Quantum Game Theory - III*

A Comprehensive Study

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This is the third and final part of the article titled Quantum Game Theory: A Comprehensive Study [1, 2]. Here, we introduce the concept of quantum two-person duel and conclude our three-part article.

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1. Two Person Duel

1.1 *Classical Version*

Let there be two players, Alice and Bob, who begin the game by pointing their guns towards each other and shooting. This shooting continues alternatively until one of them is down. The one to survive wins the game. Several pieces of literature [3–6] focus on two-person duels and/or three-person truels. A. P. Flitney and Derek Abbott [3] developed the quantum versions of the duels and truels, and W. F. Balthazar et al. [7] made an experimental realization of the quantum duel using linear optical circuits. The rules of the game are set as follows:

1. The moves by the players, i.e, firings are sequential. The first move is always by Alice, followed by Bob.
2. Firing at the air may be allowed.
3. Each of the players strictly prefers survival over non survival.
4. Let u_1, u_2 and u_3 be the utilities. The maximum payoff by a player is reached if the player reaches his/her objective of surviving at the end of the duel.

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Quantum game theory, quantum algorithms, quantum computing, quantum duel.

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- If the player eliminates the opponent and he/she survives, then the utility is $u_1 = 1$.
- If the player misses but he/she is survived, then the utility is u_2.
- If the player dies, then the utility is $u_3 = 0$.

The utilities follow the chain of inequalities:

$$u_1 \geq u_2 \geq u_3. \tag{1}$$

5. The probability of Alice missing is p_a and the probability of Bob missing is p_b. So, Alice shoots down Bob with probability $1 - p_a$ and Bob shoots down Alice with probability $1 - p_b$. Here,

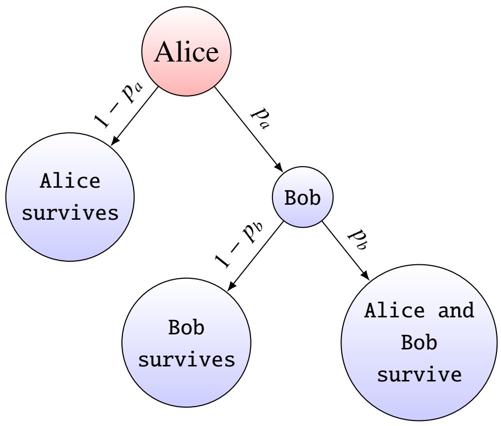
$$0 < p_a, p_b < 1. \tag{2}$$

6. The maximum number of rounds prescribed to each player is n.

7. None of the players has got any information about the strategies of the other player except the information about who survives and who does not.

In the game tree, right-hand branches are misses, and the left-hand branches are hits. If both Alice and Bob survive, there are further rounds left, and the tree gets repeated in the downward direction.

The game tree is represented by:



In the game tree, right-hand branches are misses, and the left-hand branches are hits. If both Alice and Bob survive, there are further rounds left, and the tree gets repeated in the downward direction. With both Alice and Bob having access to n rounds of shots, the expectation value of Alice's payoff is:

$$[\overline{\pi_{Alice}}]_n = 1 - p_a + (p_a)(p_b)[\overline{\pi_{Alice}}]_{n-1}. \tag{3}$$

Now, if $n \to \infty$, $[\overline{\pi_{Alice}}]_n = [\overline{\pi_{Alice}}]_{n-1}$. So, we have the expectation value of Alice as,

$$[\overline{\pi_{Alice}}]_\infty = \frac{1 - p_a}{1 - (p_a)(p_b)}. \tag{4}$$

And, the expectation value of Bob is, $[\overline{\pi_{Bob}}]_\infty = 1 - [\overline{\pi_{Alice}}]_\infty$. Now, we discuss the quantum version of this game.

1.2 *Quantum Version*

For the quantum duel, each player has access to a qubit $\in \{|0\rangle, |1\rangle\}$. The player, if alive, is represented by $|1\rangle$ and if dead, is represented by $|0\rangle$. The quantum state of the game is represented by:

For the quantum duel, each player has access to a qubit $\in \{|0\rangle, |1\rangle\}$. The player, if alive, is represented by $|1\rangle$ and if dead, is represented by $|0\rangle$.

