Classroom

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

An Alternative Approach to Write the General Solution of a Class of Second-order Linear Differential Equations*

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The objective of this article is to introduce an analytical approach to solve a class of second-order linear homogeneous ordinary differential equations (ODEs) with variable coefficients and derive a formula to write the general solution for the same. We will also solve some well-known and special ODEs.

1. Introduction

Second-order linear ODEs states numerous real-life incidents and describes several physical phenomena; particularly in the theory of electric circuit and establishing a connection with vibrations in mechanics. These are of great importance as their solution builds the basic phenomena of electromagnetics, wave motion, heat conduction, fluid mechanics, stress analysis, and aerodynamics, among others, to develop a respective theory and solve the concerning problems. One can not deny the necessity to solve second-order linear ODEs in the aforementioned fields. In addition, the study of these ODEs has led to the development of several deep and strong ideas in pure and applied mathematics.

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The second-order linear homogeneous ODEs, in an interval I, are of the form,

$$y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0, \qquad (1.1)$$

where $y = y(x)$ is the associated analytical solution and the coefficients $a_1(x)$, $a_2(x)$ are some real-valued functions of the independent variable x or constants in I. In general, there are no restrictions on $a_1(x)$, $a_2(x)$ as the problems arise naturally in real-life as well as in science and engineering. Unfortunately, there is no common method to find the general solution of the ODEs of the form (1.1). If one of the solutions of an ODE of the form (1.1) is known or given, then the other solution can be obtained using the method of variation of parameter [1]. However, there does not exist any analytical method to find even one solution of the ODE of the form (1.1), except in a few restrictive conditions. Some well-known methods and procedures, such as reduction of order, change of independent variables, method of undetermined coefficients, etc., are submitted to find the general or one of the solutions of few particular classes of the ODEs of the form (1.1) in [1, 2].

Keywords
Linear differential equation, second-order ordinary differential equation, ODEs.

In [3], the authors have submitted an alternative form to write the general solution of the general second-order non-homogeneous linear ODE. In this, the respective particular solution is computed from two different integrals. For this, it is pre-assumed that at least one solution of the corresponding second-order homogeneous linear ODE of the form (1.1) is known or given. In [4], the authors have presented an approach to find one of the solutions of the ODEs of the form (1.1), using transformations and repeated iterated integration. For this, it is strictly assumed that $a_1(x)$ and $a_2(x)$ are polynomials (i.e., continuously differentiable and integrable functions). In addition, some algorithms and procedures are also introduced to solve the ODEs of the form (1.1) by imposing some specific restrictions over the coefficients $a_1(x)$, $a_2(x)$ as well as on the solution $y = y(x)$, see [5–7] for details.

In this article, we derive a formula for writing the general solution of a class of second-order linear ODEs of the form (1.1), without knowing or finding one of the associated solutions, introducing an alternative analytical approach.

2. Main Results

Theorem 2.1. *Let us consider the second-order linear homogeneous ODEs of the form* (1.1) *in an interval I. If* $a_1(x), a_2(x)$ *be non-zero differentiable function in I, such that*

$$a_2{}^2(x) = a_1(x)a_2{}'(x) - a_2(x)a_1{}'(x). \tag{2.2}$$

Then, the general solution of the ODEs of the form (1.1) *in I is given as*

$$y(x) = c_1 e^{-\int^x \frac{a_2(s)}{a_1(s)}ds} + c_2 e^{-\int^x \frac{a_2(s)}{a_1(s)}ds}.\int^x \frac{a_2(t)}{a_1(t)}\, e^{\int^t \left(\frac{a_2(s)}{a_1(s)} - a_1(s)\right)ds} dt, \tag{2.3}$$

where c_1 *and* c_2 *are arbitrary real constants.*

Proof. Without loss of generality, we can assume that $y \in C^3(I)$ and f is a function of x in I such that

$$y''(x) = -f(x). \tag{2.4}$$

On substituting (2.4) in (1.1), we get

$$a_1(x)y'(x) + a_2(x)y(x) = f(x). \tag{2.5}$$

Again, (2.5) can be re-written as,

$$y'(x) + \frac{a_2(x)}{a_1(x)}y(x) = \frac{f(x)}{a_1(x)}, \tag{2.6}$$

which is a first-order linear ODE. Therefore,

$$y(x) = c_1 e^{-\int^x \frac{a_2(s)}{a_1(s)}ds} + e^{-\int^x \frac{a_2(s)}{a_1(s)}ds}.\int^t \frac{f(t)}{a_1(t)} e^{\int^t \frac{a_2(s)}{a_1(s)}ds} dt, \tag{2.7}$$

where c_1 is an arbitrary real constant. On differentiating (2.5) both sides with respect to x, we get

$$a_1(x)y''(x) + (a_1{}'(x) + a_2(x))y'(x) + a_2'(x)y(x) = f'(x). \tag{2.8}$$

Using (2.4), (2.8) reduces to

$$y'(x) + \frac{a_2{}'(x)}{a_1{}'(x) + a_2(x)} y(x) = \frac{f'(x) + a_1(x) f(x)}{a_1{}'(x) + a_2(x)}. \tag{2.9}$$

