Care For Some Anyons, Anyone?*

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Based on the nature of the quantum statistics they follow, the quantum particles in the universe can be divided into two broad categories, the bosons, and the fermions. Indistinguishability leads to the invariance of the wave functions of the bosons and the fermions, up to a sign, under pairwise exchange. For fermions, Pauli’s exclusion principle makes it impossible to put more than one identical particle in the same state. One consequence of this is the existence of different elements in the periodic table. On the other hand, many identical bosons can occupy a single state leading to exotic phases of matter like the Bose–Einstein condensates. In two dimensions, it is also possible to realize special quantum particles called the ‘anyons’, the particles that are neither bosons nor fermions! The world-lines\(^1\) representing the exchange of anyons appear to be different from those of the bosons and the fermions and they show ‘braiding’ (like braiding hairs) leading to more sophisticated quantum statistics. Though the space around us is three-dimensional (at least), we can create special artificial two-dimensional spaces in the form of surfaces and interfaces where anyons can exist, and cleverly designed experiments can even detect them. In this article, we attempt to understand the fundamentals of anyons and find out how anyons can emerge in a two-dimensional fractional quantum Hall system.

1. Introduction

Let us first look at the basic quantum mechanical description of bosons and fermions. Let us assume that we have two identical

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\(^1\)World-line of a particle is the trajectory of the particle on a 4-dimensional space-time diagram.
quantum particles at positions $r_1$ and $r_2$. Let the two-body wave-
function of the system comprising these two particles be of the
form $\Psi(r_1, r_2)$. Note that here we are trying to assign labels (i.e.,
1, 2) to the particles. But, in quantum mechanics, identical par-
ticles are intrinsically indistinguishable. This means, when we
exchange the labels of the two particles, we must get the exact
same physical state as before. In terms of the wavefunctions, one
can write

$$\Psi(r_1, r_2) = \zeta \Psi(r_2, r_1),$$  \hspace{1cm} (1)

where $\zeta$ is a complex number and $|\zeta| = 1$.

Since $r_1$ and $r_2$ are arbitrary, one can also write

$$\Psi(r_2, r_1) = \zeta \Psi(r_1, r_2).$$  \hspace{1cm} (2)

It is clear that (1) and (2) together must imply

$$\Psi(r_1, r_2) = \zeta^2 \Psi(r_1, r_2).$$  \hspace{1cm} (3)

Therefore,

$$\zeta^2 = 1.$$  \hspace{1cm} (4)

Solving (4), we get $\zeta = +1$ or -1.

This suggests that the wavefunction of the quantum particles can
be either symmetric ($\zeta = +1$) or anti-symmetric ($\zeta = -1$) with
respect to exchange leading to two distinct classes of particles
called the bosons ($\zeta = +1$) and the fermions ($\zeta = -1$).

This story can also be written in a slightly different language.
Rather than exchanging labels, we will move both the particles
and exchange their positions. When one exchanges two identical
quantum particles, the basic demand should be that the probabil-
ity density after exchange remains unchanged. Mathematically,

$$|\Psi(r_1, r_2)|^2 = |\Psi(r_2, r_1)|^2.$$  \hspace{1cm} (5)
This can be achieved if,

$$\Psi(r_1, r_2) = e^{i\alpha} \Psi(r_2, r_1). \quad (6)$$

where, $e^{i\alpha}$ is just a phase factor.

Now, exchange the particles again. From the frame of one particle, the process of two exchanges appears as if the other particle is going around it. Thus, we can say that two exchanges are equivalent to a rotation.

If two exchanges are equivalent to a rotation and the particles are identical, then there should be no change in the wavefunction. So we get,

$$\Psi(r_1, r_2) = e^{i2\pi\alpha} \Psi(r_1, r_2) \implies e^{i\pi\alpha} = 1 \implies \alpha = 0, 1,$$

which gives bosons and fermions as before.

Now, there is one fundamental problem with this line of argument. And that is where the fun begins. Let us try to understand the problem first. You may have noted that in the above discussion, it looks like we have forced $\alpha$ to be either 0 or 1, thinking that a $2\pi$ rotation should not change the wavefunction. But, what if the wavefunction actually changes after a rotation of $2\pi$? Recall that if the wavefunctions before and after rotation differed by an overall arbitrary phase factor or by a unitary transformation, we would still get the same probability density!

In order to understand this problem better, let us first understand the meaning of the exchange of identical particles. In fact, (5) describes the same particle configuration in different ways (permutations of position labels for $N$ identical particles for an $N$-particle wavefunction) because the particles are identical and hence indistinguishable. When we do an exchange, we just change the labels. But, that has no physical meaning, and (5) just reflects the redundancy in notation (labeling). A more formal approach to statistics of exchange would be to study the topology, i.e., the paths followed during such exchanges. You may imagine that if such paths for two consecutive exchanges are not topologically
Figure 1. Clockwise exchange of anyons in three dimensions.

Figure 2. Clockwise and anticlockwise exchange of anyons in two dimensions.

equivalent, that may lead to a difference in the two-particle wavefunction after a rotation in 2D. We will discuss this in more detail below.

Let us first work in 3D. Let's exchange particles in a circular fashion as shown in Figure 1. This is a clockwise exchange. But in 3D, we have another dimension, so we can flip the rotation on its plane, to get the anticlockwise rotation. Hence, in three dimensions, the clockwise and anticlockwise rotations are topologically equivalent. Hence, regardless of the path (clockwise or anticlockwise) chosen for performing the exchange, the wavefunctions will remain the same, and hence in 3D, only two types of particles can exist, i.e., bosons ($\alpha = 0$) and fermions ($\alpha = 1$).

In 2D, the situation is more complex. In 2D, clockwise and an-
anticlockwise rotations cannot be transformed into one another due to the absence of the third dimension. This is illustrated in Figure 2. In other words, clockwise and anticlockwise rotations in 2D are topologically inequivalent and hence, the wavefunctions before and after two consecutive exchanges may differ. Therefore, after a single exchange, the wavefunctions can satisfy (6) for $\alpha$ in the range $[0, 1]$ [1]. Such identical particles, whose exchange statistics (basically the difference in the wavefunction after single exchange) are dictated by $\alpha$ values that are not 0 or 1 are known as \textbf{anyons}, in particular, \textbf{abelian anyons} [2]. More technically, the line of argument introduced above can also be rephrased in a group-theoretic language, making use of the path connectedness of SO(3) rotation group to the identity, and lack of this in SO(2) rotation group [3].