$$|\psi\rangle = |Alice\rangle \otimes |Bob\rangle. \tag{5}$$

In the classical version, all the players were played separately, but in the quantum version, the qubits representing the 'dead' or 'alive' states are combined together by tensor products, so that the local unitary operations are applied to the combined state. The initial state of the game is represented as:

$$G_Q^0 = |\psi_0\rangle = |11\rangle. \tag{6}$$

because, both the players are alive and is equivalent to $|\psi_0\rangle = |1\rangle \otimes |1\rangle$. Each player is allowed to operate only on their opponent's qubit. If the opposite is allowed, it is called a *quantum Russian roulette* [8, 9]. The action of firing at an opponent is equivalent to flipping the opponent's qubit state from $|1\rangle$ to $|0\rangle$ using a local unitary operator. The operator of $SU(2)$ group that denotes Alice 'firing' at Bob, with probability of success equal to $1 - a = \sin^2(\frac{\theta_1}{2})$, is:

$$
\begin{aligned}
[A\hat{lic}e]_{Bob}(\theta_1,\alpha_1,\beta_1) = {} & [e^{i\alpha_1}\cos(\frac{\theta_1}{2})\,|11\rangle \\
& + ie^{i\beta_1}\sin(\frac{\theta_1}{2})\,|10\rangle]\,\langle 11| \\
& + [e^{-i\alpha_1}\cos(\frac{\theta_1}{2})\,|10\rangle \\
& + ie^{-i\beta_1}\sin(\frac{\theta_1}{2})\,|11\rangle]\,\langle 10| \\
& + |00\rangle\langle 00| + |01\rangle\langle 01|\,.
\end{aligned} \tag{7}
$$

The operator of $SU(2)$ group that denotes Bob 'firing' at Alice, with probability of success equal to $1 - b = \sin^2(\frac{\theta_2}{2})$, is :

$$
\begin{aligned}
[B\hat{o}b]_{Alice}(\theta_2,\alpha_2,\beta_2) = {} & [e^{i\alpha_2}\cos(\frac{\theta_2}{2})\,|11\rangle \\
& + ie^{i\beta_2}\sin(\frac{\theta_2}{2})\,|01\rangle]\,\langle 11| \\
& + [e^{-i\alpha_2}\cos(\frac{\theta_2}{2})\,|01\rangle \\
& + ie^{-i\beta_2}\sin(\frac{\theta_2}{2})\,|11\rangle]\,\langle 01| \\
& + |00\rangle\langle 00| + |10\rangle\langle 10|\,.
\end{aligned} \tag{8}
$$

In the above equations $\theta_1, \theta_2 \in [0, \pi]$ is fixed and is related to the shooting skill of both the players and the parameters, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [-\pi, \pi]$ are arbitrary phase factors which controls the outcome of a poorly performing player. In equation (48), the last two terms imply that Alice can do nothing if her qubit is in the 'dead' state. The same happens for equation (49), where the last two terms imply that Bob can do nothing if his qubit is in the 'dead' state. Also, it can be noticed that in equation (48), there is a term $|11\rangle\langle 10|$ that revives Bob, and in equation (49), there is a term $|11\rangle\langle 01|$ that revives Alice.

The game is carried on for a maximum of n rounds for each of the player. After the final state is generated, the payoff function to each of them is calculated. The player with the higher payoff wins. The game state after n rounds is given by:

$$
G_Q^n = |\psi_n\rangle = ([B\hat{o}b]_{Alice}[A\hat{lic}e]_{Bob})^n\,|\psi_0\rangle\,. \tag{9}
$$

Where $G_Q^0 = |\psi_0\rangle$ is the initial game state. The most natural initial game state is where both Alice and Bob are alive, given by equation (47). There can be other unnatural initial game states where a player can be dead to start with or in a superposition of dead and alive [10]. Now, keeping in mind the chain of inequalities concerning the utilities, given by equation (42), the expected payoff function to Alice is:

$$\overline{\pi_{Alice}} = u_1|\langle 10|\psi_n\rangle|^2 + u_2|\langle 11|\psi_n\rangle|^2 + u_3|\langle 01|\psi_n\rangle|^2. \quad (10)$$

Considering $u_2 = 0.5$, the equation becomes,

$$\overline{\pi_{Alice}} = |\langle 10|\psi_n\rangle|^2 + 0.5|\langle 11|\psi_n\rangle|^2. \quad (11)$$

the expected payoff function to Bob is:

$$\overline{\pi_{Bob}} = 1 - \overline{\pi_{Alice}} = |\langle 01|\psi_n\rangle|^2 + 0.5|\langle 11|\psi_n\rangle|^2. \quad (12)$$