If the condition (2.2) holds, then the L.H.S. of (2.6) and (2.9) are same. Therefore, R.H.S. of (2.6) and (2.9) must also be same. Thus,

$$\frac{f'(x) + a_1(x) f(x)}{a_1{}'(x) + a_2(x)} = \frac{f(x)}{a_1(x)}. \tag{2.10}$$

Using the condition (2.2), (2.10) can be written as,

$$f'(x) + \left(a_1(x) - \frac{a_2{}'(x)}{a_2(x)} \right) f(x) = 0, \tag{2.11}$$

which is again a first-order linear ODE. Therefore,

$$f(x) = c_2 a_2(x) e^{-\int^x a_1(s) ds}, \tag{2.12}$$

where c_2 is an arbitrary real constant. On combining (2.7) and (2.12), we get (2.3).

Verification: The solution given in (2.3) can be written as $y(x) = c_1 y_1(x) + c_1 y_2(x)$, where

$$y_1(x) = e^{-\int^x \frac{a_2(s)}{a_1(s)} ds} \tag{2.13}$$

and

$$y_2(x) = e^{-\int^x \frac{a_2(s)}{a_1(s)} ds} . \int^x \frac{a_2(t)}{a_1(t)} \, e^{\int^t \left(\frac{a_2(s)}{a_1(s)} - a_1(s) \right) ds} dt. \tag{2.14}$$

Using successive differentiation and the condition (2.2) in (2.13) and (2.14) respectively, we get

$$y_1{}'(x) = -\frac{a_2(x)}{a_1(x)} e^{-\int^x \frac{a_2(s)}{a_1(s)} ds},$$

$$y_1{}''(x) = 0,$$

$$y_2{}'(x) = \frac{a_2(x)}{a_1(x)} e^{-\int^x a_1(s) ds} - \frac{a_2(x)}{a_1(x)} e^{-\int^x \frac{a_2(s)}{a_1(s)} ds} . \int^x \frac{a_2(t)}{a_1(t)} \, e^{\int^t \left(\frac{a_2(s)}{a_1(s)} - a_1(s) \right) ds} dt,$$

and $y_2{}''(x) = -a_2(x) e^{-\int^x a_1(s) ds}$.

Clearly, $y_1(x)$ and $y_2(x)$ satisfy (1.1). Hence, $y_1(x)$ and $y_2(x)$ are the solutions of the ODEs of the form (1.1). Again, the Wronskian of the solutions $y_1(x)$ and $y_2(x)$ is

$$W(y_1, y_2)(x) = \frac{a_2(x)}{a_1(x)}.e^{-\int^x \frac{a_2(s)}{a_1(s)}ds}.e^{-\int^x a_1(s)ds} \neq 0, \tag{2.15}$$

as $a_1(x), a_2(x)$ are non-zero functions of x in I. Hence, $y_1(x)$ and $y_2(x)$ are the linearly independent (or fundamental) solutions of the ODEs of the form (1.1) and the solution given by (2.3) is the respective general solution. This completes the proof. □

Remark 2.2. In the proof of Theorem 2.1, the assumption $y \in C^3(I)$ is used in (2.8) as $f'(x) = -y'''(x)$. However, this assumption is sufficient but not necessary as $a_1(x)$, $a_2(x)$ are non-zero differentiable functions in I and the right hand side of (2.12) asserts that $f(x)$ or $y''(x)$ is differentiable in I.

Remark 2.3. If the assumptions of Theorem 2.1 hold, then $a_1(x)$ and $a_2(x)$ are linearly independent functions in I. If $a_1(x)$ and $a_2(x)$ are linearly dependent functions in I, then the condition (2.2) holds iff $a_2(x) = 0$ in I. Let $a_2(x) = 0$, (2.6) and (2.9) imply that

$$f'(x) + \left(a_1(x) - \frac{a_1{}'(x)}{a_1(x)}\right) f(x) = 0. \tag{2.16}$$

Thus,

$$f(x) = k_1 a_1(x) e^{-\int^x a_1(s)ds}, \tag{2.17}$$

where k_1 is an arbitrary real constant. Therefore, from (2.6) and (2.17), we get

$$y(x) = k_1 \int^x e^{-\int^t a_1(s)ds} dt + k_2, \tag{2.18}$$

where k_2 is also an arbitrary real constant, i.e., for $a_2(x) = 0$, the solution obtained by the introduced approach (in the proof of Theorem 2.1) coincide with the exact solution of the ODEs (1.1) in I.