As has been shown above, the definition of anyons depends on the global phase ($\xi$). But in elementary lectures on quantum mechanics, we learn that only measurement probabilities are important and global phases don’t contribute to the statistics of observables. Hence, we should ask some important questions here. First, do anyons exist in reality, or are they just theoretical constructs? Second, shouldn’t the global phase depend on the way a quantum system evolves? Are global phases observable? We will start by answering the second question first using the concept of the Berry phase in quantum mechanics and the Aharonov–Bohm (AB) effect. We will then move on to show the possibility of the existence

\textbf{Figure 3.} World-line for anticlockwise exchange of anyons in two dimensions (A braid diagram).
of anyons in two-dimensional electron gases exhibiting fractional quantum Hall effect.

A. Berry Phase in Quantum Mechanics

When we talk about the dynamics of a quantum system, the Hamiltonian plays a crucial role. The Hamiltonian is usually of the form,

\[ H(q_i, R_j), \]

where, \( q_i \) are the dynamical degrees of freedom, and \( R_j \) gives the collection of external parameters that can be varied (controlled), such as an external magnetic field applied to a system. Prior to the discovery of the Berry/Pancharatnam phase, it was thought that evolution under such a Hamiltonian would cause the quantum state to acquire just a dynamical phase, which could be eliminated under an appropriate gauge transformation. In 1984, M. V. Berry [4] studied the dynamics under cyclic adiabatic evolution and found the presence of a phase that was geometric or topological in nature and could not be eliminated by a simple gauge transformation. In this context, it should be noted that S. Pancharatnam, in 1956, had introduced such a phase in his study of coherence between two polarization states. The work of Pancharatnam is nicely discussed in another Resonance article by Rajaram Nityananda [6]. For simplicity, we shall discuss Berry’s approach to the geometric phase [4], [7]. As we will see, the Berry phase is a global phase that accumulates due to the change of external parameters of the Hamiltonian as opposed to the phase that accumulates due to the time evolution of a quantum state.

Let us begin our calculation of the Berry phase by setting the external parameters \( R_j \) such that the system at time \( t = 0 \) is in the energy eigenstate. We make an important assumption here—**the eigenstate is unique**. We now choose to vary parameters \( R(t) \) so the instantaneous eigenstates are \( |n(R(t))\). We will vary the parameters so that by the end, we are at the same configuration of the parameters, i.e., we will traverse a loop in the space of the
parameters. The phase accumulated at the end of the loop, apart from the dynamical phase will be the Berry phase. The parameters are varied slowly so that we can use the ‘adiabatic theorem’. This theorem states that if we place a system in a non-degenerate energy eigenstate and vary the parameters sufficiently slowly, the system will remain in the energy eigenstate of the instantaneous Hamiltonian \( \langle H(R(t)) \rangle \) at any instant of time \( t \), and won’t transition to a higher or lower energy eigenstate. However, during the variation, the energy eigenstate of the Hamiltonian at different \( R \) values could become degenerate. This may lead to the level crossing and the adiabatic theorem may fail. Such level crossing (the state ending up in an eigenstate different from the initial one) is rare. If the energies do become degenerate, the transition matrix element for a transition to another eigenstate could be forbidden or so small that in the time the parameters are changed, the transition is not feasible. This leads to the system remaining in the initial energy eigenstate when the parameters are changed slowly. In practice, the adiabatic theorem works.

The evolved state can be generally written as,

\[
|\psi(t)\rangle = e^{-i\theta(t)}|n(R(t))\rangle. \tag{7}
\]

Using the Schrödinger’s equation, we get,

\[
H(R(t))|\psi(t)\rangle = i\hbar \frac{d}{dt}|\psi(t)\rangle, \tag{8}
\]

\[
E_n(R(t))|n(R(t))\rangle = \hbar \left( \frac{d}{dt} \theta(t) \right) |n(R(t))\rangle + i\hbar \frac{d}{dt} |n(R(t))\rangle. \tag{9}
\]

Taking the inner product with \( \langle n(R(t))| \) and assuming normalized state, we get,

\[
\frac{E_n(R(t))}{\hbar} - i\langle n(R(t))| \frac{d}{dt} |n(R(t))\rangle = \frac{d}{dt} \theta(t), \tag{10}
\]

\[
\theta(t) = \int_0^t \frac{E_n(R(t'))}{\hbar} dt' - i \int_0^t \langle n(R(t'))| \frac{d}{dt} |n(R(t'))\rangle dt'. \tag{11}
\]

The first term gives the phase accumulated due to time evolution (dynamical phase), and the second gives the negative of the Berry
Figure 4. Particle moving around an infinitely long solenoid.

phase $e^{i\gamma_n}$, which arises due to the trajectory covered in the parameter space.

\[
\gamma_n = i \int_0^T \langle n(R(t')) | \frac{d}{dt} | n(R(t')) \rangle dt'
\]

\[
= i \int_0^T \langle n(R) | \nabla_R | n(R) \rangle \frac{dR}{dt} dt'
\]

\[
= i \int_{R(0)}^{R(t)} \langle n(R) | \nabla_R | n(R) \rangle dR
\]

\[
= i \int_{R(0)}^{R(t)} A_n dR
\]  

(12)

where, $\tilde{A}_n(R)$ is the Berry potential. For a closed trajectory $C$, the Berry phase turns out to be gauge independent which gives us an impression that it could be experimentally observed,

\[
\gamma_n = i \oint_C \tilde{A}_n dR. 
\]  

(13)

Now, that we know the relation between the trajectory in the parameter space and a global geometric phase known as the Berry phase, let us see if we can actually observe the phase experimentally.

