The Baire category theorem has numerous applications in real and functional analysis. In this note, we discuss an interesting application in the context of metric spaces. We characterize all subsets of a general metric space that appear as the set of continuity points for functions on that space. Furthermore, we juxtapose our results with the Tietze extension theorem, which beautifully leads us in another direction towards the study of extensions of continuous functions. Most of the results we discuss here already exist scattered in the literature. In this note, we strive to present them in a coherent form.

Introduction

Given $A \subseteq \mathbb{R}$ (for instance $A = \mathbb{Q}$ is a natural choice), is there a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous exactly on $A$ and not outside $A$? The answer, in general, is no. But then we can ask for what all sets $A \subseteq \mathbb{R}$ we can do that? The same question can be asked in a general setup for a function $f : X \to Y$ where $X, Y$ are general metric spaces. Let’s stick to $\mathbb{R}$ for now and consider $A = \mathbb{Q}^c$ (the set of irrationals) to question: is there a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous on $\mathbb{Q}^c$ and discontinuous on $\mathbb{Q}$? The answer is yes. An example of such a function is given by the famous Thomae function

$$f(x) = \begin{cases} 
0 & : x \in \mathbb{Q}^c \\
\frac{1}{n} & : x = \frac{m}{n} : \gcd(m, n) = 1; m, n \in \mathbb{Z}
\end{cases}.$$

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Does there exist a function on \( \mathbb{R} \) which is continuous on rational numbers and discontinuous on the set of irrational numbers?

But what about functions with the property with the roles of rationals and irrationals interchanged? Does there exist a function \( f : \mathbb{R} \to \mathbb{R} \), which is continuous on \( \mathbb{Q} \) and discontinuous on \( \mathbb{Q}^c \)? This is something the reader may be tempted to construct. But the answer is, unfortunately no. We shall see why that is the case and how to answer the general question. This is where the Baire category theorem enters.

1. The Baire Category Theorem

The Baire category theorem is standard fare in introductory real analysis courses. But, here, we shall use a weaker form of the theorem. Before stating the theorem and its other forms, let us quickly go through the notions required to understand its statement.

Definition [Dense Set]: Let \( X \) be a metric space. \( A \subseteq X \) is said to be dense in \( X \) if and only if the closure of \( A \) does not contain any open set.

Definition [Nowhere Dense]: Let \( X \) be a metric space. \( A \subseteq X \) is said to be nowhere dense in \( X \) if and only if \( \text{int}(\bar{A}) = \emptyset \).

Definition [Meagre/First Category]: Let \( X \) be a metric space. \( A \subseteq X \) is said to be meagre if and only if it can be written as countable union of nowhere dense sets.

Now we are ready to state the Baire category theorem.

Theorem 1 [Baire category]. Let \( X \) be a complete metric space. Then:

1. A countable intersection of dense open sets is dense.
2. A Meagre set has empty interior.
3. The complement of a meagre set is dense. (also called residual set).

With some effort, one can verify that the above three assertions are, in fact, equivalent. Below we state a weaker form of BCT, which will be directly useful for us.
**Corollary** [Weak BCT]. Let $X$ be a non-empty complete metric space. Then $X$ cannot be written as a countable union of nowhere dense sets.

The weak BCT can be derived from Theorem 1. using the fact that the countable intersection of dense open sets is non-empty. Thus we are not entirely using the fact that the countable intersection of dense open sets is dense, making the latter weaker.

2. **$G_δ$ and $F_σ$ sets**

The notions of $G_δ$ and $F_σ$ sets are the building blocks of our theory of characterization of the set points of continuity.

**Definition** [$G_δ$ set]: Let $X$ be a metric space. $A \subseteq X$ is called a $G_δ$ set if and only if it can be written as a countable intersection of open sets.

**Definition** [$F_σ$ set]: Let $X$ be a metric space. $A \subseteq X$ is called a $F_σ$ set if and only if it can be written as countable union of closed sets.

