Visual Estimation of Sum of First n Triangular, Tetrahedral and Pentatope Numbers*

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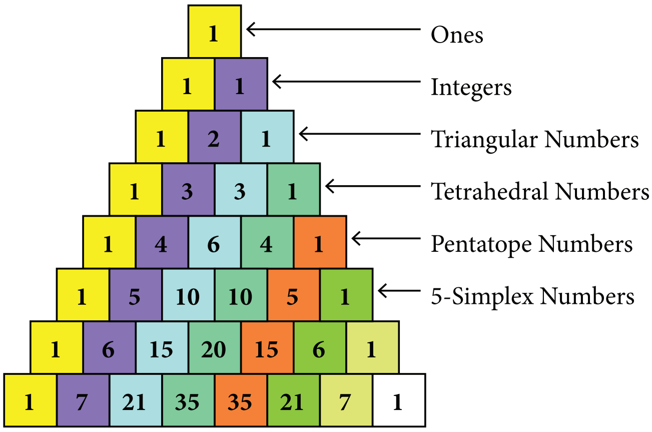
Triangular, tetrahedral and pentatope numbers appear along the third, fourth and fifth lines respectively of Pascal's triangle either from right side or from left.



In this article, we prove formulas for the sum of first n triangular numbers, first n tetrahedral numbers, and first n pentatope numbers visually and wordlessly.

The nth triangular number

$$t_n = \frac{n(n+1)}{2}.$$

The nth tetrahedral number

$$T_n = \frac{n(n+1)(n+2)}{6}.$$

The nth pentatope number

$$P_n = \frac{n(n+1)(n+2)(n+3)}{24}.$$

Claim 1 : Sum of first n triangular number is

$$T_n = \frac{n(n+1)(n+2)}{6}.$$

Claim 2 : Sum of first n tetrahedral number is

$$P_n = \frac{n(n+1)(n+2)(n+3)}{24}.$$

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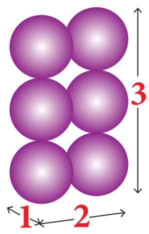
Keywords
Triangular numbers, tetrahedral numbers, pentatope numbers.

Claim 3 : Sum of first n pentatope numbers is

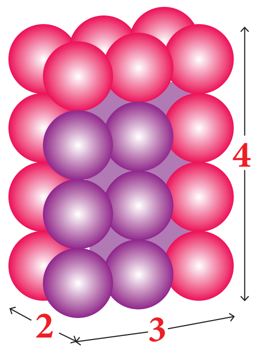
$$\sum_{i=1}^{n} P_i = \frac{n(n+1)(n+2)(n+3)(n+4)}{120}.$$

Visual proof of claim 1:

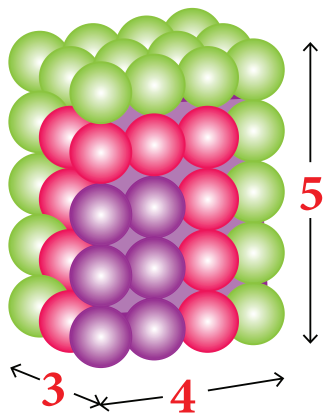
$$(t_1, t_2, t_3, t_4, \ldots) = (1, 3, 6, 10, \ldots).$$



$$6(t_1) = (1)(1+1)(1+2)$$



$$6(t_1 + t_2) = (2)(2+1)(2+2)$$



$$6(t_1 + t_1 + t_3) = (3)(3 + 1)(3 + 2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$6(t_1 + t_2 + t_3 + \cdots + t_n) = (n)(n + 1)(n + 2)$$

$$\Rightarrow \boxed{T_n = t_1 + t_2 + t_3 + \cdots + t_n = \frac{n(n+1)(n+2)}{6}}$$

Note 1:

Mathematics behind this visual thinking

$$6t_n = (n)(n + 1)(n + 2) - (n - 1)(n)(n + 1)$$

$$6t_1 = 1.2.3 - 0.1.2$$

$$6t_2 = 2.3.4 - 1.2.3$$

$$6(t_1 + t_2) = 2 \cdot 3 \cdot 4$$

$$6t_3 = 3 \cdot 4 \cdot 5 - 2 \cdot 3 \cdot 4$$

$$6(t_1 + t_2 + t_3) = 3 \cdot 4 \cdot 5$$

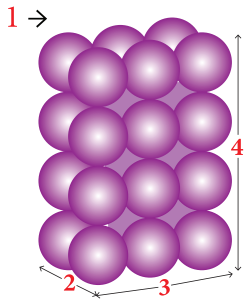
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$6(t_1 + t_2 + t_3 + \cdots + t_n) = (n)(n + 1)(n + 2)$$

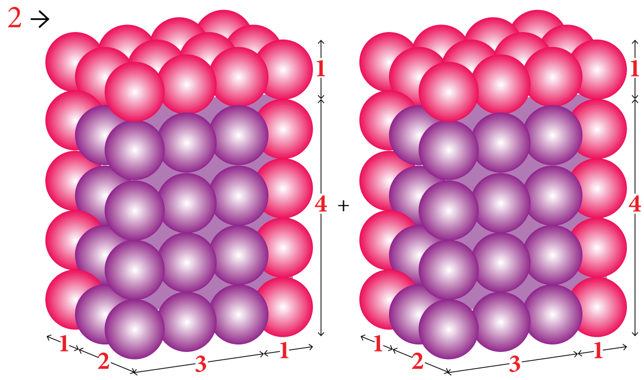
$$\Rightarrow \boxed{T_n = t_1 + t_2 + t_3 + \cdots + t_n = \frac{n(n+1)(n+2)}{6}}$$

Visual proof of claim 2:

$(T_1, T_2, T_3, T_4, \ldots) = (1, 4, 10, 20, \ldots).$



$$24\, T_1 = 1 \cdot 2 \cdot 3 \cdot 4 = 1(1{+}1)(1{+}2)(1{+}3)$$



$$24\,(T_1 + T_2) = 2 \cdot 3 \cdot 4 \cdot 5 = \mathbf{2}(\mathbf{2}{+}1)(\mathbf{2}{+}2)(\mathbf{2}{+}3)$$

$$24(T_1 + T_2 + T_3) = 3.4.5.6 = \mathbf{3}(\mathbf{3} + 1)(\mathbf{3} + 2)(\mathbf{3} + 3)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$24(T_1 + T_2 + T_3 + \cdots + T_n) = n(n + 1)(n + 2)(n + 3)$$

$$\Rightarrow \boxed{P_n = T_1 + T_2 + T_3 + \cdots + T_n = \frac{n(n+1)(n+2)(n+3)}{24}}$$

Note 2:

Mathematics behind this visual thinking

$$24T_n = n(n + 1)(n + 2)(n + 3) - (n - 1)n(n + 1)(n + 2)$$

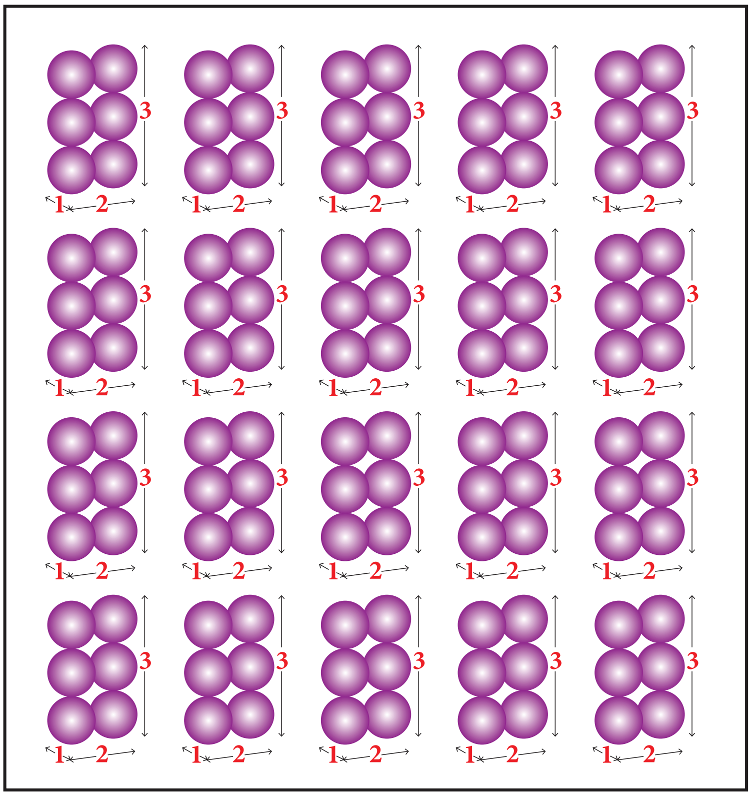
$$\begin{aligned}
\Rightarrow\ 24T_1 &= 1.2.3.4\\
24T_2 &= 2.3.4.5 - 1.2.3.4\\
\hline
24(T_1+T_2) &= 2.3.4.5\\
24T_3 &= 3.4.5.6 - 2.3.4.5\\
\hline
24(T_1+T_2+T_3) &= 3.4.5.6\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