B. Aharonov–Bohm Effect

Let us try to imagine an experimental setup. This setup is known as an Aharonov–Bohm setup, and the effect that we are going to
discuss is called the Aharonov–Bohm (AB) effect. Let’s begin with an infinitely long solenoid (Figure 4). When a current is passed through the solenoid, a magnetic field is produced, which is zero outside the solenoid. Now take a charged particle of charge \( q \) localized at a position \( \vec{X} = 0 \). Corresponding to the magnetic field, we can define a vector potential \( \vec{A} \). The form of \( \vec{A} \) is gauge-dependent. Let us choose suitable gauge such that \( \vec{A}(\vec{X} = 0) = 0 \).

Now, from this point, we let the charged particle moves slowly around the solenoid. The corresponding Hamiltonian can be written as,

\[
H = \frac{1}{2m} \left(-i\hbar \nabla - q\vec{A}(\vec{X})\right)^2 + V(\vec{X}),
\]

where, \( V(\vec{X}) \) is the localizing potential for localizing the particle at every point \( (\vec{X}) \) during this traversal. Here, we can see that \( \vec{A}(\vec{X}) \) acts as the external parameter of the Hamiltonian and simply by varying \( \vec{X} \) (the position of the particle) the parameter space can be traversed. With this Hamiltonian, the Schrödinger equation can be written and solved to obtain the wavefunction,

\[
\psi(\vec{X}) = \exp \left( \frac{iq}{\hbar} \int_0^{\vec{X}} \vec{A}(\vec{X}) \cdot d\vec{X} \right) \psi(\vec{X} = 0).
\]

Now bring the particle back to the same position (where it started the journey) after covering a closed path \( C \) around the solenoid. You can imagine that the traversal was done very slowly such that the adiabatic theorem remains valid, and after the path got closed, the wavefunction would acquire a Berry phase. We use (13) to write the Berry phase as

\[
e^{i\gamma} = e^{\frac{iq}{\hbar} \int_0^{\vec{X}} \vec{A}(\vec{X}) \cdot d\vec{X}}.
\]

In general, if a particle of charge \( q \) goes around a flux \( \Phi \), then the AB phase accumulated by the wavefunction is \( e^{i\Phi} = e^{\frac{2i\pi q}{\hbar} \Phi} \). where, \( \Phi \) is the magnetic flux enclosed by the loop and \( \Phi_0 = \frac{\hbar}{e} \) is the ‘fundamental quantum of flux’. One can see that the Berry phase thus obtained is not dependent on the choice of gauge and depends on the strength of the applied magnetic field. Hence, this can be observed experimentally [8].

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Now, let us investigate how this phase affects the physics of the charged particle. Consider Figure 4 again. Let’s now work in angular coordinates with the particle moving at a constant radius \( r \). We have,

\[
\oint \mathbf{A} \cdot d\mathbf{r} = \oint \mathbf{B} \cdot d\mathbf{s} = \Phi 
\]

(17)

\[
A_\phi = \frac{\Phi}{2\pi r}.
\]

(18)

The Hamiltonian and the energy solutions are now given by,

\[
H = \frac{1}{2m} \left(-i\hbar \partial_\phi - qA_\phi\right)^2,
\]

(19)

\[
\psi_n = \frac{1}{\sqrt{2\pi r}} e^{in\phi},
\]

(20)

\[
E_n = \frac{\hbar^2}{2mr^2} \left(n + \frac{\Phi}{\Phi_0}\right)^2, \quad n \in \mathbb{Z}.
\]

(21)

From this, we see that, when we make \( \Phi = \Phi_0 \), the form of the spectrum remains the same. This goes to show that the Hamiltonians at the two \( \Phi \) values have the same set of eigenstates. When \( \Phi : 0 \rightarrow \Phi_0 \) the energy eigenstate goes from the ground state to the first excited state of the initial Hamiltonian. Though for the respective Hamiltonians, the energy of the eigenstates would have interchanged, keeping the adiabatic theorem in mind. This is known as ‘spectral flow’. Taking into account the AB phase accumulated at the end, we see something interesting. The energy eigenstate (with respect to the initial Hamiltonian) increases by \( m \) whenever the flux takes the value \( m\Phi_0 \). We will use the idea of the Berry phase and spectral flow to understand certain physical situations that will lead to the understanding of the anyons.

C. Hall Effect

Imagine a current flowing in \( X \)-direction (due to some applied potential) on a conducting strip on the XY-plane of width \( w \) (in \( Y \)-direction), subjected to a magnetic field \( B \) in \( Z \)-direction. It is known that in such a system, a voltage called the Hall voltage—\( V_H \)—develops in the transverse \( (Y) \) direction. Classically, the
Lorentz force acting on the charge carriers is canceled by the force of the electric field due to the Hall voltage, thereby reaching a steady-state condition. For electrons of charge $e$ moving with velocity $v$, we write,

$$e v B = \frac{e V_H}{w}$$

$$\implies \frac{V_H}{I} = \frac{B}{ne} = R_H,$$

where, $n$ is the density of electrons, so the current $I = nev$ and $R_H$ is known as the Hall resistance.

Measurement of the Hall effect and the analysis of the same using the classical model described is extremely useful as this directly gives the charge carrier density and the nature of charge carriers (electrons or holes) in a system. From this classical model of the Hall effect, we see that the Hall resistance should increase linearly with the strength of the applied magnetic field ($B$). Furthermore, the magnetic field has no effect on the longitudinal resistance measured in the X-direction ($R_{xx}$). However, this classical theory has a limit. This limit was first encountered experimentally by Klaus Von Klitzing and collaborators in 1980. In the seminal work [10], the group studied the Hall effect on a silicon metal-oxide-semiconductor field-effect transistor (Si-MOSFET)
at extremely low temperatures (below 1.5 Kelvin). Klitzing et al., increased the magnetic field to a very high value of 18 T, when the MOSFET was kept at 1.5 Kelvin. Surprisingly, the Hall resistance, instead of following a straight line with the magnetic field, displayed distinct plateaus. Between the two plateaus, the Hall resistance was seen to jump exactly by one quantum of conductance \( (e^2/h) \). The Hall resistivity was seen to vary as,

\[
\rho_{xy} = \left( \frac{h}{e^2} \right) \frac{1}{\nu},
\]

where, \( \nu \) took on precisely integer values. That is why this effect is called integer quantum Hall effect (IQHE). A difference between the classical and quantum case is shown in Figure 5.