Clearly, $A$ is $G_δ$ if and only if $A^c$ is $F_σ$. Also note that a set can be simultaneously both $G_δ$ and $F_σ$. Let’s see a few examples to increase our comfort level with these notions.

**Example**

(i) Obviously any open set is $G_δ$ and any closed set is $F_σ$.

(ii) Any closed set is $G_δ$. This result is not immediately obvious. But observe if $A \subseteq X$ is closed $A = \bigcap_{n=1}^{\infty} \{x \in X : d(x,A) < \frac{1}{n}\}$. Now as $(x \in X : d(x,A) < \frac{1}{n})$ is open for all $n \in \mathbb{N}$, we can conclude that any closed set is $G_δ$.

(iii) Similarly, any open set is $F_σ$.

(iv) Any countable set is $F_σ$. We can just take countable union of singletons. We know singletons are always closed.

(v) Indeed $G_δ$ and $F_σ$ are two different classes of sets. To verify that observe $Q^c$ is a $G_δ$ set but not a $F_σ$ set. In general any dense $G_δ$ set with empty interior is not a $F_σ$ set.
Having made acquaintance with $G_\delta$ and $F_\sigma$ sets we can move to our most important part, that is, the characterization of subsets of $X$ that can arise as the set of points of continuity of a function on $X$.

3. Set of Points of Continuity

Let $f : X \to Y$ be a function from a metric space $X$ to another metric space $Y$. In order to have a quantitative idea of the size of a bounded set $A$ in $X$, we define the notion of diameter of $A$ as follows,

$$\text{diam}(A) := \sup\{d(x, y) : x, y \in A\}.$$ 

Recall that $f$ is said to be continuous at $x_0 \in X$ if and only if for every $\epsilon > 0$ there is $\delta > 0$, such that whenever $d_X(x_0, x) < \delta$ we have $d_Y(f(x_0), f(x)) < \epsilon$. In other words for every $n \in \mathbb{N}$, there is an open set $U \subseteq X$ such that $x_0 \in U$ and $f(U) \subseteq B_{\frac{1}{n}}(f(x_0))$. Equivalently, $f$ is continuous at $x_0$ if and only if for all $n \in \mathbb{N}$, there is an open set $U \subseteq X$ such that $x_0 \in U$ and $\text{diam}(f(U)) < \frac{\epsilon}{n}$.

Let $\mathcal{C}_f$ be the set of points of continuity of $f$, that is,

$$\mathcal{C}_f := \{x \in X : f \text{ is continuous at } x\}.$$

Let $\mathcal{D}_n := \{x \in X : \text{there is an open set } U \subseteq X \text{ such that } x \in U \text{ and } \text{diam}(f(U)) < \frac{1}{n}\}$. From the definition of continuity we considered above, it is easy to verify that,

$$\mathcal{C}_f = \bigcap_{n=1}^{\infty} \mathcal{D}_n.$$ 

**Lemma.** For every $n \in \mathbb{N}$, the set $\mathcal{D}_n$ is open in $X$.

**Proof.** Let $n \in \mathbb{N}$, and $x$ be an arbitrary point in $\mathcal{D}_n$. From the definition of $\mathcal{D}_n$, there is an open set $U \subseteq X$ such that $x \in U$ and $\text{diam}(f(U)) < \frac{1}{n}$. For any $y \in U$ we can have $\text{diam}(f(U)) < \frac{1}{n}$, thus making $y$ an element of $\mathcal{D}_n$. Thus $x \in U \subseteq \mathcal{D}_n$ and hence an interior point of $\mathcal{D}_n$. Thus we conclude that $\mathcal{D}_n$ is open.

**Theorem 2.** Let $X$ be a metric space. For every metric space $Y$, and a function $f : X \to Y$, the set $\mathcal{C}_f \subseteq X$ is a $G_\delta$ set.
**Proof.** Since $\mathcal{C}_f = \bigcap_{n=1}^{\infty} \mathcal{C}_n$ and $\mathcal{C}_n$ is open for all $n \in \mathbb{N}$, from Lemma 3, it follows that $\mathcal{C}_f$ is a $G_\delta$ set.