$$24(T_1+T_2+T_3+\cdots+T_n) = n(n+1)(n+2)(n+3)$$

$$\Rightarrow \boxed{P_n = T_1+T_2+T_3+\cdots+T_n = \frac{n(n+1)(n+2)(n+3)}{24}}$$

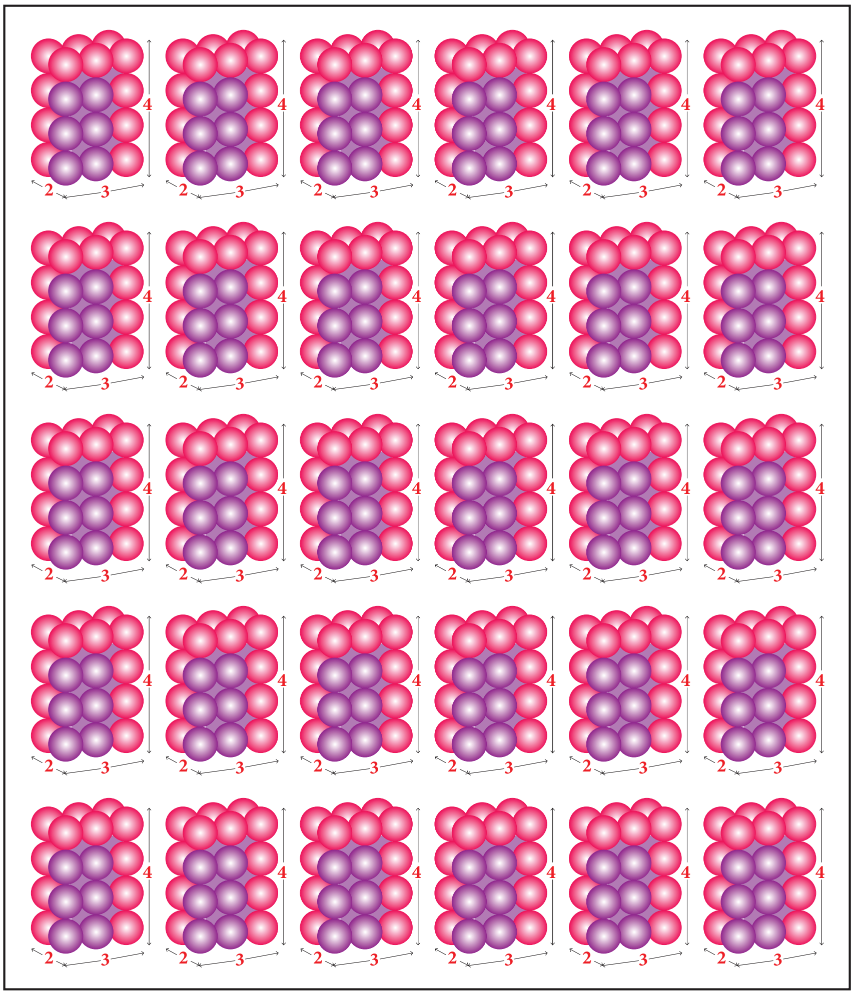
Visual proof of claim 3:

$(P_1, P_2, P_3, P_4, \ldots) = (1, 5, 15, 35, \ldots)$.