This discovery led to Klaus Von Klitzing winning the 1985 Nobel Prize. The ratio \( \left( \frac{\hbar}{e^2} \right) \) is now known as the Von Klitzing constant which is a new practical standard for electrical resistance [11].

These experimental observations suggested that there was something more interesting than what the classical theory taught us was going on. What we have missed in our classical model is obviously quantum mechanics. We have also ignored any interaction between the electrons. As we will see later in this article, many-body Hamiltonian and wavefunctions lead to a new phenomenon called the integer quantum Hall effect (IQHE), and the presence of interactions in them lead to the emergence of fractional quantum Hall effect (FQHE) and the anyons.

\textbf{D. Landau Levels}

Before discussing the physics of the quantum Hall systems, we need to review another quantum story, the story of Landau levels. Let us consider a free electron under the effect of a magnetic field. The Hamiltonian is given by (14) with \( g = -e \) and we take \( V = 0 \).

As we will be discussing particles going round and round, we’ll work in the symmetric gauge, i.e.,
\[ \vec{A} = -\frac{y B}{2} \hat{x} + \frac{x B}{2} \hat{y}. \]

This preserves the rotational symmetry about the origin. Now let’s define momentum as \( \vec{p} = \vec{p} + e \vec{A} \), and the ladder operators as \( a = \frac{\vec{e} \cdot \vec{m}}{\sqrt{2} e h B} \), with \([a, a^\dagger] = 1\).

Thus, it can be shown that the Hamiltonian can be reduced to a form,

\[ H = \hbar \omega_B \left( a^\dagger a + \frac{1}{2} \right). \]

Here, \( a \) and \( a^\dagger \) are operators similar to the annihilation and creation operators, respectively, as in a harmonic oscillator problem. \( \omega_B = \frac{2B}{m} \) is the cyclotron frequency. Therefore, the Hamiltonian is of the form of the one-dimensional Harmonic oscillator. Hence, the energy spectra \( E_n \) and eigenstates \( |n\rangle \) are given by,

\[ E_n = \hbar \omega_B \left( n + \frac{1}{2} \right), \quad n \in \mathbb{N} \]

\[ a|0\rangle = 0 \]

\[ a|n\rangle = \sqrt{n}|n - 1\rangle \]

\[ a^\dagger |n\rangle = \sqrt{n + 1}|n + 1\rangle. \]

The discrete energy levels are equally spaced, and the spacing depends on the magnetic field. These energy levels are known as the Landau levels. The Landau levels are responsible for the quantum Hall effect. The discreteness of the energy spectrum is reflected in the quantization of the quantum Hall conductance.

2. Quantum Hall Effect

We have already introduced the quantum Hall effect in section I(C). Now, we will look at the physics that gives rise to the quantum Hall effect. Let us first analyze the classical situation a bit.
Figure 6. The (fractional) quantum Hall effect. The figure was reproduced with permission from [12].

more and see how it relates to quantization. Since we are practically dealing with charged particles confined to two spatial dimensions and in presence of a magnetic field, it is natural to expect that Landau levels will form. It is easy to see that for a finite-size system, the degeneracy of each Landau level is $\Phi/\Phi_0$, where $\Phi$ is the net magnetic flux piercing through the sample, and $\Phi_0$ is the fundamental quantum of flux. This means, $\Phi/\Phi_0$ number of electrons will fill one Landau level. Now, if we compare the Hall resistance as calculated from the classical model, and (22), we see that they are equal when,

$$n = \frac{B}{\Phi_0} \nu.$$

This is the carrier density needed to fill $\nu$ Landau levels, and hence, here we see that $\nu$ is the filling fraction of Landau levels. When $\nu$ Landau levels are filled, there is a gap of $\hbar \omega_B$ with $\omega_B = eB/m$ in the energy spectrum. Therefore, for small electric fields and when $k_BT \ll \hbar \omega_B$, the electrons don’t move and $R_{xx} = 0$. This is indeed what is seen experimentally. The longi-
tudinal resistance is no longer constant with the magnetic field. Whenever the Fermi level of the system falls between two Landau levels, $R_{xx}$ becomes zero. We will refer to the energy gap between the two Landau levels as the ‘mobility gap’.

When the Fermi level of the system falls in the mobility gap, the system should, in principle, be an insulator as there are no states at the Fermi energy. However, when the bulk of the sample is an insulator, the edges can have special states called the ‘edge states’ or the ‘extended states’ due to the confinement potential present near the edges. The current is carried by such extended states. These states also have special quantum properties. For example, these states are chiral in nature. That means, in one edge of the sample, the current can flow only in one direction. These have very important consequences. We will use the fact that the extended states exist. We will also take for granted that there are localized states due to the charge carriers getting trapped by impurities in a real sample. But, we will not discuss the role of impurities and the emergence of the extended states in detail as that will be a major digression from the topics of our interest i.e., fractional quantum Hall effect and the anyons.

Before proceeding to the fractional quantum Hall effect, let us try to get a deeper understanding of the quantum processes going on in a quantum Hall system. For that, we will follow the seminal work by Laughlin [13], [14]. As usual, let us take a rectangular conducting sheet in the X-Y plane, which is finite in the Y direction (width $L_y$), provide a magnetic field $B$ in Z-direction, and

**Figure 7.** Edges of the rectangular sheet are identified. Extended states give rise to current in presence of flux threading through the cylinder.