With Theorem 3., we have arrived at an interpretable necessary condition for a subset of a metric space $X$ to be the set of points of continuity for a function on $X$. We are now in a position to answer our original question of the existence of functions $f : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{C}_f = \mathcal{Q}$. In §4., (i), we will show that $\mathcal{Q}$ is **not** a $G_\delta$ set in $\mathbb{R}$ as an application of weak BCT. This will tell us that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{C}_f = \mathcal{Q}$, thereby answering our original question.

Since our goal is to characterize sets in metric spaces that arise as sets of points of continuity of a function, it is natural to wonder whether the converse to Theorem 3. holds. In other words, given a $G_\delta$ subset $A \subseteq X$, is there a metric space $Y$ and a function $f : X \to Y$ such that $\mathcal{C}_f = A$? It is easy to see that $A$ must contain all isolated points of $X$ if the answer is to be affirmative. In Theorem 3., we show that if $A$ contains all isolated points of $X$, we may just choose $Y = \mathbb{R}$ and we can find a function $f : X \to \mathbb{R}$ such that $\mathcal{C}_f = A$. We first need a technical lemma to be used in the proof of Theorem 3.

**Lemma.** Let $X$ be a metric space without isolated points. Then there is a dense subset $A \subseteq X$ such that $A^c$ is also dense.

**Proof.** Note that it is sufficient to show that there exist two disjoint dense subsets $A, B \subseteq X$. We define \textit{\epsilon-far} property for a subset $S \subseteq X$. We say $S \subseteq X$ is \textit{\epsilon-far} if and only if

1. For any two distinct points $x, y \in S$, $d(x, y) \geq \epsilon$.

2. $S$ is maximal with respect to the above property.

Now, by Zorn’s lemma such \textit{\epsilon-far} sets exist for every $\epsilon > 0$. Let $S_1, S_2, \ldots, S_n$ be disjoint subsets of $X$ such that $S_k$ is $\frac{1}{k}$-far for all $k \in \{1, 2, \ldots, n\}$. Then as $X$ is a metric space without isolated points the complement of $S_1 \cup S_2 \cup \cdots \cup S_n$ is non-empty and has no isolated points. Therefore, there is a set $S_{n+1}$ disjoint from $S_1 \cup S_2 \cup \cdots \cup S_n$, which is a $\frac{1}{n+1}$-far subset of $X$. Thus we got...
a sequence of disjoint subsets of $X$, $\{S_n\}_{n=1}^{\infty}$ having $k$-th term a $\frac{1}{k}$-far subset. Define

$$A = \bigcup_{n=1}^{\infty} S_{2n} \text{ and } B = \bigcup_{n=1}^{\infty} S_{2n-1}.$$ 

Clearly, $A$ and $B$ are disjoint. It is also not hard to check that $A$ and $B$ are actually dense in $X$. Thus we are done!

Now we are ready to prove our desired theorem.

**Theorem 3.** Let $X$ be a metric space. Let $A \subseteq X$ be a $G_\delta$ set containing all isolated points. Then there is a function $f : X \to \mathbb{R}$ such that $C_f = A$.