Remark 2.4. If the assumptions of Theorem 2.1 hold, then the fundamental solutions $y_1(x)$ and $y_2(x)$ are given by (2.13) and

(2.14) respectively. Thus, the solution $y_2(x)$ can be re-written as

$$y_2(x) = y_1(x). \int^x \frac{1}{y_1(t)} \frac{a_2(t)}{a_1(t)} e^{-\int^t a_1(s)ds} dt. \tag{2.19}$$

Now, suppose $y_1(x)$ is one of the known or given fundamental solution of the ODEs of the form (1.1), then using the method of variation of parameter, the other fundamental solution $y_2(x)$ is given by

$$y_2(x) = y_1(x). \int^x \frac{1}{(y_1(t))^2} e^{-\int^t a_1(s)ds} dt. \tag{2.20}$$

From the condition (2.2), we get

$$\frac{a_1(x)a_2{}'(x) - a_2(x)a_1{}'(x)}{a_1{}^2(x)} = \left(\frac{a_2(x)}{a_1(x)}\right)^2,$$

$$i.e., \quad \left(\frac{a_2(x)}{a_1(x)}\right)' / \left(\frac{a_2(x)}{a_1(x)}\right) = \left(\frac{a_2(x)}{a_1(x)}\right),$$

$$i.e., \quad ln\left(\frac{a_2(x)}{a_1(x)}\right) = \int^x \frac{a_2(s)}{a_1(s)} ds + lnk, \tag{2.21}$$

where k is an arbitrary real constant.

Therefore,

$$\frac{a_2(x)}{a_1(x)} = ke^{\int^x \frac{a_2(s)}{a_1(s)} ds}. \tag{2.22}$$

From (2.13) and (2.22), we get

$$\frac{a_2(x)}{a_1(x)} = \frac{k}{y_1(x)}. \tag{2.23}$$

Therefore, (2.23) assert that the solutions given by (2.19) and (2.20) are scalar multiples of each other. Hence, in respect of the second fundamental solution of the ODEs of the form (1.1), the introduced approach (used in the proof of Theorem 2.1), together with the condition (2.2), coincide with the method of variation of parameters.

Moreover, from (2.15) and (2.22), we get

$$W(y_1, y_2)(x) = k\, e^{-\int^x a_1(s)ds}, \tag{2.24}$$

i.e., the respective fundamental solutions satisfy the the Abel's identity [1].

3. Solved Examples

In this section, we solve some renowned and interesting second-order linear ODEs to demonstrate the importance of the obtained results. For convenience, c_i, $i = 1, 2, \ldots, 6$ are used as arbitrary real constants of integration throughout in this section.

Example 3.1. The Caucy–Euler's ODEs of second-order are given by

$$x^2 y''(x) + mxy'(x) + ny(x) = 0, \quad x \neq 0, \tag{3.25}$$

where m and n are some scalars. The ODEs of the form (3.25) can be written as

$$y''(x) + \frac{m}{x} y'(x) + \frac{n}{x^2} y(x) = 0, \quad x \neq 0, \tag{3.26}$$

which is of the form (1.1) with $a_1 = \frac{m}{x}$ and $a_2 = \frac{n}{x^2}$. If $m = -n$ and $n \neq 0$, then the necessary conditions of Theorem 2.1 hold. Hence, for $m = -n$ and $n \neq 0$, the general solution of the ODEs of the form (3.25), using (2.3), is given by

$$y(x) = c_1 x + c_2 x^n.$$

In addition, for $n = 0$, the general solution of the ODEs of the form (3.25), using (2.18), is given by

$$y(x) = c_3 x^{1-m} + c_4.$$

Example 3.2. Let us consider a non-homogeneous second-order linear ODE,

$$xy''(x) - (1 + x)y'(x) + y(x) = x^2 e^x, \quad x \neq -1, 0. \tag{3.27}$$

The ODE (3.27) can be re-written as

$$y''(x) - \left(1 + \frac{1}{x}\right) y'(x) + \left(\frac{1}{x}\right) y(x) = xe^x, \quad x \neq -1, 0. \tag{3.28}$$

The associated homogeneous ODE of the ODE (3.28) satisfy the necessary conditions of Theorem 2.1. Therefore, the general solution, $y_h(x)$, of the respective homogeneous ODE, using (2.3), is given as

$$y_h(x) = c_1(1 + x) + c_2 e^x.$$

Again, using the method of variation of parameters, the particular solution, $y_p(x)$, of the ODE (3.28) is given as

$$y_p(x) = e^x\left(1 - \frac{x^2}{2} + \frac{x^3}{3}\right).$$

Hence, the general solution of the ODE (3.27) is written as

$$y(x) = y_h(x) + y_p(x) = c_1(1 + x) + c_2e^x + e^x\left(1 - \frac{x^2}{2} + \frac{x^3}{3}\right).$$

