In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

In this article, we consider 35 special properties of a triangle with an angle of 60°. These properties relate to the most different elements of the triangle: sides, angles, area, circumradius and inradius, various singular points of the triangle, and their mutual location. Some of them are quite elementary and can be used in the study of geometry by high school students. Others are more complex and involve concepts outside of the school geometry course (Fermat point, isodynamic point, symmedian, and others). They can be helpful in teaching mathematics teachers at teacher training colleges and universities and can be used for mathematical enrichment and the expansion of the ‘toolbox’ available to them.

You can hardly find another example of a polygon having as many different properties as the ‘king of triangles’—an equilateral triangle. Maybe only a square can compare with it in this. In the book [1] about 200 pages, are devoted to the properties of an equilateral triangle and its role in culture, history, and everyday life of human society. And although the equilateral triangle has an ancient history, you can find more and more publications that reveal

**Keywords**

Triangle with an angle of 60°, triangle properties, Fermat point, isodynamic point.

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*Vol.26, No.8, DOI: https://doi.org/10.1007/s12045-021-1213-1*
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However, if we waive the requirement that all the angles of the triangle be $60^\circ$, but keep the requirement that at least one of the angles be $60^\circ$, then it turns out that such a triangle also has many, and sometimes unexpected, properties. This article presents 35 such properties.

What prompted us to consider a triangle with an angle of $60^\circ$? At one of the geometry lessons, the schoolgirl received the task to prove that if in the triangle $ABC$ ($BA = BC$), the bisectors $AE$ and $CF$ intersect at point $I$, then $IE = IF$. The schoolgirl easily coped with the task and asked if the opposite statement was true, that is if it was true that in case the segments of the bisectors $IE$ and $IF$ were equal, then the triangle was isosceles. Using GeoGebra, the students noticed that if $\angle B = 60^\circ$, then the triangle can have a shape far from isosceles. So we decided to investigate the properties of such a triangle. As a result, we found many interesting properties that we present below.

**The Properties of the Triangle**

Consider an arbitrary triangle $ABC$ with an angle of $60^\circ$. Denote three internal angles of the triangle by $\alpha, \beta, \gamma$ ($\alpha \leq \beta \leq \gamma$). Then $\beta = 60^\circ$, $\alpha = \beta - \theta$, $\gamma = \beta + \theta$, where $\theta$ is any angle less than $60^\circ$.

Denote the sides of the triangle $ABC$ by $a, b, c, a \leq b \leq c$, i.e. the angles $\alpha, \beta, \gamma$ are opposite the sides $a, b, c$ respectively.

1. According to the law of cosines $b^2 = a^2 + c^2 - ac$.

2. According to the law of sines $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{\sqrt{3}}{2} = b \sin (60^\circ + \theta) \leq b$, i.e. $\frac{\sqrt{3}}{2} \leq b$. The equality holds if and only if $\gamma = 90^\circ$.

3. Since $b^2 = a^2 + c^2 - ac$, $4b^2 = 4a^2 + 4c^2 - 4ac = 3a^2 + 3c^2 - 6ac + a^2 + c^2 + 2ac = 3(a-c)^2 + (a+c)^2$. Since $(a-c)^2 \geq 0$, we obtain that $4b^2 \geq (a+c)^2$, i.e. $b \geq \frac{a+c}{2}$. The equality holds for equilateral triangle only.

4. From (3), it follows that in a triangle with an angle of $60^\circ$, the length of the side opposite this angle is more than the arithmetic
mean, the geometric mean, and the harmonic mean of the lengths of two other sides: $b \geq \frac{a+c}{2} \geq \sqrt{ac} \geq \frac{2}{\frac{1}{a} + \frac{1}{c}}$.

5. From (3) it follows that $2b^2 = a^2 + c^2 + (a-c)^2 \geq a^2 + c^2 \Rightarrow b^2 \geq \frac{a^2 + c^2}{2}$.

6. $\frac{a^2 + c^2}{2} = \frac{(a+c)(a^2 + c^2 - ac)}{2} = \frac{ac}{2} \cdot b^2 \leq b \cdot b^2 = b^3$, i.e. $b^3 \geq \frac{a^2 + c^2}{2}$.

