Conway, Knots and Groups*

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John Conway was one of the most versatile mathematicians in modern times, who made important contributions to several areas of mathematics. In this article, we highlight his contributions to two areas—*knot theory* and *group theory*.

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1. Conway and Knot Theory

Conway made important contributions to topology, especially to knot theory. After some background, we highlight two of these. An excellent introduction to knot theory that does not require too much background is the book by Colin Adams [1].

1.1 *Knots and Links*

A *knot* in topology is essentially a knotted string with the two ends of the string glued together, except we ignore the thickness of the string. We regard two knots as the same if one can be transformed into the other without cutting (and re-gluing) the string. More formally, a knot K is a smooth embedding of the circle S^1 into R^3. We say knots K_1 and K_2 are *isotopic* if there is a family of smooth embeddings of the circle starting with K_1 and ending with K_2. Requiring embeddings at all intermediate times corresponds to not allowing the string to be cut and re-glued, or to cross itself.

A convenient way to study knots is by considering their projection onto a plane. So long as the projection is taken in a so-called *regular* direction, we get a smooth curve in the plane with finitely many crossings—a result called *Sard's lemma* says that most directions are regular. A *knot diagram* is such a projection with an indication at each crossing of which strand is above—which

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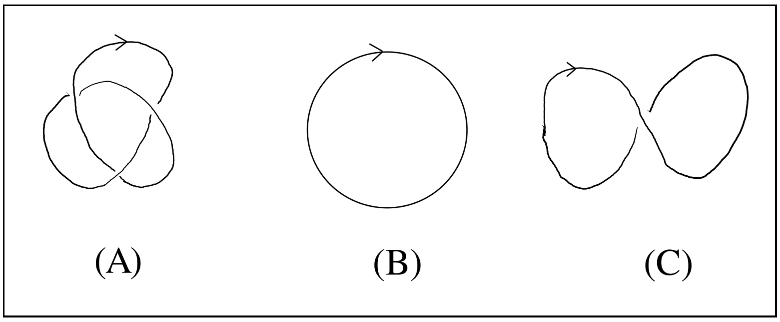
Figure 1. Some knots.



strand is above can be shown visually as in *Figure* 1. Note that we chose orientations for knots and indicated them in the knot diagram.

Observe that of the three knots depicting in *Figure* 1, the knots (B) and (C) are, in fact, *unknots*, in particular, they are isotopic. The first knot (labelled A) is called the *trefoil* knot. Intuition suggests that this is not isotopic to the unknot, but this is not easy to show.

1.2 *Linking Number*

A link is like a knot except it can be made of many strings. Thus, a link is a collection of smooth disjoint embeddings of finitely many circles—the case with just one circle corresponds to knots.

It is tricky to show that the trefoil and the unknot are different, so we begin with a simpler but similar problem—showing that the so-called *Hopf link* is different from the two-component unlink. A link is like a knot except it can be made of many strings. Thus, a link is a collection of smooth disjoint embeddings of finitely many circles—the case with just one circle corresponds to knots. Like knots, we can project these onto a plane to obtain a *link diagram*.

In *Figure* 2 we see the link projections of some links. The first is the unlink, i.e., two separate circles. It is easy to see that the third is also an unlink—one circle is below the other and can be separated from it. On the other hand, the two circles in the middle picture are linked together—this is called the *Hopf link*. The *linking number* makes this precise.

Observe that we have chosen orientations for each of the circles

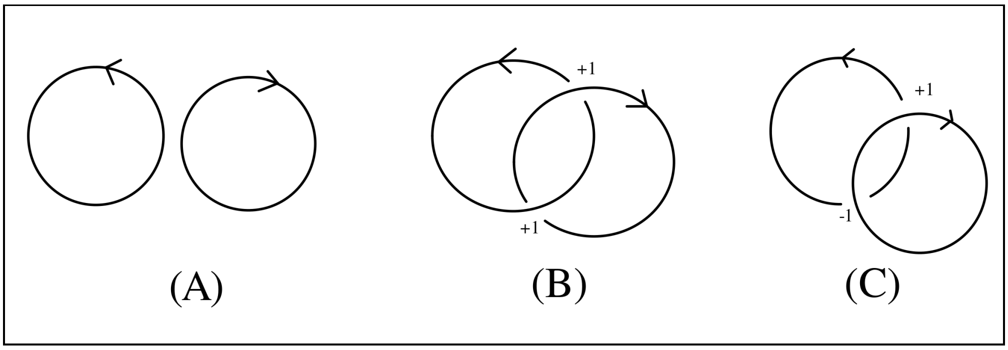


Figure 2. Unlinks and the Hopf link.

