John Horton Conway combined his passion for mathematics with an unquenchable enthusiasm for games. In addition to inventing a bewildering variety of games, he was deeply interested in analyzing them to determine how one should play to win.

Introduction

John Horton Conway was recognized for his wide-ranging contributions to mathematics, spanning group theory, knot theory, number theory, and coding theory. In addition to this “serious” side, he loved analyzing games. Not for him the game theory that mathematically models how buyers and sellers make strategic decisions to maximize their returns in a marketplace. Rather, he was interested in the games we play for entertainment. He was a prolific inventor of such games, and he devoted a significant amount of effort to their analysis to determine which player should win in a given situation and how he should play to achieve this.

Conway’s interest in games was a natural fit for Martin Gardner’s famous column on recreational mathematics, ‘Mathematical Games’, that ran monthly in Scientific American from 1957 to 1980. Several of Conway’s discoveries appeared in Gardner’s column and hence reached a much wider audience, conferring on him a minor celebrity status rare among mathematicians.

Along the way, Conway met two other mathematicians who shared his interest in games, Elwyn Berlekamp from the University of California, Berkeley and Richard Guy from the University of Calgary. Together, they set about writing an exhaustive treatise on the

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The games considered by Conway are those traditionally played for entertainment, needing nothing more than pen and paper. Typically, these games involve two players who take turns to move.

Clearly, there should be well-defined rules about who moves next, what the possible moves are, and what the outcome of each move will be. Moreover, a game should be guaranteed to end after a finite sequence of moves with a clear win for one of the players—that is, no draws. This last constraint rules out many standard games like tic-tac-toe and chess where ties are possible, as well as cases like perpetual check in chess, which results in a draw through an infinite play.

The usual assumption is that the game ends when some player is stuck and cannot move. The player who gets stuck loses. If this condition is inverted so that the player who gets stuck wins, we get the misère version of the game.

In some common games, the winner is decided differently. For instance, in dots-and-boxes, we have a grid of dots. Each play connects two adjacent dots. Whenever a box is formed, the player “claims” that box. After the grid is complete, the player who owns more boxes wins. This game is finite but is not decided by who makes the last move. Such games can also be brought within the theory developed by Conway and his co-authors.
2. Analyzing Games

How does one analyze a game? Let’s consider a very simple game where players take turns to count down from a number $n$. They play alternately and can subtract 1, 2, or 3 from the current number. The game stops when the number becomes 0. At 0, the player who has to move next loses.

If it is our turn to move and the number is 1, 2 or 3, we can reduce the number to 0 and win in one move, so these game positions are winning for us. If our opponent moves at 4, whatever he does will result in a winning position for us, so the opponent must lose. Thus, to force the opponent to play at 0, it is sufficient to force the opponent to play at 4.

By the same logic, from 5, 6, or 7, we can force the game to 4, from which we win. And, from 8, the opponent has no option but to leave us in a winning position.

Continuing this analysis, we see that all the multiples of 4 are bad for the player who has to move, and all other numbers are winning for the player who has to move. So if the game starts at 29, say, the first player moves to 28 and wins. If the game starts at 32, however, the first player will lose no matter what move he chooses, provided, of course, the opponent plays sensibly!

We can generalize this game to have more than one number. Suppose we start with two numbers $n_1$ and $n_2$. To move, a player picks one of the numbers and subtracts 1, 2, or 3 from that number. The game ends when both numbers become 0.

Why constrain ourselves to two numbers? We could have $k$ numbers, $\{n_1, n_2, \ldots, n_k\}$. To make things more interesting, we can allow subtraction by any number. Think of each number $n_j$ as a pile of $n_j$ coins. In one move, we can choose any pile and remove any number of coins from that pile.

This fully general version of the countdown game is called Nim. The simple analysis that we gave for the single number countdown game can be extended to Nim, though the combinatorics becomes immensely more complicated. In the 1930’s, R.M. Sprague
and P.M. Grundy independently came up with a way of associating an index with each game position that determines how the game will evolve. This index is now referred to as a nimber\(^1\). Unfortunately, we do not have the space here to explain how nimbers are defined and calculated.

Sprague and Grundy proved an even more remarkable theorem. Nim is an impartial game in that both players have exactly the same options at every move—it is not like chess, where one player moves only white pieces, and the other moves only black pieces. The Sprague–Grundy theorem says that any impartial game can be converted into a suitable version of Nim and analyzed using nimbers.