Considering $G_Q^0 = |\psi_0\rangle = |11\rangle$, after application of $[\hat{Alice}]_{Bob}$ on G_Q^0, we get

$$[\hat{Alice}]_{Bob}|11\rangle = e^{i\alpha_1}\cos(\frac{\theta_1}{2})|11\rangle + ie^{i\beta_1}\sin(\frac{\theta_1}{2})|10\rangle. \quad (13)$$

Next, applying $[\hat{Bob}]_{Alice}$, we get the game state after one round, i.e,

$$\begin{aligned} G_Q^1 &= [\hat{Bob}]_{Alice}[\hat{Alice}]_{Bob}|11\rangle \\ &= |\psi_1\rangle \\ &= e^{i(\alpha_1+\alpha_2)}\cos(\frac{\theta_1}{2})\cos(\frac{\theta_2}{2})|11\rangle \\ &+ e^{i(\alpha_1+\beta_2)}\cos(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2})|01\rangle \\ &+ ie^{i\beta_1}\sin(\frac{\theta_1}{2})|10\rangle. \end{aligned} \quad (14)$$

If the probability amplitudes are calculated for each player, after the first round, it will give similar results as that of a classical duel, i.e,

$$|\langle 11|\psi_1\rangle|^2 = \cos^2(\frac{\theta_1}{2})\cos^2(\frac{\theta_2}{2}). \quad (15)$$

$$|\langle 01|\psi_1\rangle|^2 = \cos^2(\frac{\theta_1}{2})\sin^2(\frac{\theta_2}{2}). \tag{16}$$

and,

$$|\langle 10|\psi_1\rangle|^2 = \sin^2(\frac{\theta_1}{2}). \tag{17}$$

So the expected payoffs,

$$\overline{\pi_{Alice}} = \sin^2(\frac{\theta_1}{2}) + 0.5\cos^2(\frac{\theta_1}{2})\cos^2(\frac{\theta_2}{2}). \tag{18}$$

and

$$\overline{\pi_{Bob}} = \cos^2(\frac{\theta_1}{2})\sin^2(\frac{\theta_2}{2}) + 0.5\cos^2(\frac{\theta_1}{2})\cos^2(\frac{\theta_2}{2}). \tag{19}$$

If the game is proceeded in the next round, the state of the game after the second round is:

$$\begin{aligned} G_Q^2 &= ([\hat{Bob}]_{Alice}[\hat{Alice}]_{Bob})^2\,|11\rangle \\ &= [\hat{Bob}]_{Alice}[\hat{Alice}]_{Bob}\,|\psi_1\rangle \\ &= |\psi_2\rangle \\ &= [e^{2i(\alpha_1+\alpha_2)}\cos^2(\frac{\theta_1}{2})\cos^2(\frac{\theta_2}{2}) \\ &- e^{i\alpha_1}\cos(\frac{\theta_1}{2})\sin^2(\frac{\theta_2}{2}) \\ &- e^{i\alpha_2}\sin(\frac{\theta_1}{2})\cos(\frac{\theta_2}{2})]\,|11\rangle \\ &+ i[e^{i(2\alpha_1+\alpha_2+\beta_2)}\cos^2(\frac{\theta_1}{2})\cos(\frac{\theta_2}{2})\sin(\frac{\theta_2}{2}) \\ &+ e^{i(\alpha_1+\beta_2-\alpha_2)}\cos(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2})\cos(\frac{\theta_2}{2}) \\ &- e^{i\beta_2}\sin(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2})]\,|01\rangle \\ &+ i[e^{i(\alpha_1+\alpha_2+\beta_1)}\cos(\frac{\theta_1}{2})\cos(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) \\ &+ e^{i(\beta_1-\alpha_1)}\sin(\frac{\theta_1}{2})\cos(\frac{\theta_1}{2})]\,|10\rangle\,. \end{aligned} \tag{20}$$

Now, after calculating the probability amplitudes, the expected payoff to Alice turns out to be,

$$\overline{\pi_{Alice}} = 0.5(1 + |\langle 10|\psi_2\rangle)|^2 - |\langle 01|\psi_2\rangle)|^2). \tag{21}$$

and the expected payoff to Bob,

$$\begin{aligned} \overline{\pi_{Bob}} &= 1 - \overline{\pi_{Alice}}. \\ &= 0.5 - |\langle 10|\psi_2\rangle)|^2 + |\langle 01|\psi_2\rangle)|^2. \end{aligned} \tag{22}$$

Figure 1. A window for clicking on two-person duel.