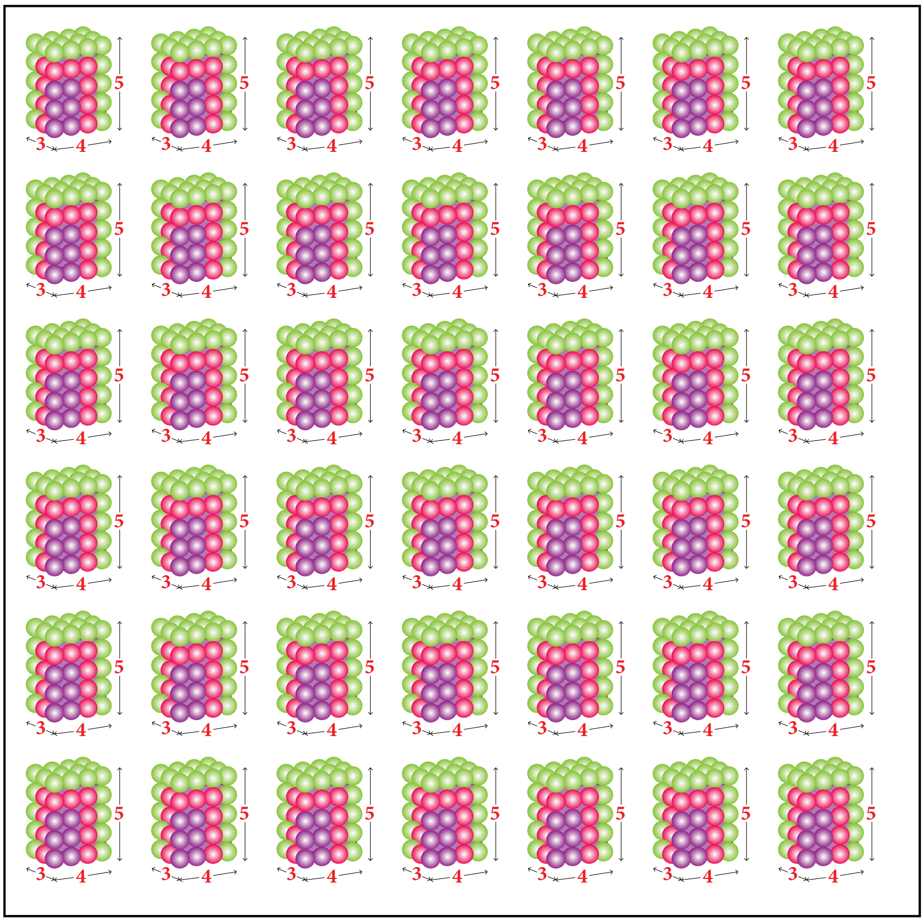
4×5

$$120P_1 = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 1(1+1)(1+2)(1+3)(1+4).$$



5 × 6

$$120(P_1 + P_2) = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 2(2 + 1)(2 + 2)(2 + 3)(2 + 4).$$



6×7

$$120(P_1 + P_2 + P_3) = 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 3(3 + 1)(3 + 2)(3 + 3)(3 + 4).$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$120(P_1 + P_2 + \cdots + P_n) = n(n + 1)(n + 2)(n + 3)(n + 4)$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} P_i = \frac{n(n + 1)(n + 2)(n + 3)(n + 4)}{120}}$$

Acknowledgement

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Suggested Reading

[1] Roger B. Nelsen, *Proofs without Words: Exercise in Visual Thinking*, The Mathematical Association of America, 1993.

[2] Roger B. Nelsen, *Proofs without Words II: More Exercise in Visual Thinking*, The Mathematical Association of America, 2000.

[3] Roger B. Nelsen, *Proofs without Words III: More Exercise in Visual Thinking*, The Mathematical Association of America, 2015.