3. Partizan Games

What about games that are not impartial, where each player has a different set of moves. Winning Ways refers to these as partizan games and develops an analysis that is similar to nimbers for impartial games.

Here is one of the simplest partizan games in the book, called Ski Jumps. A rectangular grid represents a ski slope running from west (left) to east (right). There are two players, Left and Right, who control multiple skiers labelled L and R, respectively. In one turn, Left can move a skier L any number of squares east along a row. Likewise, Right can move a skier R any number of squares west along a row. If a skier falls off the grid, he is out of the game.

In addition, one skier can jump vertically over another type directly below it, if the square below is vacant. In their typical style, the authors write that “a man jumped over is so humiliated that he will never jump over anyone else—in fact, he is demoted from being a jumper to an ordinary skier or slipper.”

Figure 1 shows some moves in Ski Jump. Jumpers are represented by L and R and slippers by lowercase ℓ and r.

As usual, the first player whose skiers all move off the grid loses because no further play is possible. Without vertical jumps, the

\(^1\)Nimbers are special types of ordinal numbers that define the values of positions in the game Nim.
best strategy is to move one square at a time to stay on the board as long as possible. We can count the moves available to each skier and decide who wins. The analysis becomes more complicated when we take jumps into account. With some clever computation, the authors infer that jumps sometimes count for a half move and sometimes for a full move, leading to a complete analysis of any game position.

4. Sprouts

One of Conway’s best-known games is Sprouts. Co-invented with Michael S Paterson, this pen and paper game featured in the July 1967 edition of Martin Gardner’s column ‘Mathematical Games’.

The game is very simple to play. You start with a collection of spots. A move connects two spots by a line or a curve that does not cross any line that has already been drawn. A line can connect a spot to itself. A spot may have at most three lines connected to it. A new spot is added on the line that has just been drawn. The game ends when no new lines can be drawn.

Figure 2 shows a short game of Sprouts starting with two spots. Solid lines are drawn by the first player and dotted lines by the second player. In the final configuration, there are two spots with only two connections, but we cannot draw a line between them without crossing another line, so no further moves are possible.
Is a game of Sprouts guaranteed to be finite? Since each line generates a new spot, this is not obvious. However, we can prove that any game of Sprouts must terminate. Suppose the game starts with $n$ spots. Each spot can have three connections, or three lives, so we have $3n$ lives, to begin with. Each line we draw kills two lives (one at each endpoint) and creates one new one (the new spot along the line still has one life remaining), so the total lives reduce by 1. After $3n - 1$ moves, we can have only one life left, the one belonging to the final spot added. Thus, any game that starts with $n$ spots must end within $3n - 1$ moves.

However, it could end sooner, due to the rules governing moves. For instance, the game in Figure 2 starts with 2 spots and must, therefore, end within $3 \cdot 2 - 1 = 5$ moves. However, as we see, the game in the figure actually ends in 4 moves. A slightly more involved argument shows that a Sprouts game must go on for at least $2n$ moves.

Conway, along with Denis Mollison, proved that any game that ends in exactly $2n$ moves will reach a configuration built out of five fundamental figures, shown in Figure 3. Characteristically, Conway describes each of these figures as a type of insect. Of course, these insects may be hard to identify in the terminal configuration after $2n$ moves since they can be deformed.

Despite the simplicity of the game, Sprouts defies easy analysis. Mollison proved that the second player has a winning strategy for the 6 spot game. His argument ran to 47 pages. A computerized analysis in the 1990s by a team at Carnegie Mellon University managed to solve the 11 spot game. Computational techniques have pushed the boundary to 44 spots as of 2011, with isolated
proofs for games with 46, 47 and 53 spots.

Recall that the misère version of a game reverses the winning criterion: the first player unable to move wins. In Sprouts, it would appear a simple matter to adapt the analysis for the normal game to this setting. This is surprisingly hard to do, and the largest misère game that has been analyzed is the 20 spot game.

The World Game of Sprouts Association holds an annual tournament online. One of the rules is that the games are “for humans only”, to disallow computer programs from competing directly.

Conway proposed an extension to Sprouts where one starts with a number of crosses instead of spots. Each line connects two free endpoints of crosses, and we draw a new cross along the line, with two free endpoints. In yet another joke, Conway called this game Brussels Sprouts. Figure 4 shows a 2 cross game of Brussels Sprouts. Despite the similarity between the two games, Brussels Sprouts has proven even harder to analyze than Sprouts.

5. Winning Ways

Ski Jumps and Sprouts are but two examples of the bewildering assortment of games discussed in the book Winning Ways. A large number of the games analyzed in the book were invented while writing the book.