in our link diagrams. To define the linking number, we associate a sign to each crossing, with a positive sign if the strand above followed by the strand below has the same orientation as the x-axis, followed by the y-axis. We then count the number of crossings with sign and divide by 2 to get the linking number.

This gives a formula for computation, which we see gives the same number, namely 0, for the first and third links, but 1 for the Hopf link. However, to conclude that the Hopf link is not isotopic to the unlink, we need to show that link diagrams that give isotopic links always have the same linking number.

1.3 *Reidemeister Moves and Knot/Link Invariants*

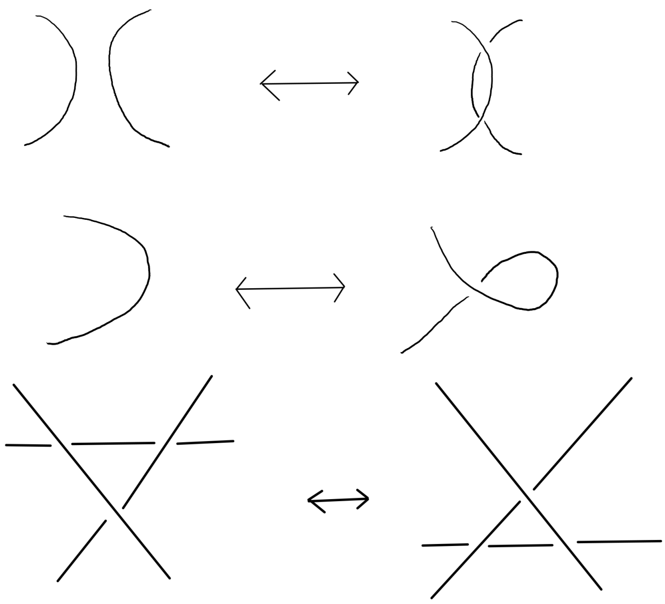
As we have seen, link diagrams of knots or links are not unique. However, two link diagrams correspond to the same knot/link if and only if they are related by a sequence of *Reidemeister moves*. These are of three kinds, as shown in *Figure* 3.

Two link diagrams correspond to the same knot/link if and only if they are related by a sequence of *Reidemeister moves*.

It is obvious that link diagrams related by Reidemeister moves correspond to isotopic knots or links. The converse can be shown by projecting the family of embeddings giving the isotopy in a regular direction for the family. The existence of a regular direction for a family once more follows from Sard's lemma.

Thus, if two links are isotopic, they are related by a finite sequence of Reidemeister moves. It is easy to see that the linking number does not change under Reidemeister moves, and hence

Figure 3. Reidemeister moves.



under isotopy. A quantity that is unchanged under isotopy is called an *invariant* of knots/links. More generally, if we associate to a link diagram a quantity (such as a number or polynomial), it is an invariant if and only if it is unchanged by Reidemeister moves.

1.4 *Alexander and Conway Polynomials*

Conway discovered that the Alexander polynomial, when suitably normalized, satisfies a so-called *skein relation*. We call this normalized version the *Conway polynomial*.

Perhaps the most fundamental (though not the most powerful) invariant of knots is the *Alexander polynomial*. This is best defined (and was originally defined) using algebraic topology, so we will only sketch the definition in a box for those with an adequate background.

Conway discovered that the Alexander polynomial, when suitably normalized, satisfies a so-called *skein relation*, which we describe below. We call this normalized version the *Conway polynomial*.