When the bulk of the sample is an insulator, the edges can have special states called the ‘edge states’ or the ‘extended states’ due to the confinement potential present near the edges. The current is carried by such extended states. These states also have special quantum properties.
subject to an electric field in X-direction. We now use the Landau
gauge $\mathbf{A} = xB\hat{y}$. We differ here from the symmetric gauge used
previously. The Landau gauge makes a rectangular geometry easier
to handle. For $N$ electrons, the Hamiltonian (14) is modified to,

$$H = \sum_{j} N \left( \frac{1}{2m} \left( p_{x_j}^2 + (p_{y_j} - eBx_j)^2 \right) + eE x_j \right),$$

and the individual electron orbital wavefunctions have the form,

$$\psi_{n,k}(x,y) \sim e^{iky}H_n(x + x_0)e^{-\frac{(x-x_0)^2}{2b^2}}$$

$$x_0 = k\frac{e}{B} + \frac{mE}{eB^2} \frac{\hbar}{B}$$

where, $H_n$ are Hermite polynomials. The many-body wavefunc-
tion will come out of a Slater determinant of the electron wave-
functions; remember that we have ignored the electron-electron
interaction here.

First, notice that when we change the vector potential by a con-
stant, such that,

$$\mathbf{A} \rightarrow \mathbf{A} + A_0\hat{y},$$

we get the current operator to be,

$$\frac{e}{m} \sum_j \left[ \frac{\hbar}{i} \frac{\partial}{\partial y} - eA_y(r_j) \right] = \frac{\partial H}{\partial A_0}.$$

As we have seen earlier, the addition of a constant vector poten-
tial does nothing but multiply a phase factor to the wavefunction;
it is a gauge transformation with no physical meaning. To give
it a physical meaning we perform a trick. As shown in Figure
7, take the rectangular sample and connect the edges parallel to
the X-axis to make a cylinder. We can do this as (22) does not
depend on the geometry of the sample. Now, add a flux through
the cylinder (say by inserting an infinitely long solenoid in the
middle) which can now be used to change the vector potential
experienced by the electron on the cylinder. As we change this flux
slowly, current flows in a loop around the cylinder due to Faraday’s law. Now, the states which are localized (due to impurities) don’t contribute to this current, they just get an AB phase factor of $exp\left(\frac{ieA}{\hbar}\right)$, where $y$ is the coordinate around the loop. But the extended states contribute to the current. Hence, to keep the wavefunctions single-valued as $y \rightarrow y + L_y$, the vector potential is quantized,

$$A = \frac{n \Phi_0}{L_y},$$

and the location of the electron shifts as $x_0 \rightarrow x_0 - \frac{eA}{B}$. As we change the flux, the Hamiltonian ($H_0$) and hence, the energy spectra ($E_0$) change. But what this also does is that it shifts the electrons in the direction of the applied electric field (corresponding to the potential $V$). There are at least two extended states at the edges of the loop. Now, if we increase the flux to $\Phi_0$, spectral flow occurs, and we get the same states but with the excited state energy interchanged with the ground state energy. Therefore, there is a net transfer of one electron from each Landau level, from one end of the loop to the other. The total change in energy is, $\Delta E = veV$. Using the Hellmann–Feynman theorem[15], we see that the current can now be expressed as the change in energy with respect to the change in flux, which gives,

$$I = \frac{\Delta E}{\Phi_0} = \frac{veV}{\Phi_0} = \frac{ve^2V}{h},$$

$$R_H = \frac{V}{I} = \left(\frac{\hbar}{e^2}\right) \frac{1}{V}.$$  

More formally, we write,

$$E' = \frac{\hbar}{e^2v} \hat{\gamma} \times \hat{f}' \, \quad (23)$$

Note that while this description gives the correct result for Hall resistance, this does not provide a detailed view of how the extended states arise at the edges and localized states form the in-
Figure 8. Flux threading through annulus creates azimuthal electric field (black) and produces a radial current (red). This gedanken experiment creates an eigenstate of a fractional charge: A quasi-hole [12].

Sculpting bulk of the sample. As mentioned before, we will refrain from discussing that here. Readers are referred to [16] for more on these aspects. We would like to simply mention that the plateaus that arise at different values of $\nu$ occur due to the ‘sufficient’ amount of impurities present in the material and the presence of localized and extended states.

In this derivation, though we have assumed that $\nu$ is integer-valued, (23) holds good even for fractional $\nu$ values—this happens in fractional quantum Hall effect (FQHE). Please see Figure 6 displaying fractional quantization in a quantum Hall effect experiment. Now we are ready to go into the details of FQHE, where anyons will emerge.

3. Fractional Quantum Hall Effect (FQHE)

To start this section, let us first assume that we have the effect, i.e., $\nu$ in (23) is fractional. Here we will be dealing with quasi-holes and quasi-particles. These ‘quasi-things’ are formally thought to be excitations of a ground state describing the FQHE. We will first show that these have fractional charges, and that is what makes $\nu$ a non-integer. Then, in the next section, we will discuss these in the context of ‘vortices’—that will help us understand the fractional statistics of these ‘quasi-things’.
1. Exciting Fractional Excitations

Consider the situation in Figure 8. The situation is similar to the one we discussed in the calculation of the AB phase, but now there are multiple electrons trapped on an annulus. Let us start from the ground state. As in the AB phase calculations, we increase the flux in the solenoid from 0 to $\Phi(t)$ adiabatically. Due to change in the flux, an azimuthally symmetric electric field $E$ (electric field makes the electrons move around the disk, i.e., in $\hat{\theta}$ direction) is experienced by the electrons. Now, there is a subtlety here. The ground state is highly degenerate. But each state differs according to the angular momentum quantum number $m$ (this will become clearer when we see Laughlin’s many-body wavefunction), and so does the position of the state from the center.

If we increase the flux slow enough so that the electrons move around the solenoid at a time period much smaller than the time it takes for the electric field to operate, level crossing does not take place. This is because angular momentum is conserved, and the transition between degenerate eigenstates with different angular momentum is forbidden. Hence, the adiabatic condition can be applied. As the flux is increased, a radial current density $J_r$ due to (23) is seen.

