^2/2\) of them), then the computational complexity would be \(O(k^{4.373})\) with the fastest algorithm and \(O(k^5)\) with the trivial approach.

**Wheatstone Bridge**

An implementation of the Lagrange multiplier method is presented here to find the equivalent resistance between nodes \(1\) and \(4\) for the Wheatstone bridge in Figure 2.

**Step 1** First, the Laplacian matrix \(L\) is constructed for the Wheatstone bridge based on the prescription in Eq. (10) to imple-
ment the method.

\[
L = \begin{pmatrix}
\frac{1}{R_1} + \frac{1}{R_3} & -\frac{1}{R_5} & -\frac{1}{R_5} & 0 \\
-\frac{1}{R_5} & \frac{1}{R_1} + \frac{1}{R_5} + \frac{1}{R_2} & -\frac{1}{R_5} & -\frac{1}{R_5} \\
0 & -\frac{1}{R_5} & \frac{1}{R_5} + \frac{1}{R_4} & -\frac{1}{R_5} \\
& & & \frac{1}{R_2} + \frac{1}{R_4}
\end{pmatrix}
\]

(22)

**Step 2** Then, the expression for \( g_{ij}(0) \) is evaluated from Eq. (21) by using the eigenvalues and eigenvectors of \( L \) for all \( i, j = 1, \ldots, n \). This can be done explicitly, both numerically and analytically.

**Step 3** By applying Eq. (20), the specific matrix entries are used to find the equivalent resistance between nodes \{1\} and \{4\} as

\[
R_{eq}^{14} = g_{11}(0) - 2g_{14}(0) + g_{44}(0)
= \frac{R_5(R_1 + R_2)(R_3 + R_4) + pR_1R_2R_3R_4}{sR_5 + (R_1 + R_3)(R_2 + R_4)},
\]

(23)

The same results will be obtained had Kirchhoff’s laws been used instead.
where \( p = \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \right) \), \( s = (R_1 + R_2 + R_3 + R_4) \).

Note that the equivalent resistances between the other pairs of nodes \( \{\eta\}, \{\sigma\} \) can be found in a similar fashion using the entries \( g_{\eta\sigma}(0) \) without any further computations. Additionally, the seemingly complex calculations between Eqs. (9)–(16) were for illustration purposes only and not necessary for finding the actual equivalent resistances.

4 Suggested Problems

(A) Show that the power minimization formula leads to the same equivalent resistance formula for two resistors in parallel with each other as obtained from Kirchhoff's laws. This would be similar to the derivation done for the resistors in series in Sec. 2.

(B) For any Laplacian matrix such as \( L \) in Eq. (9) show that

(a) \( L \) is a singular matrix with the smallest eigenvalue being 0; therefore shows that the determinant \( |L| = 0 \) and \( L \) is not invertible.

(b) \( L \) is positive-semidefinite so that its eigenvalues are greater than or equal to 0.

(C) For a circuit of your choice, find the equivalent resistance using Kirchhoff's laws. Then find the equivalent resistance for the same circuit using the Lagrange multiplier approach. Verify and show that the same answer is obtained for both methods.

5 Discussions

This article presents a recipe to find the equivalent resistance in an electrical circuit using the variational principle. This recipe provided is useful for investigating a multitude of resistor circuits and finding equivalent resistances between all the pairs of nodes in the circuit. The challenging step in such a task to find the series and
parallel components of a circuit is completely avoided. Moreover, the computational cost to find the equivalent resistances is significantly cheaper than Kirchhoff’s laws. Overall, the method is easy to implement both conceptually and numerically. To highlight its easy implementation, a demonstration in Mathematica [24] finds the equivalent resistances for a hundred different resistor circuits with unit resistances (or analogous graphs with unit weights). A natural extension of this article would be to find impedances in electrical circuits with resistors and capacitors. As a step towards that direction, a variational approach to find voltages on the nodes in such an electrical circuit has been studied by Gibson et al., [25].

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Appendix

A Lagrange Multiplier Example

To demonstrate the use of the Lagrange multiplier method consider minimizing the function

\[ g(x, y) = x^2 + y^2 - r^2 = 0, \]  \hspace{1cm} (A.1)

subject to the constraint

\[ h(x, y) = x + y - 1 = 0. \]  \hspace{1cm} (A.2)

The Lagrange multiplier methods finds the optimum value of the function

\[ f(x, y, \lambda) = g(x, y) - \lambda h(x, y) \]
\[ = x^2 + y^2 - r^2 - \lambda(x + y - 1) = 0, \] \hspace{1cm} (A.3)

by solving a set of three simultaneous equations for the unknowns \((x, y, \lambda)\). The simultaneous equations are obtained by differentiat-
**Figure 3.** A geometric visualization for the minimum value of $0$ for $x^2 + y^2 - r^2$ subject to the constraint $x + y - 1 = 0$. The minimum occurs at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ where the circle of radius $\frac{1}{\sqrt{2}}$ (in blue) is tangent to the line (in red).

