Error Propagation

Error is not a mistake in science. A mistake can be due to an incorrect entry of data in an experiment or an incorrect calculation. Error refers to the precision (uncertainty) of measurements, which one obtains by estimating the standard deviation of repetitive measurements of a certain parameter. We often use “the method of error propagation” to determine uncertainty (error) in a dependent variable from the measured uncertainty in the independent variables. Here, we discuss the origin of the error propagation equation and the assumptions considered to derive it. Intuitional notion of error propagation in statistics suggests that random relative error in the dependent variable cannot be less than the sum of those in the independent variable(s). In this article, we explain that some transcendental functions (such as trigonometric and log functions), however, do not follow this notion of error propagation because their first partial derivatives are usually small in magnitude and sometimes vanish completely at certain points. We further explain and discuss the behaviour of such a function. We have made suggestions for estimating errors in such non-linear functions.

Introduction

We often derive a parameter based on the measured values of independent parameters. For example, to estimate the volume of a box, we independently estimate the length, width, and height of the box. In nature, any estimate has an error associated with it. Likewise, measured values of length, width, and height will have errors. Standard deviations of measured values of each parameter provide the random error in the respective parameters. Most
often, we do not know the actual error in independent parameters rather, we know the random errors by measuring the parameters multiple times (crudely speaking, it should be more than 30 times). Now the question is how do we estimate the error in volume if we know the random errors in length, width, and height of the box? This brings us to the theory of error propagation. Error propagation is perhaps the most useful concept in the statistical analysis of any natural or social science data, where one derives a quantity (dependent variable) from the observations of some independent variables [1]. Let us say we want to estimate \( y \), which is dependent on the variables \( a, b, \ldots \). The most probable value of \( y \) will be

\[
\bar{y} = f(\bar{a}, \bar{b}, \ldots),
\]

(1)

where \( \bar{y}, \bar{a}, \bar{b}, \ldots \) represent mean values of \( y, a, b, \ldots \) respectively. Error in \( y \) can be estimated by considering the spread in \( y_i \) through the measurements of \( a_i, b_i, \ldots \) such that

\[
y_i = f(a_i, b_i, \ldots). \quad (2)
\]

For a large number \( n > 30 \) of measurements, error in \( y \) can be defined as

\[
\sigma_y = \sqrt{\frac{\sum (y_i - \bar{y})^2}{n}}. \quad (3)
\]

Using Taylor series expansion, we can express

\[
y_i - \bar{y} \sim (a_i - \bar{a}) \frac{\partial y}{\partial a} + (b_i - \bar{b}) \frac{\partial y}{\partial b}.
\]

(4)

Equations (3) and (4) will yield

\[
\sigma_y \sim \sqrt{\frac{1}{n} \sum \left[ (a_i - \bar{a})^2 \left( \frac{\partial y}{\partial a} \right)^2 + (b_i - \bar{b})^2 \left( \frac{\partial y}{\partial b} \right)^2 + 2 (a_i - \bar{a})(b_i - \bar{b}) \frac{\partial y}{\partial a} \frac{\partial y}{\partial b} + \ldots \right]}
\]

\[
\sim \sqrt{\sigma_a^2 \left( \frac{\partial y}{\partial a} \right)^2 + \sigma_b^2 \left( \frac{\partial y}{\partial b} \right)^2 + 2 \sigma_{ab} \frac{\partial y}{\partial a} \frac{\partial y}{\partial b} + \ldots}.
\]

(5)
Equation (5) is error propagation equation. If the errors in $a, b, \ldots$ are independent of each other, i.e., covariances ($\sigma_{ab}^2$) are zero, equation (5) becomes:

$$
\sigma_y \approx \sqrt{\sigma_a^2 \left( \frac{\partial y}{\partial a} \right)^2 + \sigma_b^2 \left( \frac{\partial y}{\partial b} \right)^2 + \ldots} \quad (6)
$$

Higher derivative terms of the series in equation (5) are generally neglected as most often their contributions to the resultant error in dependent variable becomes insignificant compared to the first partial derivatives. Having said that, error in a derived quantity depends not only on the errors in the independent variables but also on the derivatives, which act as weights. We often use equation (6) to estimate the errors in dependent variables but we should note that this equation is valid only when:

(i) $\sigma_a, \sigma_b, \ldots$ (i.e., the relative error in individual measurements is less than 10%; else one needs to consider higher order terms)
(ii) $\sigma_a, \sigma_b, \ldots$ are random errors, i.e., they are symmetric about the mean and follow the Gaussian distribution, and
(iii) $a, b, \ldots$ are independent of each other, i.e., errors in these variables are not correlated with each other. If the parameters are dependent on each other, the covariance terms must be added [2].