Example 3.3. Let us consider the ODEs,

$$\phi''(x) - \frac{4}{4x^2 - 1}\phi'(x) + \frac{8}{(2x - 1)^2(2x + 1)}\phi(x) = 0, \quad x \neq \pm\frac{1}{2}, \tag{3.29}$$

$$x\,\psi''(x) - x\cot x\,\psi'(x) + \cot x\,\psi(x) = 0, \quad x \neq \frac{n\pi}{2}, n \in \mathbb{Z}. \tag{3.30}$$

Since the ODEs (3.29) and (3.30) satisfy the necessary conditions of Theorem 2.1. Therefore, using (2.3), the respective general solutions of the ODEs (3.29) and (3.30) are given as

$$\phi(x) = c_3(2x - 1) + c_4(2x - 1)\ln\left(\frac{2x - 1}{2x + 1}\right),$$

and $\psi(x) = c_1x + c_2x\displaystyle\int \frac{\sin x}{x^2}dx \approx c_5x + c_6\left(\ln x + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n + 1)!\,(2n)}\right).$

Example 3.4. The Hermite ODEs of the second-order are given by

$$y''(x) - 2xy'(x) + 2\lambda y(x) = 0, \qquad x \neq 0, \tag{3.31}$$

where λ is a parameter. For $\lambda = 1$, the ODE (3.31) satisfy the necessary conditions of Theorem 2.1. Therefore, for $\lambda = 1$, the general solution of the ODE (3.31), using (2.3), is given by

$$y(x) = c_1x + c_2x\int \frac{e^{x^2}}{x^2}dx. \tag{3.32}$$

Again, for $\lambda = 0$, the general solution of the ODE (3.31), using (2.18), is given by

$$y(x) = c_3\int e^{x^2}dx + c_4. \tag{3.33}$$

On substituting $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ (the Taylor's series expansion about origin), in (3.32) and (3.33), we get

$$y(x) = \begin{cases} c_1 x + c_2(x^2 - 1)\sum_{n=2}^{\infty} \frac{x^{2n-1}}{n!(2n-1)}, & \text{for } \lambda = 1 \\ c_3 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)} + c_4, & \text{for } \lambda = 0 \end{cases} .$$

Example 3.5. Considering two specific classes of second-order linear ODEs in an interval J, given as

$$y''(x) - (x-\alpha)g(x)y'(x) + g(x)y(x) = 0, \quad x \neq \alpha, \qquad (3.34)$$

$$\phi''(x) + (\alpha - \beta x)\phi'(x) + \beta\phi(x) = 0, \quad x \neq \frac{\alpha}{\beta}, \beta \neq 0, \qquad (3.35)$$

where α, β are two parameters and $g(x)$ is any arbitrary non-zero differentiable real-valued function in J. Using Theorem 2.1, the respective general solutions of the ODEs (3.34) and (3.35) in J can be written as

$$y(x) = c_1(x-\alpha) + c_2(x-\alpha)\int \frac{1}{(x-\alpha)^2}\, e^{\int (x-\alpha)g(x)dx} dx,$$

and $\phi(x) = c_3(\beta x - \alpha) + c_4(\beta x - \alpha)\int \frac{1}{(\beta x - \alpha)^2} e^{\frac{x}{2}(\beta x - 2\alpha)} dx.$

4. Concluding Remarks

In this article, an analytical approach is introduced to find the general solution of a class of second-order linear homogeneous ODEs with variable coefficients. This class of second-order linear homogeneous ODEs is generated by the hypothesis of Theorem 2.1 and consists of several well-known and special second-order ODEs. Such as Cauchy–Euler's ODEs, Hermite ODEs, ODEs having a polynomial and an exponential solution, ODEs having a polynomial and a logarithmic solution, etc. The main advantage of this approach is that it provides a general formula to write the general solution of the aforesaid class of second-order linear homogeneous ODEs, without guessing or finding one of the fundamental solutions. As well, the general solution of a non-homogeneous second-order linear ODE can also be determined if the associated homogeneous ODE belongs to the specified class of second-order linear homogeneous ODEs. Moreover, Remark 2.3 and Remark 2.4 assert that the introduced approach coincides with some of the standard methods in some particular cases. In a broader sense, the introduced approach can be extended or modified to solve some more second-order, as well as, higher-order ODEs with variable coefficients. We consider it as our future objectives.

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