Using Eq. (A.3):

\[
\begin{align*}
\nabla_x f &= 2x - \lambda = 0 \\
\nabla_y f &= 2y - \lambda = 0 \\
\n\nabla_\lambda f &= -(x + y - 1) = 0.
\end{align*}
\]

From the first two of these equations, we see that $x = y = \frac{1}{\sqrt{2}}$. These values can be plugged into the third equation to get $\lambda = 1$. Combining these results, we arrive at the solution $x = y = \frac{1}{\sqrt{2}}, \lambda = 1$. These values can now be plugged back into Eq. (A.3) to fix the value of $r = \frac{1}{\sqrt{2}}$.

Geometrically, this means that the minimum value of $g(x, y)$ of $0$ occurs when the circle of radius $\frac{1}{\sqrt{2}}$ is tangent to line $x + y - 1 = 0$ at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ as shown in Figure 3.

**B Laplacian Matrix Inversion**

The modified Laplacian $L(\epsilon)$ has the same matrix entries as $L$ except the diagonal entries are changed. Hence, the unitary transformation $U$, that diagonalizes the matrix $L$, is the same as the one which diagonalizes the $L(\epsilon)$. If the eigenvalues of $L$ are $\lambda_i$ with orthonormal eigenvectors $\Psi_i = (\psi_{i1}, \psi_{i2}, \ldots, \psi_{in})$ for $i = 1, \ldots, n$, then the eigenvalues of the modified Laplacian are of the form
\[ \lambda_i + \epsilon \text{ as shown below:} \]

\[
U^T LU = \Lambda \\
\Rightarrow U^T L(\epsilon) U = U^T (L + \epsilon I) U \\
= \Lambda + \epsilon I = \Lambda(\epsilon),
\]

where \( \Lambda \) is a diagonal matrix with the entries \( \Lambda_{ij} = \lambda_i \delta_{ij} \). The matrix \( U \) is the set of eigenvectors of \( L \) that is \( U_{ij} = \psi_{ji} \). The inverse of the modified Laplacian is the symmetric Green’s function defined as

\[
G(\epsilon) = L^{-1}(\epsilon) = UA^{-1}(\epsilon)U^T. \tag{B.2}
\]

Specifically, the matrix elements are

\[
G_{ij}(\epsilon) = \sum_{k,l=1}^{n} U_{jk} \Lambda_{kl}^{-1}(\epsilon) U_{lj}^T \\
= \sum_{k,l=1}^{n} \psi_{kl} \frac{\delta_{kl}}{\lambda_k + \epsilon} \psi_{lj}^* \\
= \psi_{ij} \frac{1}{\lambda_1 + \epsilon} \psi_{ij}^* + \sum_{k=2}^{n} \psi_{kj} \frac{1}{\lambda_k + \epsilon} \psi_{kj}^* \\
= \frac{1}{n \epsilon} + g_{ij}(\epsilon),
\]

where \( g_{ij}(\epsilon) = \sum_{k=2}^{n} \psi_{kj} \frac{1}{\lambda_k + \epsilon} \psi_{kj}^* \). Since \( L \) has a zero eigenvalue, the modified Laplacian \( L(\epsilon) \) also has a zero eigenvalue. Subsequently, the inverse of the diagonal matrix of the eigenvalues of \( L(\epsilon) \), i.e., \( \Lambda^{-1}(\epsilon) \) has a zero entry denoted by \( \lambda_1 \). The corresponding orthonormal eigenvector \( \psi_1 \) has all its entries as \( \frac{1}{\sqrt{n}} \). This leads to the first term \( \frac{1}{n \epsilon} \) in the last line of Eq. (B.3).

This treatment helped us to circumvent the problem of inverting the Laplacian matrix \( L \). If we inspect the matrix entries of \( L^{-1}(\epsilon) = G(\epsilon) \) carefully, we find that we have extracted the divergent part of the matrix entries. In Eq. (B.3), in the limit of \( \epsilon \rightarrow 0 \), the divergence occurs due to the first term \( \frac{1}{n \epsilon} \), while the second term \( g_{ij}(0) \) is not divergent. The other eigenvalues \( \lambda_2, \ldots, \lambda_n \) are nonzero and the term \( \frac{1}{\lambda_k \epsilon} \) goes to a finite number in the limit. In [26], a theorem provides a rigorous proof of the above procedure.
Suggested Reading