Namely, let L_+ be a link diagram with a fixed positive crossing. We associate to this, two other link diagrams L_- and L_0 as in *Figure* 4, with the link diagrams of L_- and L_0 the same as that

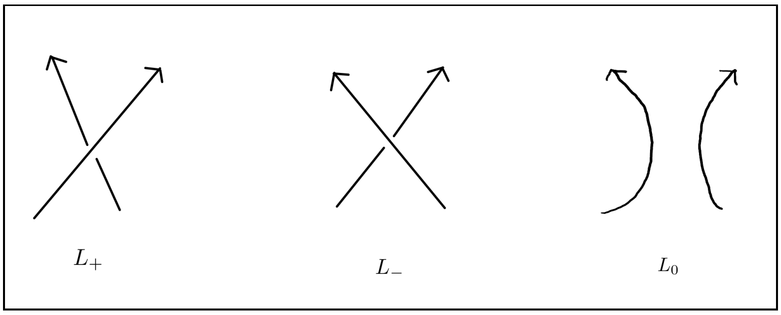


Figure 4. Link diagrams for the skein relation.

of L_+ outside the region shown. The link diagram L_- is obtained by replacing the positive crossing with a negative crossing. We *resolve* the positive crossing, using the orientations, to obtain a link diagram L_0. Namely, we delete the interior of the crossing arcs to get a diagram with 4 vertices, two of which are initial vertices and two terminal vertices of the deleted arcs. We attach to these vertices two arcs that do not cross in such a way that the initial vertex of each of the deleted arcs is connected to the terminal vertex of the other arc.

Given a triple of link diagrams related as above, Conway showed that we have the relation

$$\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0).$$

This relation allows us to compute the Conway polynomial of an arbitrary knot or link starting from a link diagram. Hence the Conway polynomial can be defined purely combinatorially. Further, one can directly prove invariance under Reidemeister moves.

The more powerful invariants of knots—the Jones polynomial and its generalization, the HOMFLYPT polynomial, are best defined in this way. While the Jones polynomial was originally discovered using planar algebras, it is not easy to show that the complicated definition is a knot invariant. Instead, one can derive the skein relation and construct the invariant from this. The HOMFLYPT polynomial is, in fact, defined using the skein relation.

Conway's work provided a bridge from the Alexander polynomial, rooted in algebraic topology, to the more powerful (though less geometric) invariants of knots.

Thus, Conway's work provided a bridge from the Alexander polynomial, rooted in algebraic topology, to the more powerful (though less geometric) invariants of knots.

Box 1. More on the Alexander Polynomial

Given a knot $K \subset \mathbb{R}^3$, the *knot group* is the fundamental group $\pi_1(R^3 \setminus K)$ of the complement of the knot in the Euclidean space $\mathbb{R}^3$ (we briefly introduce *groups* in the next section, and the fundamental group is a group associated to a space). This is essentially a complete invariant of the knot. Unfortunately, the knot group as an invariant is not directly very useful, as to whether two groups are the same (i.e., isomorphic) is algorithmically undecidable. However, we can derive from the knot group more tractable invariants.

Given a group G, we can associate to it its *abelianization* $G/[G, G]$, which is its largest abelian quotient. It is straightforward to decide when two finitely generated abelian groups are isomorphic. Unfortunately, for every knot group $G = \pi_1(R^3 \setminus K)$, the abelianization $G/[G, G]$ is isomorphic to $\mathbb{Z}$ (the abelianization of the fundamental group is the homology group $H_1(\mathbb{R}^3 \setminus K)$). We shall identify $G/[G, G]$ with the infinite cyclic group $\langle t \rangle = \{t^n : n \in \mathbb{Z}\}$, with t a formal variable.

We can, however, extend the idea of abelianization a little to get a useful invariant. Namely, let $G^{(1)} = [G, G]$ be the kernel of the abelianization homomorphism $G \to G/[G, G]$, and consider its abelianization $M = G^{(1)}/[G^{(1)}, G^{(1)}]$. This does depend on the knot.

Further, we have a well-defined action of $G/[G, G] = \langle t \rangle$ on A by conjugation (more geometrically, we consider the cover of $\mathbb{R}^3 \setminus K$ with fundamental group $[G, G]$, and M is the homology of this cover with action by deck transformations). This makes M into a module over the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials. It can be shown that the *annihilator* of M (called the *Alexander ideal*) is a principal ideal—its generator is the *Alexander polynomial* $\Delta_K(t)$.