The physical quantities mentioned above are given by,

\[
E = \frac{1}{2\pi r} \frac{\partial \Phi}{\partial t},
\]

\[
J_r = \frac{e^2 v}{\hbar} E = \frac{e^2 v}{\hbar} \frac{1}{2\pi r} \frac{\partial \Phi}{\partial t},
\]

\[
I_r = \oint J_r dS = \oint J_r r d\theta = \frac{e^2 v}{\hbar} \frac{\partial \Phi}{\partial t}.
\]

One can see that, the total current at radius $r$ does not depend on $r$. This shows that there will be a lump of charge accumulated at the interior of the annulus at time $t$. This is given by,

\[
Q(t) = \frac{e^2 v}{\hbar} \Phi(t) = ev \frac{\Phi}{\Phi_0}.
\]  

(24)

From (24) we can see that, when we increase the flux to $\Phi_0$, a lump of charge $ev$ is accumulated at the interior. Also due to
the application of the adiabatic theorem, this state must be an
eigenstate of the Hamiltonian. Furthermore, due to spectral flow,
the spectrum of this state is the same as that at $\Phi = 0$. Hence,
there must be an excited state at $\Phi = 0$ with a lump of charge $e\nu$
localized at the interior. This state is known as a ‘quasi-particle’ if
the charge is negative and a ‘quasi-hole’ if the charge is positive.
We have assumed that $\nu$ takes fractional values, and hence, these
have fractions of charges of electrons. In summary, quasi-holes
and quasi-particles are excitations above the ground state with
charges $e\nu$, and interestingly can be thought of as a fraction $\nu$
of a whole particle (hence, the word ‘quasi’).

Now, without further ado, let us get $R_H$ with fractional values of
$\nu$.

2. Laughlin Wavefunctions and Resistance

Laughlin proposed the following many-body wavefunction to take
into account the interactions between electrons. This surprisingly
was a very good variational approximation to the true ground state
and for a few particles [12, 17].

$$
\psi(z_i) = \prod_{i<j}^{N}(z_i - z_j)^m e^{-\sum_{i=1}^{N} \frac{E_F}{4\pi^2}} ,
$$

(25)

where, $z_j = x_j + iy_j$ are the positions of the particles. Here, $m$
is the relative angular momentum of a pair of particles. Here, $m$
is assumed to take odd values, and it can easily be seen that the
wavefunction describes QH state for fermions. In this wavefunc-
tion, there are $m(N−1)$ powers of a particular $z_i$. This tells us that
the maximum angular momentum of the particle is $m(N − 1)$ and
so its maximum radius is $R = l_B \sqrt{2mN}$. The corresponding area
is $A \sim 2\pi mNl_B^2$. The number of states in a Landau level is given
by $N = \frac{BA}{e\Phi_0} = \frac{A}{2\pi l_B^2} \sim mN$. This shows that the filling fraction
$\nu = \frac{1}{m}$ for this state!

Quasi-holes can also be described using similar wavefunctions.
The main idea is that at these holes, the electron density vanishes
in the electron fluid. For $M$ quasi-holes in a QH fluid at positions $\eta_j$, the wavefunction is given by.

$$\psi(z; \eta) = \prod_{l=1}^{M} \prod_{k=1}^{N} (z_k - \eta_l) \prod_{i<j} (z_i - z_j)^m e^{-\sum_{i=1}^{N} \frac{\text{Im} z_i}{\eta_i}}. \quad (26)$$

From this, it can also be seen that the quasi-holes have fractional charges. To see this, keep $m$ quasi-holes at the same place, the first factor is raised to the power of $m$, and the first $\Pi$ vanishes. If $\eta$ described the position of another electron, then this modified wavefunction would describe $N + 1$ electrons. But $\eta$ is instead a parameter that describes the deficit of electrons. Hence, $m$ holes describe the deficit of a single electron and hence, a quasi-hole is $\frac{1}{m}$ of an electron with charge $+\frac{2}{m}e = ev [17].$

Now, we easily get the value of $R_H$ for FQH fluid (when we deal with vortices, we’ll see why the notion of fluid comes up). Let us consider the situation of Figure 8 again and increase the flux to $\Phi_0$. As we have seen, this induces a spectral flow that increases the angular momentum by one, and a particle of charge $ev$ accumulates in the interior of the annulus. This means that a whole electron is transferred to the outer part of the annulus if the flux is increased to $m\Phi_0$. Hence, for the transfer of one electron, the previous analysis gives back (22) for fractional $v$. The primary difference between the previous case (IQHE) and this case (FQHE), is that in the previous case, $v$ arose from the number of Landau levels filled. But here, $v$ arises due to $\frac{1}{m}$ quasi-holes being transferred, generating a current.

A. Vortices in the Fractional Quantum Hall Fluid

In the creation of quasi-holes, we saw that we can multiply (25) with $\prod_i (z_i - \eta)$ to get a quasi-hole at position $\eta$. This factor also increases the power of $z_i$ for each electron by one, and hence, its angular momentum. This is also clear from the annulus experiment. Just when the quasi-hole is created, the electrons receive a unit of angular momentum, which we will now write explicitly
as \( \hbar \). Hence, we say that a quasi-hole carries a ‘quantum of vorticity’. This is crucial for the statistics of quasi-holes. But before going any further, let us review vortices in two dimensions.

1. The Whirlpools

To think of vortices, simply imagine a whirlpool. Water moves vigorously around the center of the whirlpool. The center itself does not seem to contain water; it is a fluid-depleted region. Now constrain the same picture to two dimensions (XY-plane) and make the angular momentum of each fluid particle the same with respect to the center. This vortex then carries a single unit of vorticity. You may want to keep this picture in mind for all further discussions.