The equalities in 4–6 holds for equilateral triangle only.

7. Let point $I$ be incenter of triangle $ABC$. Then $ID = IE$ (see Figure 1).

Indeed $AD$, $BF$, and $CE$ are the angle bisectors and $\alpha + \gamma = 180^\circ - 60^\circ = 120^\circ$. Then $\angle DIE = \angle AIC = 180^\circ - \frac{\alpha + \gamma}{2} = 120^\circ$.

Thus in the quadrilateral $BDIE$ $\angle B + \angle DOE = 180^\circ$ and so $BDIE$ is a cyclic quadrilateral (i.e. a quadrilateral with vertices upon which a circle can be circumscribed). Since the inscribed angles $\angle DBI$ and $\angle EBI$ are equal, the chord $ID$ and $IE$ are equal.

Denote by $l_b$, $h$, $m_b$ the angle bisector $BL$, the altitude $BP$, and the median $BM$ from vertex $B$ of the triangle $ABC$ (Figure 2), and by $R$ and $r$ its circumradius and inradius respectively.

Then:

8. The area $\Delta$ of the triangle $ABC$ is:

$$\Delta = 0.5 \cdot ac \cdot \sin 60^\circ = 0.5 \cdot ah_b \sin 30^\circ + 0.5 \cdot cl_b \sin 30^\circ \Rightarrow l_b = \frac{\sqrt{3}ac}{2} = \sin 60^\circ \cdot \frac{2}{a+c}$$

that is the length of the internal bisector of the angle $A$.
Figure 2. Illustrations of the properties 8–17.

from the vertex $B$ is $\sin 60^\circ$ multiplied by the harmonic mean of the sides $a$ and $c$.

9. $h_b = \frac{2A}{b} \Rightarrow h_b = \frac{\sqrt{ac}}{2b} = \frac{\sqrt{ac}}{2 \sqrt{a^2 + c^2 - ac}}$

10. $m_b^2 = \frac{2a^2 + 2c^2 - b^2}{4}$ [7, pp. 282–283]. Using property 1 we obtain

$m_b^2 = \frac{a^2 + c^2}{4}$

11. $R = \frac{b}{2 \sin 60^\circ} = \frac{b}{\sqrt{3}}$

12. $r = \frac{2A}{a + b + c} \Rightarrow r = \frac{\sqrt{ac}}{2(a + b + c)}$

13. $PL^2 = l_b^2 - h_b^2 = \frac{3a^2 - \frac{a^2}{2(c-a)}}{4} - \frac{3b^2 - \frac{b^2}{2(c-b)}}{4} = \frac{9a^2 - c^2 + b^2}{4(b^2 + c^2 - ac)} \Rightarrow PL = \frac{3(b-a)}{2(c+a) \sqrt{b^2 + c^2 - ac}}$

14. $PM^2 = m_b^2 - h_b^2 = \frac{2a^2 + 2c^2 - b^2}{4} - \frac{3b^2 - \frac{b^2}{2(c-b)}}{4} = \frac{(c-a)^2}{4(b^2 + c^2 - ac)} \Rightarrow PM = \frac{c-a}{2(c+b) \sqrt{b^2 + c^2 - ac}}$

15. $LM = PM - PL = \frac{2a^2 + 2c^2 - b^2}{4} - \frac{3b^2 - \frac{b^2}{2(c-b)}}{4} = \frac{(c-a)(c-a)}{2(c+b) \sqrt{b^2 + c^2 - ac}} \Rightarrow LM = \frac{(c-a) \sqrt{b^2 + c^2 - ac}}{2(c+b)}$

16. From properties 9 and 14, it follows that $\tan \angle PBM = \frac{c-a}{\sqrt{3ac}}$.

17. From properties 9 and 13 it follows that $\tan \angle PBL = \sqrt{3} \cdot \frac{a}{c+a}$.

Let $AD$ and $CE$ be the altitudes, points $H$ and $I$ be orthocenter and incenter of the triangle $ABC$ (Figure 3a).
18. \( ED = b/2. \)

Indeed, in right angle triangles \( BAD \) and \( BCE \), \( \angle BAD = \angle BCE = 30^\circ \). Therefore, \( \frac{BD}{AB} = \frac{BK}{CB} = \frac{1}{2} \). From this it follows that the triangles \( DBE \) and \( ABC \) with the common angle \( B \) are similar and so \( \frac{ED}{EB} = \frac{BD}{AB} = \frac{1}{2} \).