1.5 *Conway Notation*

The Conway notation is a very efficient way to enumerate knots, based on a discovery of Conway that a large class of *tangles* can be encoded very efficiently, in such a way that we can immediately determine whether two tangles are equivalent.

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For our purposes, a tangle is a part of a knot/link diagram that is enclosed in a circle, with exactly four points on the boundary, as in the examples in *Figure* 5. The tangles in the figure are what Conway called *rational tangles*, for reasons that will soon become clear.

Observe that the tangles in *Figure* 5 are labelled. The first two, labelled ∞ and 0, are the two tangles with no crossings. The third

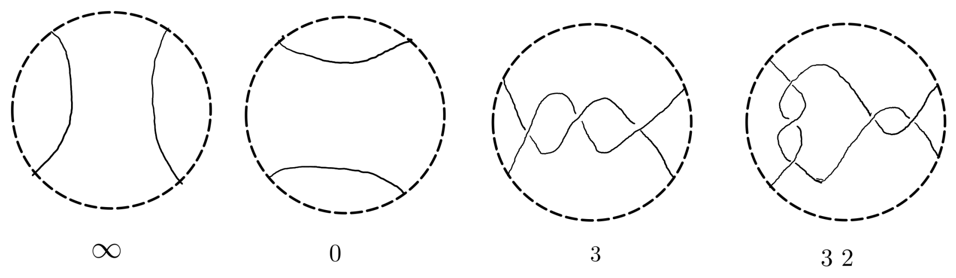

Figure 5. Some tangles.

is obtained by 3 left twists from 0 and hence labelled 3—if we had used the right twists, we would have labelled this −3. The final tangle is obtained by *reflecting* the tangle denoted 3 along the diagonal from its top-left vertex to its bottom-right vertex, followed by 2 twist—it is denoted 3 2 to reflect this.

The tangles we can construct by iterating these constructions are what Conway calls *rational tangles*. We have associated to these a sequence of integers, such as 3 2 in our example. These can be further mapped to a rational number or ∞ by taking a corresponding continued fraction—for example −2 3 2 maps to $2 + \frac{1}{3+\frac{1}{-2}}$.

Two tangles are equivalent if they are related by Reidemeister moves within the circle. Conway showed the remarkable result that two rational tangles are equivalent if and only if the corresponding rational numbers are equal. The Conway notation builds on this.

We can *multiply* two tangles in the same way as we constructed the tangle labelled 3 2, namely reflect the first tangle in the diagonal from the top-left to the bottom-right, place the second tangle to the right of the result and identify the right endpoints of the tangle on the left with the left endpoints of the tangle on the right. We can also *add* two tangles by simply placing the second to the right of the first and making similar identifications.

The tangles we obtain in this manner are called *algebraic tangles*.

We can make a tangle into a knot or link by adding arcs on the left and right—algebraic tangles give algebraic knots/links. Using the efficient enumeration of rational tangles, one can efficiently enumerate algebraic knots. Using this, in the 1960s, Conway extended (with a few hours of calculations) the previously known tables of knots (which were made in the 1800s), in the process, discovering a new knot, now called the *Conway knot*.

The Conway knot was the subject of one of the most remarkable recent pieces of work in topology. A knot $K \subset \mathbb{R}^3$ can be regarded as a curve in the boundary of the 4-dimensional ball B^4, as $\partial B^4 = S^3$ and deleting a point from S^3 gives $\mathbb{R}^3$. The knot K is said to be *slice* if it bounds a smooth disc in B^4. Whether a knot is slice is essentially a question in the topology of smooth 4-manifolds, which is the hardest and most mysterious part of topology. So such questions are deeper than most knot theory questions.

Due to advances in the topology of smooth 4-manifolds, many techniques were developed that could help decide whether a knot was slice. These were strong enough to decide whether the knot was slice for every knot with a knot diagram with up to 12 crossings except for the Conway knot (there are thousands of such knots). This was expected not to be slice, but the problem was especially hard since a closely related knot, called the Kinoshita–Terasaka knot, was slice—and invariants for the Conway and Kinoshita–Terasaka knot are usually the same.