**Proof.** To begin with, let us assume that $X$ does not have any isolated points. Since $A$ is a $G_\delta$ set, it can be written as a countable intersection of open sets. Therefore, $A^c$ can be written as a countable union of closed sets. Furthermore, $A^c$ can be written as a countable union of an increasing sequence of closed sets. Let’s say

$$A^c = \bigcup_{n=1}^{\infty} F_n \text{ and } F_1 \subseteq F_2 \subseteq \ldots.$$ 

By Lemma 3., there is a $B \subseteq X$ such that $B$ and $B^c$ both are dense in $X$. Let $g_B$ be the characteristic (indicator) function on $B$. Now, we define a function on $X$ as

$$f(x) = \left( g_B(x) - \frac{1}{2} \right) \frac{1}{n \in K} \frac{1}{2^n}, \text{ where } K = \{ n \in \mathbb{N} : x \in F_n \}.$$ 

Thus $f(x) = 0$ whenever $x \in A$. Now if $\{x_n\}_{n \geq 1}$ converges to $x \in A$ then $\{f(x_n)\}_{n \geq 1}$ converges to 0, as $x$ can’t be a limit point of any $F_k$ so even if the sequence is in $A^c$, they should be from different $F_k$’s making the limit 0. Thus $f$ is continuous at $x \in A$.

Now let $x \in int(A^c)$, then every neighbourhood of $x$ will contain points taking different signs of $f$. Thus if $x \in int(A^c)$ then $f$ is not continuous at $x$. Now if $x \in A^c$ but $x$ is not an interior point of $A^c$, then every neighbourhood of $x$ will contain a point where the $f$ is 0. Thus again $f$ is not continuous at $x$. Therefore if $x \in A^c$ then $f$ is not continuous at $x$. So we conclude $C_f = A$. 

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Next we consider the case where $X$ is a general metric space. Let $H$ denote the set of isolated points of $X$. Note that $X' := X \setminus H$ is an open subset of $X$ and $A' := A \cap X'$ is a $G_δ$ set in the metric space $X'$ which has no isolated points. From the preceding discussion, there is a function $f' : X' \to \mathbb{R}$ such that $C_{f'} = A'$. We may arbitrarily assign real numbers to points in $H$ to extend $f'$ to a function $f : X \to \mathbb{R}$. Note that $C_f = A' \cup H = A$.

Thus now we have both directions. We summarize our results below in the form of a theorem.

**Theorem.** Let $X$ be metric space. Let the set of isolated points of $X$ be denoted by $H$. Then

$$\{C_f : f \text{ is a function from } X \text{ to } Y, \text{ for some metric space } Y\} = \{A \subseteq X : H \subseteq A \text{ and } A \text{ is } G_δ\}.$$ 

4. Applications

The theory of characterization of set of points of continuity have various applications. Lets see few of them to make ourselves comfortable.

1. Let’s begin with our favourite one. We already discussed that there can’t be any function $f$ on $\mathbb{R}$ having $C_f = \mathbb{Q}$. It is sufficient to show that $\mathbb{Q}$ is not a $G_δ$ set. To show this, the BCT comes to the rescue. Let assume the contradiction i.e $\mathbb{Q}$ is a $G_δ$ set. Then

$$\mathbb{Q} = \bigcap_{n=1}^{\infty} O_n \implies \mathbb{Q}' = \bigcup_{n=1}^{\infty} O_n' \implies \mathbb{R} = \bigcup_{q \in \mathbb{Q}} \{q\} \cup \bigcup_{n=1}^{\infty} O_n'.$$

Now as $O_n$ are open and dense sets (it is dense because $\mathbb{Q} \subseteq O_n$), $O_n'$ are nowhere dense sets. Also $\{q\}$ are clearly nowhere dense sets. But then $\mathbb{R}$ is written as countable union of nowhere dense sets, which is impossible by weak BCT. Thus $\mathbb{Q}$ is not a $G_δ$ set.