Here, some quantum effects of interference are evident. The values of θ_1, θ_2 decide which cosine terms Alice/Bob plans to maximize. Next, we will consider a case, where, after the first round is played, Alice shoots at the air instead of shooting at Bob, i.e, in the second round $[A\hat{lice}]_{Bob} = I$. So, equation (61) becomes,

$$\begin{aligned} G_Q^2 &= [B\hat{o}b]_{Alice}[B\hat{o}b]_{Alice}[A\hat{lice}]_{Bob} |\psi_1\rangle \\ &= [e^{i(\alpha_1+2\alpha_2)} \cos(\frac{\theta_1}{2}) \cos^2(\frac{\theta_2}{2}) \\ &+ ie^{i\alpha_1} \cos(\frac{\theta_2}{2}) \sin^2(\frac{\theta_2}{2})] |11\rangle \\ &+ [ie^{i(\alpha_1+\alpha_2+\beta_2)} \cos(\frac{\theta_1}{2}) \cos(\frac{\theta_2}{2}) \sin(\frac{\theta_2}{2}) \\ &+ e^{i(\alpha_1-\alpha_2+\beta_2)} \cos(\frac{\theta_1}{2}) \cos(\frac{\theta_2}{2}) \sin(\frac{\theta_2}{2})] |01\rangle \\ &+ ie^{i\beta_1} \sin(\frac{\theta_1}{2}) |10\rangle . \end{aligned} \tag{23}$$

And the analyses proceeds further as the previous cases.

1.3 *Simulation Results*

A Python file named *QDuel.py* can be written that generates a graphical user interface (GUI) for handling different simulations concerning quantum duel. On running the software, a window appears. See *Figure* 1.

Clicking on the red button another window appears. See *Figure* 2.

This window consisting of the blue button asks the user to initialise the qubits. Clicking on this blue button provides the user with various initial quantum states to start with. See *Figure* 3.

Figure 2. A window for initializing the qubits.

Figure 3. A window allowing the user to start with various initial quantum states.

Figure 4. A window allowing the user to plot 4 types of graphs.

The superposed state of a player between 'alive' and 'dead' state is represented by, $|q_s\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. Selecting one of the six initial states as $|\psi_0\rangle$, a new window pops up, that lets us plot 4 types of graphs. See *Figure* 4.

1. Clicking on *Plot I*, a new window pops up. See *Figure* 5.

   The 'PLOT' button allows us to plot Alice's and Bob's expected payoffs as functions of α_1 and α_2 in a two round duel, i.e, $n = 2$. See *Figure* 6. To simulate games where more rounds are played, n is increased accordingly.

   In *Figure* 6, for $|\psi_0\rangle = |11\rangle$ and for parameters $a = \frac{2}{3}, b = \frac{1}{2}, \beta_1 =$

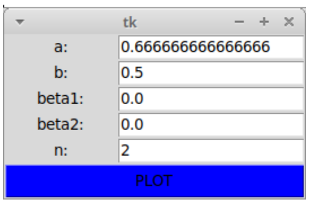


Figure 5. A window that pops up after clicking on Plot I.

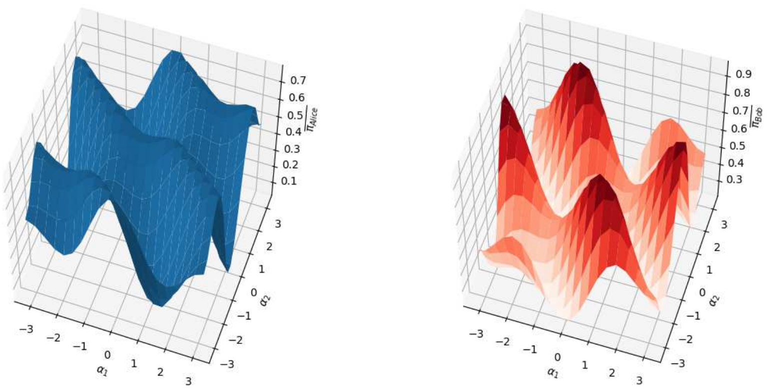


Figure 6. The expected payoffs as functions of α_1 and α_2 in a two round duel.

$0, \beta_2 = 0$, it can be noticed that, Alice's expected payoff is maximized for $\{\alpha_1 = \pm\frac{\pi}{3}, \alpha_2 = \mp\frac{2\pi}{3}\}$ or $\{\alpha_1 = \pm\pi, \alpha_2 = 0\}$ and Bob's expected payoff is maximized for $\{\alpha_1 = 0, \alpha_2 = \pm\pi\}$ or $\{\alpha_1 = \pm\frac{2\pi}{3}, \alpha_2 = \mp\frac{\pi}{3}\}$. Situations where more rounds are played are more complex.

