In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

**Claim:** In this note, we discuss the convergence of following class of infinite continued fractions.

Let us define

\[ x_{(m,n)} = m + \frac{n}{m + \frac{n}{m + \ldots \infty}} \]

where ‘\(m\)’ be any positive real number and ‘\(n\)’ be any non-zero real number.

Then

(i) For \(m > \sqrt{n}\); 
(a) \(x_{(m,n)} = \frac{m + \sqrt{m^2 + 4n}}{2}\); when \(n > 0\),

for \(m > 2 \sqrt{n}\); 
(b) \(x_{(m,n)} = \frac{m + \sqrt{m^2 + 4n}}{2}\); when \(n < 0\),

for \(m = 2 \sqrt{n}\); 
(c) \(x_{(m,n)} = \frac{m}{2}\); when \(n < 0\).

**Keywords**  
Continued fractions, convergence.

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The results which we prove in this note will make extensive use of the following well-known theorem and the subsequent two simple deductions.

1. **Theorem A (p.84 [1])**: Let $g$ be continuous on the closed interval $[a, b]$ with $g : [a, b] \rightarrow [a, b]$. Then $g$ has a fixed point $p \in [a, b]$. Furthermore, if $g$ is differentiable on the open interval $(a, b)$ and there exists a positive constant $k < 1$ such that $|g'(x)| \leq k < 1 \ \forall \ x \in [a, b]$, then the fixed point in $[a, b]$ is unique.

2. **Deduction 1**: Let $g(x) = m + \frac{n}{x}; \ x \in \mathbb{R} \setminus \{0\}$.
   \[
   |g'(x)| < 1 \Rightarrow \left| \frac{n}{x^2} \right| < 1 \Rightarrow \left| x^2 > |n| \right. \quad \text{(I)}
   \]

3. **Deduction 2**: $x_{(m,n)} = m + \frac{n}{x_{(m,n)}}$
   \[
   \Rightarrow x_{(m,n)}^2 - mx_{(m,n)} - n = 0; \quad x_{(m,n)} = \frac{m \pm \sqrt{m^2 + 4n}}{2} \quad \text{(II)}
   \]

(i)-(a) $m^2 + 4n > 0$, $n > 0$ and $m > \sqrt{n}$.

\[
x_{(m,n)} = \frac{m + \sqrt{m^2 + 4n}}{2}
\]

**Example 1.**

\[
2 + \frac{3}{2 + \frac{3}{2 + \ldots \infty}} = 3.
\]

**Pattern of convergence →**
**Note 1:** Let \( n > 0 \) and \( m > \sqrt{n} \) be any positive real number. In this case, it is easy to verify that:

\[
\frac{m - \sqrt{m^2 + 4n}}{2} < \sqrt{n} < \frac{m + \sqrt{m^2 + 4n}}{2}.
\]

Also \( g_1(\sqrt{n}) = m + \sqrt{n} > \sqrt{n} \) and \( \lim g_1(x) = m > \sqrt{n} \).

So \( g_1([\sqrt{n}, \infty)) \subset [\sqrt{n}, \infty), \) where we assume \( g_1 : [\sqrt{n}, \infty) \to [\sqrt{n}, \infty) \)
defined by

\[
g_1(x) = m + \frac{n}{x}; \quad x \in [\sqrt{n}, \infty).
\]

In this case, considering above fact, convergence follows from *Theorem A and Deduction 1*.

Also it is worth mentioning that the fixed point (point of convergence) in this case is unique.

![Graph showing the convergence of the sequence](image)

(i)-(b) \( m^2 + 4n > 0, \) \( n < 0 \) and \( m > 2 \sqrt{|n|} \).

\[
x_{(m,n)} = \frac{m + \sqrt{m^2 + 4n}}{2}.
\]

**Example 2.**

\[
5 - \frac{4}{5 - \ldots \infty} = 4.
\]

\*Pattern of convergence*

**Note 2:** Let \( n < 0 \) and \( m > 2 \sqrt{|n|} \) be any positive real number. Like previous deduction, it is easy to verify that:

\[
\frac{m - \sqrt{m^2 + 4n}}{2} < \sqrt{|n|} < \frac{m + \sqrt{m^2 + 4n}}{2}.
\]

\( \text{IV} \)
Also \( g_2(\sqrt[n]{m}) > \sqrt[n]{m} \) and \( \lim_{x \to \infty} g_2(x) = m > 2 \sqrt[n]{m} \), so, \( g_2([\sqrt[n]{m}, \infty)) \subseteq [\sqrt[n]{m}, \infty) \), where we assert \( g_2 : [\sqrt[n]{m}, \infty) \to [\sqrt[n]{m}, \infty) \) defined by 

\[
g_2(x) = m + \frac{n}{x}; x \in [\sqrt[n]{m}, \infty). \]

In this case, convergence follows from Theorem A and Deduction 1.

As before, in this case, the fixed point (point of convergence) obtained is also unique.

\[(i)-(c) \quad m^2 + 4n = 0, \quad n < 0 \text{ and } m = 2 \sqrt[n]{m}. \]

\[x_{(m,n)} = \frac{m}{2}. \]

**Example 3.**

\[
2 - \frac{1}{2} = 1. \\
2 - \frac{1}{2 - \frac{1}{2 - \ldots \infty}}
\]

**Pattern of convergence →**

\[y = x \]

\[g_3 = m + \frac{n}{x} \]

\[(m, m + \frac{n}{m}) \]

\[\begin{align*}
\text{Note 3:} & \text{ Let } n < 0 \text{ and } m = 2 \sqrt[n]{m} \text{ be any positive real number.} \\
& \text{Like previous deduction, it is easy to verify that;} \\
& \quad \frac{m}{2} = \sqrt[n]{m}. \quad \text{(IV)}
\end{align*}\n
Also \( g_3(\sqrt[n]{m}) = \sqrt[n]{m} \) and \( \lim_{x \to \infty} g_3(x) = m = 2 \sqrt[n]{m} \), so, \( g_3([\sqrt[n]{m}, \infty)) \subseteq [\sqrt[n]{m}, \infty) \), where we assert \( g_3 : [\sqrt[n]{m}, \infty) \to [\sqrt[n]{m}, \infty) \) defined by 

\[
g_3(x) = m + \frac{n}{x}; x \in [\sqrt[n]{m}, \infty). \]

In this case, convergence follows from Theorem A and Deduction 1.
As before, in this case, the fixed point (point of convergence) obtained is also unique.

\[ m^2 + 4n < 0 \]
\[ x_{(m,n)} \text{ diverges.} \]

**Example 4.**

\[ \begin{array}{c}
2 - \frac{3}{3} \\
2 - \frac{3}{2 - \ldots \infty}
\end{array} \]

diverges.

**Note 4:** In this case, divergence follows from the fact that the quadratic equation involved here, i.e. \( x^2 - mx - n = 0 \) has negative discriminant, and so no real root (point of convergence) is possible here.

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**Suggested Reading**