As seen from equation (6), the errors in the independent variables are squared and summed to estimate the errors in the dependent variables. Hence, intuitively it might be suggested that the relative error in the dependent variable can never be less than that in any of the independent variables. This can be proved for all linear and polynomial functions. It is often thought that the estimate of the final derived quantity cannot be more precise than the parameters used to derive it [3]. Let us now take the example of the box that we described above. Let us say the length, width, and height of the box have 3%, 4% and 5% percentage relative random errors, respectively. Using equation (6), we estimate that the error in volume will be around 7%, which is higher than the error in any of the independent parameters. On the contrary, we explain below that the relative random error in the dependent variable can be less than such errors in the independent variable(s).
1. Propagation of Errors

1.1 Polynomial Functions

Let us take a variable
\[ y = x^n; n > 1, \quad (7) \]
where \( x \) has an error of \( \sigma_x \). Using equation (6), error in \( y \),
\[ \sigma_y = nx^{n-1}\sigma_x \]
\[ \frac{\sigma_y}{y} > \frac{\sigma_x}{x} \text{ for } n > 1. \]

1.2 Transcendental Functions

Let us now take a trigonometric function
\[ y = \sin x, \quad (8) \]
where \( x \) has an error of \( \sigma_x \). Using equation (6), error in \( y \) is
\[ \sigma_y = \sigma_x \cos x \]
\[ \frac{\sigma_y}{y} = \sigma_x \cot x. \]

In the domain \([0, 2\pi]\), the relationship between \( \frac{\sigma_y}{y} \) and \( \frac{\sigma_x}{x} \) is not fixed (Table 1). We can see that the relative error in the dependent function \( y \) is not always larger than that for the independent variable \( x \). The relative error in the dependent function depends not only on the relative error in the independent function but also

<table>
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<tr>
<th>Sub-domain ( x )</th>
<th>Error relation in variables</th>
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<tr>
<td>0 – ( \pi/4 )</td>
<td>( \frac{\sigma_y}{y} = \frac{\sigma_x}{x} )</td>
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on the derivative (slope) of the function itself. Derivatives of trigonometric functions are of oscillatory nature; similar to the nature of the functions themselves (Figure 1). The most perplexing thing happens in at $\pi/2$ and $3\pi/2$, when relative error in the dependent function becomes completely zero regardless of the associated error in the independent function because function’s derivative is zero at these two points (Figure 1). We also notice that logarithmic functions have similar behaviour. So the question arises: Is it wrong to derive errors equivalent to zero at some phases of the dependent function? Is there something hidden in the function that we do not understand? We have tried another way to calculate the error in the same function. We computed Taylor expansion of $\sin x$, i.e. $(y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...)$, and then we applied the same error propagation formula as described in equation (6). We got the same result as we got without expansion as shown in Figure 1. At times, one considers the range of angle for finding errors, e.g., if $x = 90^\circ \pm 10^\circ$ for a function $\sin x$, then the difference between $\sin 80^\circ$ and $\sin 100^\circ$ would provide error. But this approach of propagating errors is incorrect when the errors are random. First-order Taylor series expansion (i.e., equation 6) is heavily used in error propagation because the estimated error associated with the first-order derivative is much higher compared to that in the second and higher order derivatives. So other terms can be neglected. However, many times, functions are nonlinear with vanishing first derivatives. So the first-order Taylor series expansion cannot meet practical needs, and thus one needs to consider higher-order terms. As we saw in $y = a \sin x$, at $\pi/2$ and $3\pi/2$, relative error in $y$ becomes zero regardless of the associated error in the independent function. In such cases, considering higher order terms (or at least including second-order derivative) in equation (4) and following equation
(5) - (6), one can obtain