2. Now on clicking on *Plot II* another window pops up. See *Figure* 7.

   The "PLOT" button allows the user to plot Alice's and Bob's expected payoffs as functions of the number of rounds played in a repeated quantum duel. The parameters α_1 and α_2 affect the values but the parameters β_1 and β_2 do not. We set $\alpha 1 = \alpha_2 = \beta_1 = \beta_2 = 0$. Also, $a = \frac{2}{3}$ and $b = \frac{1}{2}$. The plots generated are given in *Figure* 8

3. Clicking on *Plot III*, a new window pops up. See *Figure* 9.

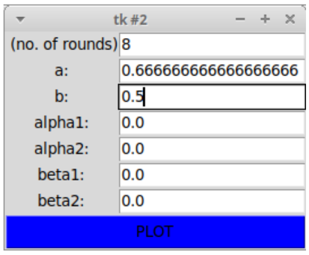


Figure 7. A window for entering the parameter values.

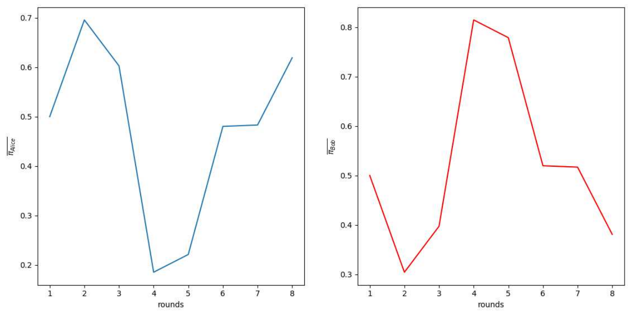


Figure 8. The expected payoffs as functions of the number of rounds played in a repeated quantum duel.

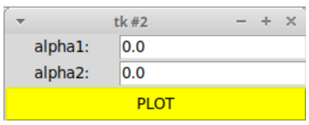


Figure 9. A window for entering the parameter values α_1 and α_2.

The 'PLOT' button allows the user to plot the improvement in Alice's expected payoff as a function of a and b, if Alice chooses to fire at the air in her second shot, in a two round game. The parameters set are, $\alpha_1 = \alpha_2 = 0$. The plot generated is given by *Figure* 10

A paradoxical result can be noticed. If Alice is a poor shot, i.e for $a \geq \frac{4}{5}$ and Bob is an intermediate shot, i.e, $b \equiv \frac{1}{2}$ then Alice has more chance of winning, if she shoots at the air during her second attempt.

4. Finally, clicking on *Plot IV*, we get the same window like that for

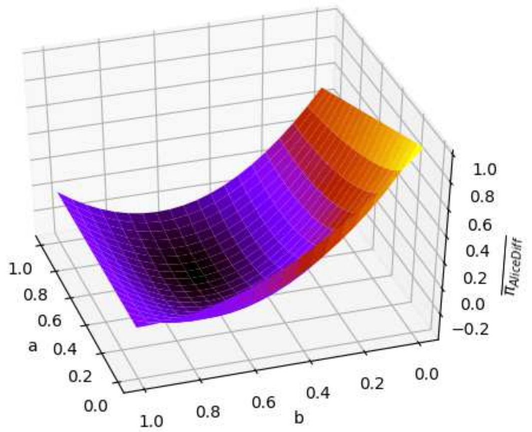


Figure 10. The improvement in Alice's expected payoff as a function of *a* and *b*, if Alice chooses to fire at the air in her second shot.

Plot III. Further clicking on the 'PLOT' button allows the user to plot the improvement in Bob's expected payoff, when Bob shoots in the air during his second attempt.

An unexpected inference drawn from quantum duel is that an equilibrium can be reached by refraining from shooting down the opponent even if further rounds are available to be played. This is never the case in a classical duel.

2. Conclusion

Starting with elementary introductions to game theory and quantum computing, this three-part article works around implementing quantum operations as strategic moves in otherwise classical games. Quantum versions of three game-theoretic models have been dealt with, and software simulating the results have been written and open-sourced. The theoretical models and the computational frameworks will help the students to toy around with more complex scenarios in this topic and make further analyses. Other quantum game-theoretic models which can draw special interests for analysing are quantum battle of the sexes [11], quantum Monty Hall problem [12] and quantum Parrondo's game [13]. Students can also look into QGameTheory [14], an R package that has been developed to simulate several quantum game theory models, available at the CRAN repository.

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Suggested Reading

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