2. Is there a function $f : \mathbb{R} \to \mathbb{R}$ for which $C_f = [0, 1]$? Since any closed subset of $\mathbb{R}$ is a $G_δ$ set, by Theorem 3., we know that such a function exists.
5. Comparison with the Tietze Extension Theorem

We may rephrase the discussion about the set of points of continuity in a manner that makes it amenable to some natural generalizations. Let \( A \subseteq X \) be a \( G_\delta \) set containing all the isolated points of \( X \). From Theorem 3., we know that it is possible to find a a function \( f : X \to \mathbb{R} \) such that \( \mathcal{C}_f = A \). What if, at the outset, we are given a continuous function \( f : A \to \mathbb{R} \) and we are interested in extending it to the whole of \( X \) without adding a single extra point of continuity? In other words, is it possible to extend \( f \) to a function \( F : X \to \mathbb{R} \) such that \( \mathcal{C}_F = A \)? On the other extreme of such parsimony (in adding points of continuity), for a closed subset \( A \) of \( X \) (which is an example of a \( G_\delta \) set), we have the Tietze extension theorem, which says that there is a continuous extension of \( f \), that is, an extension \( F \) of \( f \) such that \( \mathcal{C}_F = X \).

**Theorem 4 [Tietze extension for metric spaces].** Let \( X \) be a metric space and \( C \) be a closed subset of \( X \). Given a continuous function \( f : C \to \mathbb{R} \), there is a continuous function \( F : X \to \mathbb{R} \) such that \( F_C = f \).

Comparing our main result, Theorem 3., with the Tietze extension theorem leads us to four well-motivated statements.

1. Given a closed subset \( A \) of a metric space \( X \) and a continuous function \( f : A \to \mathbb{R} \), is there a continuous function \( g : X \to \mathbb{R} \) whose restriction to \( A \) is \( f \)? (This is the metric space version of Tietze extension theorem.)

2. Given a closed subset \( A \) of a metric space \( X \) and a continuous function \( f : A \to \mathbb{R} \), is there a function \( g : X \to \mathbb{R} \) such that \( \mathcal{C}_g = A \) and its restriction to \( A \) is \( f \)? (This is true for metric spaces without isolated points. For metric spaces with isolated points the statement is true if and only if the closed subset contain all isolated points. Why? We shall see later.)

Now trying to do the same for \( G_\delta \) sets,

3. Given a \( G_\delta \) subset \( A \) of a metric space \( X \) and a continuous function \( f : A \to \mathbb{R} \), is there a continuous function \( g : X \to \mathbb{R} \) whose
restriction to $A$ is $f$? (This is not true in general. Why? Counterexample is rather easy. What all sets we know which are $G_δ$ but not closed? This is something the reader can try first.)

4. Given a $G_δ$ subset $A$ of a metric space $X$ and a continuous function $f : A \to \mathbb{R}$, is there a function $g : X \to \mathbb{R}$ such that $C_g = A$ and its restriction to $A$ is $f$. (Similar to 2, this is also true for metric spaces without isolated points. In a way, 2 is a corollary of this. Also this statement is a stronger version of something we know closely. Can we join the dots? Again, we shall see the proof later.)

In the context of the above statements, we care about two questions in general. Firstly, for what all sets can we have a ‘continuous extension’? Now by Tietze extension, we know that for closed sets, we can have a continuous extension. Unfortunately, we can’t generalize it for $G_δ$ sets. As a counterexample, we can think of $f : (0, 1) \to \mathbb{R}$ as $f(x) = \frac{1}{x}$, which is continuous but it can’t be continuously extended to $\mathbb{R}$. Coming to the second question, for what all sets can we have a ‘discontinuous extension’? We should clarify what we mean by ‘discontinuous extension’.

**Definition [Maximally Discontinuous Extension]:** Let $X$ be a metric space and $A \subseteq X$. Given $f : A \to \mathbb{R}$ continuous. We say $f$ is maximally discontinuously extended to a function $g : X \to \mathbb{R}$ if $C_f = A$ and $g_A = f$. In that case, we also say $g$ is a *maximally discontinuous extension* of $f$ and $f$ is *maximally discontinuously extendable*.