Water moves vigorously around the center of the whirlpool. The center itself does not seem to contain water; it is a fluid-depleted region. Let’s take a vortex with a center at a point \( R \) and the relative angular momentum of each particle as \( l \). When the vortex is static, the velocity field of the background fluid is given by \( \bar{v}(\vec{r}) = \frac{l}{m} \times \frac{(\vec{r} - \vec{R})}{|\vec{r} - \vec{R}|^3} \) [18]. When the vortex moves, the velocity field is given by the sum of the motion around the center and the center’s relative velocity with the background fluid, \( \vec{R} \). If \( \vec{R} \) is in X-direction, then there is a velocity difference in the Y-direction and hence, a pressure difference leading to the Magnus force in Y-direction given by,

\[
\vec{F}_{\text{Mag}} = 2\pi n l \hat{\epsilon} \times \hat{\nu},
\]

where, \( n \) is the number density of the fluid. Now, here is something interesting. When we compare this force to the Lorentz force, we see that putting \( eB = 2\pi n l \) gives the Magnus force. To put this in a quantum mechanical picture, we use the AB phase. The AB phase, when an electron moves, is given by \( \exp \frac{ie}{\hbar} \left( \int d\vec{S} \cdot \vec{B} \right) \) so then the analogous expression for vortices moving should be \( \exp \frac{2\pi i}{\hbar} \left( \int nd\vec{S} \right) \). Now, if the vorticity of the vortex is \( l = \hbar \), the phase accumulated by a vortex traversing a closed loop is \( 2\pi \) times the number of fluid particles encircled by the vortex as
it goes around the loop. A flux quantum is analogous to a single fluid particle. Note that we have restricted our attention to neutral fluids.

To make the analogy with the AB phase more precise, imagine a vortex confined to move on a ring. Now, imagine a radial current flowing into the circle. In analogy with flux increasing in AB phase calculations, as in Figure 4, this current exerts a force on the vortex and accelerates it, leaving it with increased angular momentum. The energy, which was a function of flux in AB phase calculation, is now a function of fluid particles introduced. This helps us understand why the flux quantum is analogous to a single fluid particle.

The previous description compares and contrasts with something which is more or less a classical picture. Can we do better and calculate the Berry potential of the vortices? To do this, we use the Born–Oppenheimer approximation and say that the dynamics of a vortex depend on the dynamics of the position of its center \( R \) alone. Hence, we define \( |\psi(R)| \) and get the Berry potential from the kinetic energy as,

\[
A_B = \text{Im}(\langle \psi | \nabla_R | \psi \rangle).
\]

When we considered Laughlin wavefunctions to get quasi-holes, we multiplied the wavefunction with \( \prod_i z_i \) type factor to get the quasi-hole wavefunction. \( \prod_i z_i \) can hence, be thought of as an operator which acts on a quantum Hall state without quasi-holes to create a quasi-hole. Similarly, we define an operator

\[
\hat{\psi} = \prod_i e^{i\rho \cdot (r_i - R)} G(|r_i - R|),
\]

which acts on a featureless state to give a vortex. \( G \) is such that \( G(0) = 0 \), and hence, gives a depleting density at the center of the vortex. The exponent is responsible for giving each particle of the flux an angular momentum of \( \hbar \) with respect to \( R \). If \( \rho(r) \) is the fluid density, then using this type of operator we get,

\[
A_B(R) = \int dr \rho(r) \frac{\hat{\psi} \times (r - R)}{|r - R|^2}.
\]
This shows that the vortex sees fluid particles like electrons see flux tubes. This makes it clear that quantum Hall states can be treated like fluids.

2. Moving Whirlpools Around Whirlpools

Before going to quasi-holes, a subtlety needs to be taken into account. When a vortex in fluids goes around another vortex, adiabaticity never holds. This is because the collective charge density mode in these fluids is gap-less. Hence, if there is no adiabatic condition, we cannot talk about overall phase change in the wavefunction due to the encircling of vortices. This is the primary difference between these fluids and the quantum Hall fluids.

Let us now consider quasi-holes. As the quasi-hole moves relative to this quantum Hall fluid, it experiences a force proportional and perpendicular to its velocity, given by (27) resulting from the AB phase. If we make the angular momentum $\hbar$ and use $\omega_0 = \nu \langle n \rangle$ is number density), we get the force acting on a quasi-hole as,

$$\vec{F} = ev(\vec{v} \times \vec{B}),$$

which looks as if the Lorentz force is acting on a charge $e\nu$. This shows that quasi-holes are collective degrees of freedom like vortices. Hence, we can now think of quasi-holes as vortices in an FQH fluid. Now, let’s move a quasi-hole around another quasi-hole. We saw that when vortices move in a closed-loop, the phase that it accumulates is $2\pi$ times the number of fluid particles it encloses. But now, due to the fractional charge, we can think of a quasi-hole as $\nu$ fraction of the QH fluid particle, which is an electron. Interestingly, as a quasi hole is $\nu$ fraction of a fluid particle, and a fluid particle is analogous to a flux quantum, a quasi-hole can be thought of as $\nu$ flux quantum with charge $e\nu$. Hence, when a quasi-hole moves around another quasi-hole, the phase it accumulates is $2\pi\nu$. And since a winding of one quasi-hole around another is just two interchanges, we find that an interchange of the position of two quasi-holes multiplies the wave function by a phase of $\pi\nu$. Now, you recall the definition of the anyons given in
section 1. You can clearly see, the quasi-holes are neither bosons nor fermions. They satisfy the criteria for being anyons! [12]

Hence, we found anyons in the fractional quantum Hall state.

**B. Composite Fermions**

Using quasi-holes, we were able to develop a theory for filling fractions given by $\nu = \frac{1}{m}$. However, this is not the complete story. Figure 6 shows more exotic filling fractions such as $\frac{2}{5}, \frac{1}{3}$ and so on. The composite fermion theory gives us a handle on such fractions. Here we will see how we can deal with the exotic filling fractions and how quasi-holes of this theory are also anyons.