Note 1. The same considerations are true in the case of \( \angle C \geq 90^\circ \) (see Figure 3b).

19. The four points \( A, I, H, C \) lie on the same circle.

\( \angle AHC = 90^\circ + 30^\circ = 120^\circ \). \( \angle AIC = 120^\circ \) (see property 7). Thus \( \angle AHC = \angle AIC \) and the points \( A, H, I, C \) lie on the same circle (see Figure 3a).

Note 2. If \( \angle C > 90^\circ \), then point \( H \) is in exterior of triangle \( ABC \) and in this case \( \angle CHD = 60^\circ \Rightarrow \angle CHD + \angle CIA = 180^\circ \) and so points \( A, H, I, C \) lie also on the same circle (Figure 3b). If \( \angle C = 90^\circ \), the points \( C \) and \( H \) are coincide and together with points \( I \) and \( A \) lie on the same circle.

20. Let \( O \) be the circumcenter of the triangle \( ABC \) and \( BL \) be the bisector of angle \( B \) (see Figure 4). Then \( BH = 2R \cos \beta = 2R \cos 60^\circ = R = BO \). Thus triangle \( HBO \) is isosceles. Point \( H \) is the isogonal conjugate of point \( O \) [8] (Recall that point \( V \) is the isogonal conjugate of point \( W \) respected to a triangle \( ABC \) if \( V \) can be constructed by reflecting the lines \( AW, BW, \) and \( CW \) about the bisectors of the angles \( A, B, C \) [7, p.153–158; 8]). So

\[
\text{Figure 3. (a) } ED = b/2. \\
\text{The points } A, I, H, C \text{ lie on the same circle. (b) The case of } \angle C \geq 90^\circ.
\]

\[
\text{Let } O \text{ be the circumcenter of the triangle } ABC \text{ and } BL \text{ be the bisector of angle } B \text{ (see Figure 4). Then } BH = 2R \cos \beta = 2R \cos 60^\circ = R = BO.
\]
**Figure 4.** Point $H$ is the isogonal conjugate of point $O$.

$BL$ is the bisector of angle $HBO$ and, therefore, it is also the altitude of the triangle $HBO$, i.e., $BL \perp HO$.

21. From the above paragraph, we also obtain that $BN$ is the median of the triangle $O$, i.e., $N$ is the midpoint of $HO$, and thus, $N$ is the nine-point center of the triangle $ABC$ (the center of the nine-point circle, that is also called Euler’s circle or Euler’s circle or Feuerbach’s circle [9, pp.20–22]).

22. This property is related to the Fermat point (or Torricelli point) of the triangle $ABC$. Recall that this is such a point that the sum of its distances to the vertices of the triangle is the minimum possible. In the case of our triangle $ABC$ with all angles less than $120^\circ$ [7, pp.218–221], this is a point $F$ such that $\angle AFB = \angle BFC = \angle AFC = 120^\circ$.

Denote the sum of the distances of point $F$ to the vertices by $d$. Let’s construct three equilateral triangles $ABC’$, $ACB’$, $BCA’$ on the sides of the triangle $ABC$ outward as shown in the Figure 5.

Then $AA’$, $BB’$ and $CC’$ intersect at point $F$ and $AA’ = BB’ = CC’ = d$ [7, pp.218–220; 9, pp.82–83].

Since $\angle ABA’ = 120^\circ$, according to the law of cosines for the triangle $ABA’$ we obtain that $(A’A)^2 = (A’B)^2 + (BA)^2 + A’B \cdot BA$.