We can surely say something here. If $f : A \to \mathbb{R}$ is discontinuously extendable, then $A$ is a $G_δ$ set containing all isolated points of $X$. To avoid unnecessary confusion, let’s stick to metric spaces without isolated points. We know how to tackle the case of metric spaces with isolated points separately. Now the second question which arises naturally is, if given any $G_δ$ set $A \subseteq X$ and a continuous function $f : A \to \mathbb{R}$ is $f$ is discontinuously extendable? Now Theorem 3 says if the $f$ is constantly 0 on $A$ then $f$ is discontinuously extendable. We will see that it holds for any general function $f$. This validates statement 4 (subsequently statement
Let $X$ be a metric space and $A \subseteq X$ be a $G_δ$ subset containing all the isolated points of $X$. For every continuous function $f : A \to \mathbb{R}$, there is a maximally discontinuous extension of $f$ to $X$.

2) for metric spaces without isolated points, which is, in fact, in some sense a stronger version of Theorem 3.

**Theorem 5.** Let $X$ be a metric space and $A \subseteq X$ be a $G_δ$ subset containing all the isolated points of $X$. For every continuous function $f : A \to \mathbb{R}$, there is a maximally discontinuous extension of $f$ to $X$.

**Note**

1. $f(x) = 0$ for all $x \in A$.

2. $A$ is closed in $X$.

3. $A$ is a dense $G_δ$ subset.

4. $A$ is $G_δ$.

The fourth case is the strongest one, and it directly implies the other three cases. So why do we care to prove the other three separately? Because we will need all the previous cases to prove the 4th case. A good analogy of this can be found in real analysis. Lagrange’s theorem directly implies Rolle’s theorem but in order to prove Lagrange’s theorem, we need Rolle’s.

**Proof.** Case 1. It’s just the proof of the Theorem 3. Observe that there we actually constructed a function $g : X \to \mathbb{R}$ (written as $f$ in Theorem 3.) such that $C_δ = A$ and $g(x) = 0$ for all $x \in A$.

Case 2. Now for this case, we have $A \subseteq X$ to be closed. This should remind us of Tietze extension theorem. Given $f : A \to \mathbb{R}$ continuous and $A$ closed by Tietze extension theorem, we can have continuous extension of $f g_1 : X \to \mathbb{R}$. Again by case 1, we can have a function $g_2 : X \to \mathbb{R}$ such that $g_2(x) = 0$ for all $x \in A$ and $C_δ = A$. Define $g : X \to \mathbb{R}$ as $g = g_1 + g_2$. Now, as $g_2 = 0$ in $A$ we can say $g(x) = g_1(x) = f(x)$ for all $x \in A$. Thus $g_A = f$.

Also it’s easy to see that $C_δ = A$. We are done!

Case 3. Given a dense $G_δ$ subset $A \subseteq X$ and a continuous function $f : A \to \mathbb{R}$. $A$ is a $G_δ$ set and so $A^c$ is a $F_σ$ set. Thus $A^c$ can be
written as countable union of disjoint closed sets. Let’s say,

\[ A^c = \bigcup_{n=1}^{\infty} F_n \text{ and } F_1 \subseteq F_2 \subseteq \ldots \text{ are closed.} \]

Now we can write \( A^c \) as the disjoint union of \( F_1, F_2 \setminus F_1, F_3 \setminus F_2, \ldots \). Say,

\[ A^c = \bigcup_{n=1}^{\infty} C_n \text{ where } C_1, C_2, \ldots \text{ are disjoint sets,} \]

such that \( \bigcup_{i=1}^{n} C_i = \bigcup_{i=1}^{n} F_i \) for every \( n \in \mathbb{N} \). Define

\[ K(x) = n, \text{ if } x \in C_n. \]

It is easy to see that if \( \{x_n\} \) is a sequence in \( A^c \) converging to a point in \( A \), then \( K(x_n) \to \infty \).

Now as \( A \) is dense in \( X \), we can talk about \( \lim_{x \to b} f(x) \) for \( b \in A^c \). The limit may or may not exist. Define

\[ B_1 = \{ b \in A^c : \lim_{x \to b} f(x) \text{ exists} \} \text{ and } B_2 = \{ b \in A^c : \lim_{x \to b} f(x) \text{ does not exist} \}. \]

Note that \( f \) may be continuously extended to \( A \cup B_1 \) by defining \( f(b) := \lim_{x \to b} f(x) \) for \( b \in B_1 \).