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2. Groups

We begin with some basic background on groups. An excellent and elementary introduction to groups is the book by M A Armstrong [2]. For an excellent account of the topics to which Conway contributed (which we sketch here), we recommend [4].

A group is a set G together with a way of combining elements of G, which satisfies certain axioms—for example integers form a group with addition being the operation. The axioms that a group must satisfy are as below. Here G is a set, and $\cdot$ denotes the operation of combining elements so that for $a, b \in G$, $a \cdot b$ is an element of G.

1. For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity).

2. There exists an element $e \in G$ such that for all $a \in G$, $e \cdot a = a \cdot e = a$ (identity).

3. For all $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ (inverse).

Remarkably, these simple axioms encode a rich structure theory and many remarkable examples, especially when the set G is finite. At the same time, groups are ubiquitous, appearing everywhere in mathematics and beyond. Conway had an important role in uncovering the structure of finite groups, and in finding remarkable connections with other areas of mathematics.

As we have mentioned, integers with addition as the operation form a group. More typical examples, especially from our point of view, are given by *symmetry*. For example, consider an equilateral triangle A symmetry of a triangle (or any other figure) is a transformation that takes the triangle to itself, where, in this case, the transformations we consider are rigid motions. There are 6 symmetries of the equilateral triangle: 3 reflections about axes joining vertices to opposite sides, rotations by 120$^{\rm o}$ and 240$^{\rm o}$, and the identity transformation. We can combine these symmetries by applying one transformation followed by the other. Thus we have a set with 6 elements and an operation combining them. This forms the *group* of symmetries of a triangle, which we call S_3 (as the reader would guess, we have groups S_n for $n \in \mathbb{N}$).

Similarly, the group of symmetries of an isosceles triangle form a group of order 2, which we denote C_2, with $C_2 = \{e, \tau\}$ with $\tau \cdot \tau = e$ and $e \cdot a = a \cdot e = a$ for $a = e, \tau$. Indeed this is the same as the group of symmetries of a line segment, with τ corresponding to the symmetry which interchanges the endpoints.

More precisely, the groups of symmetries of an isosceles triangle and of an interval are *isomorphic*, i.e., there is a one-to-one correspondence between their elements so that product of two elements in the first group corresponds to the product of the corresponding elements in the second group. We regard isomorphic groups as equal, and by classification of a class of groups, we mean giving a list of groups of the class up to isomorphism.

As the notation suggests, C_2 is one of an infinite class of groups. For $n \geq 3$, symmetries of a regular polygon with n sides that are given by *rotations* form the group C_n. Thus, C_n consists of rotations by angles $\frac{k}{n} \cdot 360^{\mathrm{o}}$, with $k = 0, 1, \ldots, n-1$.

The symmetries of a scalene triangle form a group with just one element, the identity. Such a group is called the trivial group.

Similarly, the group S_3 is one of an infinite family of groups, with S_4 being the group of symmetries of a regular tetrahedron. To define these, we ignore the geometry and observe that symmetries of an equilateral triangle, with vertices A_1, A_2 and A_3, is determined by their images A_{i_1}, A_{i_2} and A_{i_3}, and hence the permutation $i_1 i_2 i_3$. In general permutations of length n form the group S_n.

The symmetries of a scalene triangle form a group with just one element, the identity. Such a group is called the trivial group.

2.1 *Group Extensions and Simple Groups*

We can view S_3 as being built from smaller groups. Namely, if we draw a triangle on a piece of paper and apply the transformations, then the reflections flip the paper and the rotations and identity do not. If we combine the transformations that do not flip the paper, then clearly their composition also does not flip the paper. Hence, these elements form a group with 3 elements, called A_3 (observe that A_3 is the same as C_3 described above). Further, we can map the elements of S_3 to C_2, with those that flip mapping to τ and those that do not to the identity. Further, we can see (by considering cases of elements that flip and do not flip, for instance) that the image in C_2 of the product of two elements in

S_3 is the product of their images—we say the map is a *homomorphism*. Finally, observe that the elements that map to e are exactly those belonging to A_3.