Conway talks of inventing games on a daily basis in order to illustrate aspects of the theory they were trying to build. Most of these games were discarded after some discussion. Over a period of time, Conway notes that they had two large collections: games without names and names without games. Though they tried to
match these as they went along, inevitably each new game generated a new name, and over time, the games that did not make it left their orphaned names behind.

The names of the games discussed in the book form entertaining reading. There's Hackenbush and Toads-and-Frogs, the White Knight and Wyt Queens, Kayes and Gulles, Treble Cross, and Lasker's Nim.

Volume 1 of the book is largely devoted to the simplest two player games, where each player makes exactly one move in each turn.

Volume 2 proceeds to more complex games where the game can have many components and a player can, in one move, make multiple changes across different components.

Volume 3 consolidates the theory of the first two volumes and applies it to “real” games, old and new. These are grouped by types: games involving turning over coins, games involving drawing lines (like Sprouts), as well as some board games.

Finally, Volume 4 turns to variations of solitaire, games played by one player. There is a detailed dissection of peg solitaire, where you start with an arrangement of pegs and eliminate them one by one by jumping, till you have only one left. There is also an analysis of construction puzzles, such as the Soma Cube of Piet
Hein. And, for the final flourish, there is the Game of Life.

6. The Game of Life

The Game of Life is easily the most famous game invented by Conway. This is a zero-player game that plays on its own, according to the rules set down by Conway.

Life is played on an infinite board of squares, or cells. Each cell has 8 neighbours, the cells that surround it. At any given time, a cell is either dead or live. The game starts with some finite set of cells live and the rest dead. The game proceeds in time steps, according to the following three rules.

**Birth:** A cell that is dead at time \( t \) becomes live at time \( t + 1 \) if exactly 3 of its neighbours are live at time \( t \).

**Death by overcrowding:** A cell that is live at time \( t \) and has 4 or more live neighbours at time \( t \) will be dead at time \( t + 1 \).

**Death by exposure:** A cell that is live at time \( t \) and has 0 or 1 live neighbours at time \( t \) will be dead at time \( t + 1 \).

These rules imply that a live cell at time \( t \) will survive at time \( t + 1 \) if and only if it has 2 or 3 live neighbours at time \( t \).

*Figure 5* shows how the game develops starting with a line of 5 live cells. In each configuration, a solid black circle denotes a live cell that will survive in the next step, an unfilled circle denotes a live cell that will die in the next step, and a dot indicates a dead cell that will become live in the next step. At the sixth step, the configuration reaches a pattern that Conway calls *traffic lights*, that flip forever between the horizontal and the vertical.

Traffic lights are an example of an oscillating pattern. There are also patterns that remain stationary—what Conway calls *still lifes*. Some examples are shown in *Figure 6*.

Another phenomenon is a cluster of live cells that moves as a unit across the board. Conway called these *gliders*. *Figure 7* shows one of the simplest gliders.
The Game of Life corresponds to an abstract computational device called a cellular automaton. Conway showed that the Game of Life, when viewed as a cellular automaton, is a universal computing device. In other words, with a suitable encoding, it can simulate a universal Turing machine.

Conway showed that the Game of Life, when viewed as a cellular automaton, is a universal computing device. In other words, with a suitable encoding, it can simulate a universal Turing machine. Because of this, the Game of Life is undecidable. There is no
algorithm that can determine whether a target configuration will be reached from a given starting configuration.

Another interesting question for the Game of Life, and for cellular automata in general, is to identify so-called Garden of Eden configurations. These are configurations that can never be reached during the course of a computation, so the only way for such an arrangement of cells to occur is to set it up as the starting state of the system.

7. Conclusion

John Conway’s fascination with games created an engaging body of work that has enlivened the world of recreational mathematics. However, his work in this domain is not just “fun and games”.

Conway’s study of nimbers and other numerical quantities associated with games led directly to his formulation of the surreal numbers, a new class of numbers that subsumes all the real numbers as well as all the levels of infinities, at the larger end of the spectrum and all the infinitesimals, at the smaller end.

As we have already seen, the Game of Life is not only an amazingly complex system that produces endless surprises, but also an example of a universal computing device that can shed light on different aspects of computer science.

Suggested Reading

...You get surreal numbers by playing games. I used to feel guilty in Cambridge that I spent all day playing games, while I was supposed to be doing mathematics. Then, when I discovered surreal numbers, I realized that playing games IS math.

-- John Horton Conway