\[ \sigma^2_y \sim \sigma^2_a \left( \frac{\partial y}{\partial a} \right)^2 + \sigma^2_b \left( \frac{\partial y}{\partial b} \right)^2 + ... \\
+ \frac{\partial^2 y}{\partial a \partial b} \sum \frac{(a_i - \bar{a})(b_i - \bar{b})}{n} + \frac{\partial^2 y}{\partial a^2} \sum \frac{(a_i - \bar{a})^3}{n} + \frac{\partial^2 y}{\partial b^2} \sum \frac{(b_i - \bar{b})^3}{n} + ... \\
+ \frac{\partial^2 y}{\partial a \partial b} \left( \frac{\partial^2 y}{\partial a \partial b} \right)^2 + \frac{1}{2} \frac{\partial^2 y}{\partial a^2} \sum \frac{\partial^2 y}{\partial a^2} + \frac{1}{2} \sum \frac{\partial^2 y}{\partial b^2} + ... \\
\]

Thus, we recommend to use equation (9) for sine and cosine functions. In addition, even for other functions when random errors in independent variables are greater than 10%, equation (9) should be used [4]. For simplicity, if \( y = f(a) \), equation (9) becomes

\[ \sigma_y = \sqrt{\sigma^2_a \left( \frac{\partial y}{\partial a} \right)^2 + \sigma^2_b \left( \frac{\partial y}{\partial b} \right)^2 + \frac{1}{2} \frac{\partial^2 y}{\partial a^2} \sum \frac{(a_i - \bar{a})^3}{n} + \frac{1}{2} \sum \frac{\partial^2 y}{\partial b^2} + ... } \]

2. Applications of Transcendental Functions

The use of trigonometric functions is very common in astrophysics when one observes celestial objects, such as measuring the line of sight velocity of celestial objects. The line of sight velocity of any object as seen from the observer’s point of view is called the radial velocity (RV) of the object [5]. The presence of any companion, gravitationally bound to a star, will produce relative shifts (periodic oscillations) as an additional perturbation to the regular RV of the star. This RV can be measured by looking at the Doppler shift of the stellar spectra when compared with a theore-
ical zero-velocity frame [6]. Since the gravitationally bound companion (planet or another star) will follow Kepler’s laws while in motion, the RV curve measured will be represented by a sinusoidal function. The slope of the sinusoidal RV curve changes sharply. It is thereby very crucial to make accurate measurements around such points where the function varies sharply. Thus, to minimize errors one should obtain more data around such crucial orbital phases in order to make the results statistically robust.

Even for simple calculations that undergraduate students encounter while doing lab experiments, blind application of error propagation can lead to complex and confusing results as described above. We suggest that students to carefully identify the terms (when the variable depends on multiple parameters) which contribute more to error in the dependent variable. One should focus more to minimize the errors in those terms by making repetitive measurements of the same. If the random errors are due to instrumental uncertainties, they might be reduced by improving measurement techniques. If these errors result from statistical fluctuations due to a limited data set then they must be improved by taking more measurements. For example, the resistance $R$ of a cylindrical conductor is proportional to its length $L$ and inversely proportional to its cross sectional area $A = \pi r^2$. Relative error in $R$ will be $\sqrt{\left(\frac{\sigma_L}{L}\right)^2 + 4\left(\frac{\sigma_r}{r}\right)^2}$, where $\sigma_L$ and $\sigma_r$ are errors in $L$ and $r$, respectively. Hence efforts should be made to make precise measurements of $r$ as it contributes four times of the error in $L$ to the overall error in $R$.

3. Summary

In this exercise, we have demonstrated that random relative error in the dependent variable can be less than the sum of those in the independent variable(s). We have discussed functions in which the error decreases as it propagates. We also discussed the reasons behind the decrease in error. Partial derivative provides the slope of a function with respect to one variable and thus conveys the nature of variation of the function, which further tells us how much
Figure 1. Variation in $\sin x$, relative error in $x$ and $y$ for a constant error in $x$ ($\sigma_x = 1^\circ$).

error the derivative can cause due to its variations. The rate of variation controls the rate at which the errors propagate, i.e., less slope—small errors. In $[0, 2\pi]$ domain, trigonometric functions show their varying nature and error increases rapidly at certain crucial phases. We have made recommendations for experimentalists for careful observations to minimise errors at crucial phases. We further recommend taking higher order derivatives into consideration for estimating the errors in non-linear functions.

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