In this situation, we say that S_3 is an *extension* of C_2 by A_3. In some sense we can study the group $G = S_3$ by studying the smaller groups $Q = C_2$ (the quotient) and $K = A_3$ (the kernel).

Groups that cannot be expressed as extensions in a non-trivial way are called *simple*. Hence a crucial part of understanding the structure of groups is to understand *simple groups*. Thus, especially when studying finite groups, we can think of simple groups as Lego pieces which we can combine to form more elaborate groups (as with Lego pieces, there are many different groups G that can be constructed from K and Q).

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2.2 *Finite Simple Groups*

One of the grandest theorems in mathematics is the *classification of finite simple groups*. Indeed many pieces of this need proofs that run to hundreds of pages.

We have already encountered some simple groups. The cyclic groups C_p are simple if and only if p is a prime. Further, we have analogues of $A_3 \subset S_3$ inside S_n for all n. We denote these A_n (alternating groups). All the alternating groups except A_4 are simple (however, A_2 and A_3 do not give new examples, being the trivial group and C_3, respectively).

The statement of the classification builds upon an earlier important work—the classification of simple Lie groups. Lie groups are groups of symmetries, but ones that come in smooth families, such as rotations of a sphere. One of the great results of 19th century mathematics was a classification of simple Lie groups. It was shown by Killing and Cartan that these consist of a few familiar groups of symmetries together with five *exceptional* Lie groups.

Chevalley and others constructed finite analogues of the Lie groups, called *finite groups of Lie type*. There are 16 families of these.

In addition to the 18 families mentioned above (C_p, A_n and the

16 families of finite groups of Lie type), 26 other simple groups, called *sporadic simple groups*, were discovered over the course of many decades. The classification of finite simple groups is the statement that a finite simple group is (exactly) one of the following.

- A group C_p with p a prime.
- An alternating group A_n with $n \geq 5$ (the group $A_3 = C_3$ is excluded to avoid double-counting).
- A finite group of Lie type.
- One of the 26 sporadic simple groups.

Three of the 26 sporadic groups were discovered by Conway (in collaboration with Thompson), and are in fact called Conway groups Co_1, Co_2 and Co_3. These are all obtained from another group, called the Conway group Co_0 (the group Co_0 is not simple). The Conway group Co_0 is the group of symmetries of a collection of points in $\mathbb{R}^{24}$ called the Leech lattice, which are centres of an extremely dense way to pack spheres into space (we do not have similarly dense packings except in dimensions 1, 2 and 8).

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2.3 *The Monstrous Moonshine Conjectures*

The largest of the sporadic simple groups is named the *monster group*—this has about 8×10^{53} elements.

A very fruitful way to study the monster, or for that matter any group, is to consider its *representations*—essentially ways in which a group can be expressed as *linear* symmetries. Indeed the representations of the monster were described, using what is called a *character table*, even before the monster was constructed. Fortunately, the character table has a mere 194 rows and columns (in contrast to the multiplication table, with one row and one column for each element of the monster).

Any representation can be uniquely written as a sum (in the appropriate sense) of so-called *irreducible representations*. McKay and Thompson discovered a remarkable coincidence between the dimensions of the irreducible representations of the monster group and coefficients of *modular forms*.

Modular forms are central to many areas of mathematics, including number theory and complex analysis. These are functions on complex numbers that (in an appropriate sense) respect symmetries of the complex plane that correspond to certain important groups (such as the group, called $Sl_2(\mathbb{Z})$, consisting of 2×2 integer matrices with determinant 1). In particular, the symmetries ensure that modular forms can be written in terms of the sin and cos functions, i.e., have Fourier series. McKay and Thompson found that the coefficients of the Fourier series for a particularly important modular form correspond to the dimensions of representations. They conjectured that these coincidences come from an infinite-dimensional representation, which is the sum of finite-dimensional ones at various levels.

Conway and Norton greatly generalized this conjecture and backed it by a lot of calculations. The dimension of a representation is the *trace* of the identity element. Conway and Norton conjectured (and supported with calculations) a statement expressing functions of the traces in terms of coefficients of Fourier series of modular forms. These conjectures were named *monstrous moonshine*. Building on the work of others, these were proved in the early 1990s by Borcherds.

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Suggested Reading

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