1. *Getting to FQHE*

We will follow a field theoretic approach as given by A. Stern [12]. The Hamiltonian of this theory is given by,

$$H = \frac{1}{2m} \int d^2r \left| \left( i \vec{\nabla} - \vec{A} \right) \psi(r) \right|^2 + H_{\text{int}},$$

(28)

where, $H_{\text{int}}$ is the Coulomb interaction part and $\psi(r)$ is the electronic annihilation operator. This multi-body Hamiltonian makes sense as, upon normal ordering, we get,

$$H = \frac{1}{2m} \int d^2r \left| \left( i \vec{\nabla} - \vec{A} \right) \psi^\dagger(r) \psi(r) \right| + H_{\text{int}},$$

(29)

giving the kinetic operator coupled with the electronic number operator [19]. Now to get to the composite fermion theory, we need to do a Chern–Simons transformation to get the composite fermion annihilation operator given by

$$\psi_{\text{cf}}(\vec{r}) = \psi(\vec{r}) \exp ip \int d^2r' \rho(\vec{r}') \text{arg}(\vec{r} - \vec{r}')$$

where, $p$ is an integer and $\rho = \psi^\dagger(r) \psi(r)$ which gives,

$$H = \frac{1}{2m} \int d^2r \left| \left( i \vec{\nabla} - \vec{A} + \partial^2 \psi^\dagger(r) \psi(r) \right) \right| + H_{\text{int}},$$

(30)
where, $\vec{\nabla} \times \vec{d} = p\Phi_0 \rho(\vec{r})$. This transformation attaches $p$ flux quanta to each electron, transforming it to a composite particle with charge $e$, which follows fermionic statistics. We can even see this from the Laughlin wavefunctions (25) and (26). If we interpret (25) as having $(m-1)$ quasi-holes sitting in the same place, per electron present, we see that (25) represents a wavefunction for a composite fermion with $p = (m - 1)$ here.

Using the Hartree mean-field approximation, we can replace $\vec{d}$ with its average value, mapping the problem of electrons at a magnetic field $B$ to composite fermions at a magnetic field $B^* = B - p\Phi_0 n$, where $n$ is the average density. Due to this, we get interesting results. The magnetic fields experienced by electrons and composite fermions differ, and so the filling fractions also differ,

$$n = \frac{\nu^* B^*}{\Phi_0} = \frac{\nu B}{\Phi_0}, \tag{31}$$

$$\nu = \frac{\nu^*}{1 + p\nu^*}. \tag{32}$$

From this we see that, if the composite fermion fills its lowest Landau level, i.e., $\nu^* = 1$, we get $\nu = \frac{1}{1 + p}$. IQHE for composite fermions gives rise to FQHE. The Hall resistance is given by $\frac{\hbar}{pe^2}$ for this theory. For a more detailed discussion on composite fermions, readers are referred to Jainendra K. Jain’s book on composite fermions [20], and for a simpler version, one can go through David Tong’s notes [17].

Now, let’s see how quasi-holes arise in this theory.

2. Getting to Anyons

To get the charge for quasi-things in $\nu = \frac{\nu^*}{1 + p\nu^*}$ states, we can repeat the same process on the annulus for composite fermions. Imagine adiabatically annihilating a composite fermion. This is done by first removing a charge $e$ and then introducing a flux of $p\Phi_0$, all adiabatically. The flux drives away a charge of $\frac{-\nu e}{1 + p\nu^*}$, leaving a net charge of $\frac{-\nu e}{1 + p\nu}$ as the net charge of a quasi-thing. When a composite fermion is annihilated, a single unit of charge
is removed from the Landau level but a net charge of \( q \) units (units of \( e \)) is removed from the system. This increases the magnetic flux \( \Phi^* \) by \( p\Phi_0q \) \( (\Phi^* = \Phi - p\Phi_0(N - q)) \), as \( q \) is just the number/fraction of particle units removed) making the charge in each Landau level increase by \( pq \) since, in this theory, each Landau level has a fermion per flux quantum. Hence, one Landau level lost a charge \( q \) and gained charge \( pq \), the other \( \nu^* - 1 \) levels got \( pq \) charge. Hence, the total charge satisfies,

\[
-q = -1 + pq + pq(\nu^* - 1) = -1 + pn^*q \implies q = \frac{1}{pn^* + 1}.
\]

When the composite fermion is taken out from a filled Landau level, the level acquires vorticity. When a quasi-hole goes around a closed loop, it acquires a phase depending on the number/fraction of particles it encloses. But the fraction of these particles is basically the number of \( q \) values attached to the Landau level. Hence, if a quasi-hole in the \( i^{th} \) Landau level encircles a quasi-hole in the \( j^{th} \) Landau level, the phase accumulated is,

\[
2\pi(1 - pq) = 2\pi \left( 1 - \frac{p}{pn^* + 1} \right) \text{ if } i = j,
\]

as the net \( q \) values of the level in which the quasi-hole is present is \((1 - pq)\) and,

\[
2\pi (-pq) = \left( -\frac{2\pi p}{pn^* + 1} \right) \text{ if } i \neq j,
\]

as the net \( q \) values of the level in which the quasi-hole is not present is \(-pq\). This shows that even in the composite fermion picture, the quasi-holes behave like anyons with fractional charges.

### 4. Abelian and Non-Abelian Anyons

We have discussed how quasi-particles (quasi-holes) of different theories become what are known as abelian anyons. Their wavefunction upon exchange attains a phase factor. So, when we exchange say anyon 1 with anyon 2, then anyon 2 with anyon 3, we would get the same wavefunction if instead, we exchanged the first 2 with 3, then 1 with 2. This is the abelian feature.
we would get the same wavefunction if instead, we exchanged the first 2 with 3, then 1 with 2. This is the abelian feature. But this may not be true in general. Instead of just a phase factor, the wavefunction could be multiplied by a unitary matrix upon exchange. Particles, which display this behavior are known as 'non-abelian anyons'. These are generally seen in $\nu = \frac{3}{2}, \frac{7}{2}$ type FQH states [12],[21],[22]. What we lacked in our discussion is the possibility that the eigenstates in the Berry phase calculations could be degenerate, or transitions happen between degenerate states. To understand the elegant physics of the non-abelian anyons, we would need to tackle the degeneracy issue. But, that is beyond the scope of this article.

Summary

We began our discussion with a seemingly abstract concept of anyons and then saw how its presence is revealed through fascinating thought experiments in FQH fluids. We saw how different theories at different filling fractions gave rise to anyons and studied their statistics. We then stated the possibility of more exotic anyons, namely non-abelian anyons. In summary, we can now say that anyons can be found through careful experiments and are indeed fascinating objects. In fact, non-abelian anyons are studied widely as they are thought to be good candidate particles for fault-tolerant quantum computing.

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Suggested Reading

[2] Non-abelian anyons also exist, where, the wavefunction after exchange differs by a unitary operation. But, we will not discuss non-abelian anyons here.


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