Since \( A \) is dense in \( X \), for each \( x \in A^c \) we can choose a point \( a_x \) in \( A \) such that \( d(a_x, x) < \frac{1}{K(x)} \) to define a function \( h : A^c \to \mathbb{R} \) by \( h(x) = a_x \). Now let’s try to construct our required \( g : X \to \mathbb{R} \),

\[ g(x) := \begin{cases} f(x) &; x \in A \\ f(x) + \frac{1}{K(x)} &; x \in B_1 \\ f(h(x)) &; x \in B_2 \end{cases} \]

Now to show \( \mathcal{C}_g = A \). For \( a \in A \), let \( \{x_n\} \in X \) converge to \( a \). If \( x_n \in A \) for all \( n \in \mathbb{N} \), then by continuity of \( f \), we can say \( \{g(x_n)\} = \{f(x_n)\} \) converges to \( g(a) = f(a) \). If \( x_n \in B_1 \) for all \( n \in \mathbb{N} \), then \( K(x_n) \to \infty \) as \( n \to \infty \) and using continuity \( \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} f(x_n) + \lim_{n \to \infty} \frac{1}{K(x_n)} = f(a) \). So in this case too, \( \{g(x_n)\} \to g(a) \). Finally if \( x_n \in B_2 \) for all \( n \in \mathbb{N} \), \( h(x_n) \to a \) as \( x_n \to a \) thus \( f(h(x_n)) \to f(a) \) as \( n \to \infty \).
Now to check that \( g \) is discontinuous on \( A^c \), we shall show it is discontinuous on \( B_1 \) and \( B_2 \) separately. For \( b \in B_1 \), take a sequence \( \{x_n\} \) in \( A \). \( \{g(x_n)\} \) converges to \( f(b) \) thus doesn’t converge to \( g(b) \). So, \( g \) is discontinuous on \( B_1 \). Discontinuity on \( B_2 \) is pretty easy as by definition of \( B_2 \) for any \( b \in B_2 \), \( \lim_{y \to b} f(y) \) doesn’t exist.

Case 4. Finally, let \( A \) be any general \( G_\delta \) set and let \( f : A \to \mathbb{R} \) be a continuous function. Let \( \overline{A} \) denote the closure of \( A \) in \( X \). Note that \( A \) is a dense \( G_\delta \) subset of \( \overline{A} \) in the subspace topology. Hence by case 3, \( f \) may be extended to a function \( g : \overline{A} \to \mathbb{R} \) such that \( \mathcal{C}_g = A \). Since \( \overline{A} \) is a closed subset of \( X \), by case 1, there is a function \( h : X \setminus \overline{A} \to \mathbb{R} \), which is discontinuous at every point.

Thus,

\[
F(x) := \begin{cases} 
  g(x) & ; x \in \overline{A} \\
  h(x) & ; x \in X \setminus \overline{A}
\end{cases}
\]

is a maximally discontinuous extension of \( f \).

Thus now, if we are given a metric space without isolated points \( X \) and a \( G_\delta \) subset of it (say \( A \)), any function \( f : A \to \mathbb{R} \) is discontinuously extendable on \( X \). Also, we know if any \( A \subseteq X \) has the property, then it should be \( G_\delta \).

**Note**: This property of metric spaces without isolated points can be defined for topological spaces too. Let \((X, \tau)\) be a topological space. \( X \) (a topological space) is a d.e.p space (discontinuous extension property), if each function \( f : A \to \mathbb{R} \) on a \( G_\delta \) set \( A \) has a discontinuous extension. Metric spaces without isolated points are d.e.p spaces, but not perfect topological spaces are.

**Suggested Reading**