From this, it follows the property:

$$d^2 = a^2 + c^2 + ac.$$
23. Let $AK$ and $CN$ be two cevians passed through the Fermat point $F$ (see Figure 6). Since $\angle KFN = 120^\circ$, $\angle B + \angle KFN = 180^\circ$, and so the quadrilateral $BKFN$ is cyclic. $\angle BFC = \angle BFA = 120^\circ$, $\angle KFC = \angle NFA \Rightarrow \angle BFK = \angle BFN = 60^\circ$. Thus $BK = BN$ and triangle $KBN$ is equilateral.

24. Denote $t = BK = BN = KN$ (see property 23). According to the Ptolemy's theorem [7, pp.62–63], for the quadrilateral $BKFN$, 

**Figure 5.** Point $F$ is the Fermat (Torricelli) point of the triangle $ABC$.

**Figure 6.** The quadrilateral $BKFN$ is cyclic.
Figure 7. $BG$ is the symmedian of the triangle $ABC$.

we obtain: $BK \cdot FN + BN \cdotKF = BF \cdot KN$. Then $t(FK + FN) = t \cdot BF \Rightarrow FK + FN = BF$.

25. From the above property it follows that $CN + AK = d$.

26. Using properties 10 and 22, we obtain $m_B^2 = \frac{a^2 + c^2 + ac}{4} = \frac{d^2}{4}$. Thus $m_B = \frac{d}{2}$.

27. Let $AB'C$ be equilateral triangle constructed on the side $AC$ of the triangle $ABC$ outward (Figure 7) and let point $O'$ be its center. We found that the median of triangle $ABC$ from vertex $B$ is: $BM = m_b = \frac{d}{2}$ (property 16). Then $B'O' : O'M = 2 : 1$. Since $BB' = d$ and $m_b = \frac{d}{2}$ we obtain $BB' : BM = 2 : 1 = B'O' : O'M$. According to the converse of the angle bisector theorem $BO'$ is the angle bisector of the triangle $B'BM$.

28. $\angle BCO' = 30^\circ + 60^\circ + \theta$, $\angle BAO' = 30^\circ + 60^\circ - \theta \Rightarrow \angle BCO' + \angle BAO' = 120^\circ$.
∠BAO' = 180°. Thus the quadrilateral BCO'A is cyclic. Since the chords CO' and AO' are equal, ∠CBO' = ∠ABO' and so BO' is also bisector of angle B.

29. Denote the point of intersection of AC and BB' by G (Figure 7). From properties 27–28, it follows that BG is the symmedian of the triangle ABC that is the line BG can be obtained by reflecting the line BM over the angle bisector from vertex B [7, pp.213–218]. Thus the Fermat point F lies on the symmedian from vertex B.

30. It is known that the Fermat point F is the isogonal conjugate of the internal isodynamic point S of the triangle ABC, that is of the internal point common to the three Apollonian circles of the triangle ABC [10]. Since F lies on BB' and the line BM can be obtained by reflecting BB' about the bisector of angle B (property 29), the isodynamic point S lies on the median from angle B.

31. Since ∠AFC = 120°, the point F lies on the circle passed through the points A, I, H, C (see property 19). It is known that ∠ASC = ∠B + ∠60° [1, p.296]. So ∠ASC = 120° and the point S also lies on the circle mentioned above (see Figure 8). Thus six points A, I, H, F, S, C lie on the same circle.

32. Denote AF = x, BF = y, CF = z (See Figure 5). The area S of the triangle ABC is: S = 0.5 · ac · sin 60° = 0.5 sin120° (xy + xz +
yz). Then $ac = xy + xz + yz$. According to the law of cosines
$a^2 = y^2 + z^2 + yz, c^2 = y^2 + x^2 + xy$. Since $a^2 = a^2 + c^2 + ac$
(property 22) we obtain $x^2 + y^2 + z^2 + 2(xy + xz + yz) = y^2 +
 z^2 + yz + y^2 + x^2 + xy + ac \Rightarrow 2ac = y^2 + yz + xy + ac \Rightarrow ac =
y(x + y + z) \Rightarrow ac = yd \Rightarrow y = BF = \frac{\sqrt{a}}{d}.

