An Elementary Proof of the Power Rule of Differentiation*

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The power rule of differentiation provides simple proofs of numerous important results in mathematics, statistics, engineering, and economics. Motivated by Tao [1], we found an elementary proof of the power rule of differentiation for any real index. Our approach is elementary in the sense that we do not use the product rule, quotient rule, or inverse function theorem of differentiation as derived in most of the standard texts, e.g., see [2, 3]. We derive the power rule $\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$, for $x > 0$ and any real α, without using the derivatives of e^x and $\ln x$. We also don't use the binomial theorem or telescopic property of finite sums, e.g., see [4, p.161].

The power rule of differentiation has the power to provide simple proofs of numerous important results in mathematics, statistics, engineering, and economics.

Definition 1. *For a real $x > 0$, and α any real number, we define the number x^α by the formula $x^\alpha = \lim_{n\to\infty} x^{r_n}$, where $(r_n)_{n=1}^{\infty}$ is any sequence of rational numbers converging to α.*

The above definition is well defined, see [1, Def. 6.7.2, Lemma 6.7.1]. Alternatively, we know that $x^r = e^{r\ln x}$ holds for all rationals r. By the continuity of e^x, it follows that $\lim_{n\to\infty} x^{r_n} = \lim_{n\to\infty} e^{r_n \ln x} = e^{\lim_{n\to\infty} r_n \ln x} = e^{\alpha \ln x}$, while most of the texts define x^α by the formula $x^\alpha = e^{\alpha\ln x}$, that is equivalent to the above definition 1.

Keywords
Power rule of differentiation, sequences.

Theorem 1. (**The Power Rule.**) *Let α be a real number, and let $f : (0,\infty) \to \mathbb{R}$ be defined by $f(x) = x^\alpha$. Then f is differentiable on $(0,\infty)$ and that $f'(x) = \alpha x^{\alpha-1}$.*

Proof. For $a > 0$, we have $f'(a) = \lim_{x\to a} \frac{x^\alpha - a^\alpha}{x-a} = a^{\alpha-1} \lim_{x\to a} \frac{(\frac{x}{a})^\alpha - 1}{(\frac{x}{a}) - 1} =$

*Vol.26, No.11, DOI: https://doi.org/10.1007/s12045-021-1265-2

$a^{\alpha-1}\lim_{u\to 1}\dfrac{u^{\alpha}-1}{u-1}$. To show that f is differentiable it is enough to show that $\lim_{u\to 1}\frac{u^{\alpha}-1}{u-1}$ exists and is equal to α. We prove it stepwise for different α.

Step 1. We prove the result when α is any integer. For $\alpha = 0$, the result is trivial. Let $\alpha = n$ is any natural number. We have $\lim_{u\to 1}\dfrac{u^n-1}{u-1} = \lim_{u\to 1}(1+u+u^2+..+u^{n-1}) = n$. The last limit follows by the continuity of the polynomial.
Let $\alpha = n$ be a negative integer, then $n = -m$ for some $m \in \mathbb{N}$. We have $\lim_{u\to 1}\dfrac{u^n-1}{u-1} = \lim_{u\to 1}\dfrac{u^{-m}-1}{u-1} = \lim_{u\to 1}(-\dfrac{1}{u^m})\dfrac{u^m-1}{u-1} = -m = n.$

Step 2. We prove the result when α is any rational. For $\alpha = \frac{1}{n}$, where $n \in \mathbb{N}$. We have $\lim_{u\to 1}\dfrac{u^{\frac{1}{n}}-1}{u-1} = \dfrac{1}{\lim_{u\to 1}\frac{(u^{\frac{1}{n}})^n-1}{u^{\frac{1}{n}}-1}} = \dfrac{1}{n}$. Let $\alpha = \frac{p}{q}$, where $p, q \in \mathbb{N}$. We have $\lim_{u\to 1}\dfrac{u^{\frac{p}{q}}-1}{u-1} = \lim_{u\to 1}\dfrac{(u^{\frac{1}{q}})^p-1}{u-1} = \lim_{u\to 1}\left(\dfrac{(u^{\frac{1}{q}})^p-1}{u^{\frac{1}{q}}-1}\right)\left(\dfrac{u^{\frac{1}{q}}-1}{u-1}\right) = p\dfrac{1}{q}$. From the discussion in Step 1 for negative integer, it follows that the result holds for negative rationals also.

Let $\alpha > 0$. Then there exist sequences $(r_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$ of positive rationals converging to α such that $r_n \geqslant \alpha$ and $q_n \leqslant \alpha$ for all $n \in \mathbb{N}$.

Step 3. Finally, we prove the result when α is any real. Let $\alpha > 0$. Then there exist sequences $(r_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$ of positive rationals converging to α such that $r_n \geqslant \alpha$ and $q_n \leqslant \alpha$ for all $n \in \mathbb{N}$. For $u > 1$, we have $\dfrac{u^{\alpha}-1}{u-1} \leqslant \dfrac{u^{r_n}-1}{u-1}$ and $\dfrac{u^{\alpha}-1}{u-1} \geqslant \dfrac{u^{q_n}-1}{u-1}$, hence $\lim_{u\to 1^+}\dfrac{u^{\alpha}-1}{u-1} \leqslant \lim_{u\to 1^+}\dfrac{u^{r_n}-1}{u-1} = r_n$ for all n. Hence, $\lim_{u\to 1^+}\dfrac{u^{\alpha}-1}{u-1} \leqslant \alpha$. Similarly, we can prove that $\lim_{u\to 1^+}\dfrac{u^{\alpha}-1}{u-1} \geqslant \alpha$. Thus, we have proved $\lim_{u\to 1^+}\dfrac{u^{\alpha}-1}{u-1} = \alpha$. A similar argument for $u < 1$ shows that $\dfrac{u^{\alpha}-1}{u-1} \leqslant \dfrac{u^{r_n}-1}{u-1}$ and $\dfrac{u^{\alpha}-1}{u-1} \geqslant \dfrac{u^{q_n}-1}{u-1}$, hence $\lim_{u\to 1_-}\dfrac{u^{\alpha}-1}{u-1} = \alpha$, proving the result for $\alpha > 0$.

Let $\alpha < 0$. Applying the above argument for $-\alpha > 0$, we have $\lim_{u\to 1} \frac{u^{-\alpha} - 1}{u - 1} = -\alpha$. That is $\lim_{u\to 1} (-u^{-\alpha}) \frac{u^{\alpha} - 1}{u - 1} = -\alpha$. It is easy to show that $\lim_{u\to 1} u^{-\alpha} = 1$, apply the above argument. Hence, $\lim_{u\to 1} \frac{u^{\alpha} - 1}{u - 1} = \alpha$, for $\alpha < 0$.

This completes the proof of the theorem.

□

Suggested Reading

[1] **T Tao, *Analysis I*, Third Edition, Hindustan Book Agency, 2014.**

[2] **D R Sherbert and R G Bartle, *An Introduction to Real Analysis*, Fourth Edition, Wiley 2014.**

[3] **W Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, 1964.**

[4] **T M Apostol, *Calculus, Volume 1, One-variable Calculus, With an Introduction to Linear Algebra*, Wiley, 1967.**