$a^2 = y^2 + z^2 + yz, z^2 + yz + y^2 - a^2 = 0$. Keeping in the mind that
$z > 0$, find:

\[2z = \sqrt{4a^2 - 3y^2} - y = \sqrt{4a^2 - 3} \cdot \left(\frac{\sqrt{a}}{d}\right)^2 - \frac{\sqrt{a}}{d} = \frac{2}{d} \cdot \sqrt{4(a^2 + c^2 + ac) - 3c^2 - c} = \frac{4}{d} \sqrt{(2a + c)^2 - c} = \frac{2a}{d} \Rightarrow
z = CF = \frac{4a}{d}.

Similarly obtain: $x = AF = \frac{a}{d}$.

33. Since $\angle BFK = \angle BFN = 60^\circ$, $\angle AFN = \angle CFN = 60^\circ$ (see
property 23), $FN$ and $FK$ are the angle bisectors in the triangles
$BFA$ and $BFC$ (see Figure 6). Thus for the triangle $BFA$
obtain: $\frac{AN}{NF} = \frac{BF}{FB} = \frac{c}{c + d} = \frac{a}{d}$. Similarly for the triangle $BFC$
obtain: $\frac{BN}{NF} = \frac{BF}{FC} = \frac{c}{c + d} = \frac{a}{d}$. Thus for two cevians $AK$ and $CN$
passed through the Fermat point $F$ it holds:

\[\frac{AN}{NB} = \frac{BK}{KC} = \frac{c}{a}.

34. $t = BN = BK$ (see properties 23, 24). From the above property
33 it follows that

\[\frac{t}{c} = \frac{a}{d} \Rightarrow t = \frac{ac}{d} \Rightarrow BK + BN = \frac{2a + c}{2}. \text{ Then from property 4}
obtain: $b \geq BK + BN$.

35. According to the formula for the distance between the circum-
center $O$ and any point in the plane of the triangle [11]

$OF^2 = \frac{b^2}{2s}(\sin 2\beta \cdot BF^2 + \sin 2\gamma \cdot CF^2 + \sin 2\alpha \cdot AF^2 - 2\Delta)$ (see
Figure 9).

Using the formula $\sin 2\varphi = 2 \sin \varphi \cos \varphi$ and the formulas

\[R = \frac{b}{\sqrt{3}}, \Delta = \frac{3ac}{4}, \sin y = \frac{c \sqrt{3}}{2b}, \sin a = \frac{a \sqrt{3}}{2b}, \cos a = \frac{c^2 + b^2 - a^2}{2bc}, \cos y = \frac{a^2 + b^2 - c^2}{2ba},
\]
obtain:

\[
OF^2 = \frac{b^2}{3} \left( \frac{ac}{d^2} + \frac{2a^3}{bd^2} \cos \gamma + \frac{2c^3}{bd^2} \cos \alpha - 1 \right)
\]

\[
= \frac{ab^2c + a^4 + a^2b^2 - a^2c^2 + c^4 + c^2b^2 - c^2a^2 - b^2d^2}{3d^2}
\]

\[
= \frac{b^2(ac + a^2 + c^2 - d^2) + a^4 - a^2c^2 + c^4 - c^2a^2}{3d^2}
\]

\[
= \frac{a^4 - 2a^2c^2 + c^4}{3d^2}.
\]

\[
OF^2 + BF^2 = \frac{a^4 - 2a^2c^2 + c^4}{3d^2} + \frac{a^2c^2}{d^2} = \frac{a^4 + a^2c^2 + c^4}{3d^2}
\]

\[
= \frac{(a^2 + c^2)^2 - a^2c^2}{3d^2} = \frac{b^2d^2}{3d^2} = \frac{b^2}{3} = R^2 = OB^2,
\]

that is \(BF\) is right-angled triangle and \(BF\) is perpendicular to \(FO\).

**Summary**

We found and presented 35 properties of the triangle with an angle of 60°. Some of them are of general nature, and some are related to the Fermat point and isodynamic point. Most likely, in
the future, some more new interesting properties of the considered triangle will be found.

Disclosure statement

No potential conflict of interest was reported by the authors.

Suggested Reading