John Horton Conway*

The Magical Genius Who Loved Games

John Conway—one of the most original thinkers by all accounts. The adjective 'magical genius' has been vociferously used by people who came into contact with him. *The New York Times* announcement of his passing away had the headline "John Horton Conway, a 'Magical Genius' in Math, Dies at 82". Conway passed away due to Covid-related complications. His contributions were diverse in the extreme—whether it was number theory, topology, coding theory, algebra or probability or whether it was the creation of many 'games', his originality stood out. The most well known among the games is his *Game of Life* which Madhavan Mukund describes in his article in this issue.

John Conway was born in Liverpool, England. His father Cyril was a self-taught person who made his living playing cards aided by a photographic memory. Later, Cyril was also a technician in a chemistry lab where students included George Harrison and Paul McCartney. John's mother Agnes was an avid reader of Dickens; she is said to have found John rattling off powers of 2 at the age of 4. John did his PhD work at Cambridge, England, under the supervision of Harold Davenport. Hired as an assistant lecturer at Cambridge University, John earned a reputation of being the most charismatic lecturer in the faculty. Apparently, he would often make some models or props for his games and take them along where he went. For instance, he would bring in a turnip as a prop, carving it one slice at a time into an icosahedron with its 20 triangular faces, and eat the scraps as he walked past.

John Conway was named a Fellow of the Royal Society in 1981. He moved to Princeton, USA, in 1987. There, he was usually requested to give the first-year course aimed to encourage students into taking up mathematics as the major subject. In summer camps for students, he was the star attraction where he would accept requests from students and deliver a talk on the requested topic extemporaneously.

Peter Cameron mentions the following incident in his blog: "This happened at a conference somewhere in North America. I was chairing the session at which he was to speak. When I got up to introduce him, his title had not yet been announced, and the stage had a blackboard on an easel. I said something like 'The next speaker is John Conway, and no doubt he is going to tell us what he will talk about.' John came onto the stage, went over to the easel, picked up the blackboard, and turned it over. On the other side were revealed five titles of talks. He said,

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'I am going to give one of these talks. I will count down to zero; you are to shout as loudly as you can the number of the talk you want to hear, and the chairman will judge which number is most popular.' So he did, and so I got to hear the talk I wanted to hear. RIP John, the world is a poorer place without you."

Conway was interested in almost every aspect of mathematics and often made discoveries through games and puzzles. Some of his mathematical work has been described in two articles in this issue. His SPLAG book (Sphere Packings, Lattices and Groups) and the 'Game of Life' are among the most popular even among amateur mathematicians.

In what follows, we provide a peek into some of his lesser-known discoveries so that the reader can divine for herself the virtuosity of his genius. There are all sorts of things from recreational tidbits to discoveries with deep underlying mathematics.

• In 1969, Conway came up with the *surreal number system* (aptly named by Donald Knuth). The surreal numbers form a totally-ordered field which is strictly larger than the real numbers, containing both *infinite numbers* (larger in absolute value than any positive real number) and *infinitesimal numbers* (smaller in absolute value than any positive real number). In the sense that vanishingly few real numbers are rational, very few surreal numbers are real numbers, the rest being strange numbers that we have no experience with. However, the situation is much worse because the collection of surreal numbers is far too large to be called a set in the usual (ZFC) axioms of set theory. The technical term for such an object is a *proper class*.

• Though it is easy to give 'formulae' for the day of the week corresponding to any date, Conway had an interesting observation that is easy to remember. In any given year, the following dates all fall on the same day of the week: 4/4, 6/6, 8/8, 10/10, 12/12, 5/9, 9/5, 7/11, 11/7, and the last day of February. Note that this algorithm does not depend on whether we use the American (mm/dd) or Indian (dd/mm) convention!

• Conway discovered this amazing *'look-and-say' sequence*—a discussion of this already appeared in an earlier article [1] by the second author. For the sake of completeness, let us recall this very briefly.

Start with any positive integer other than 22. Let us start with 1 say. Define the sequence which just reads out the number of times each chain of digits is repeated in turn. That is, after 1, we have 11 (meaning one 1), and after that, we have 21 (to mean two 1's) and 1211 (to mean one 2, one 1) and 111221 (meaning one 1, one 2, two 1's), etc. This is the 'look-and-say' sequence. In general, if $a_1^{k_1} a_2^{k_2} \cdots a_r^{k_r}$ with $a_i \neq a_{i+1}$, then the next term of the sequence is defined to be

$$k_1 a_1 k_2 a_2 \cdots k_r a_r.$$

For example, the sequence starting from 1 is:

$$1, 11, 21, 1211, 111221, 312211,$$

$$13112221, 1113213211, 31131211131221, \cdots$$

If d_n is the number of digits in the n-th term, then Conway discovered the remarkable fact that the ratio d_{n+1}/d_n approaches a constant λ (called *Conway's constant*), which is the unique real root of a certain integer polynomial of degree 71. Remarkably, every starting number (other than 22) leads to this same constant λ. The proof comes under the umbrella of what is now known as the cosmological theorem.

• Lagrange proved that every positive integer is a sum of four integer squares. More generally, instead of looking at the quadratic form $x^2 + y^2 + z^2 + w^2$, one can first look at the positive-definite, diagonal forms: $ax^2 + by^2 + cz^2 + dw^2$ for positive integers a, b, c, d.

Ramanujan was the first to put forth a study of universal forms when he wrote down the list of all integral, positive-definite, 4-dimensional diagonal forms which are universal! His list of 55 forms was later shown to be accurate and exhaustive excepting one form $x^2 + 2y^2 + 5z^2 + 5w^2$, which was observed to take all values excepting the value 15. Conway came up with the following general observation, which he proved along with his student William Alan Schneeberger. Consider any quadratic form

$$f(x_1, \cdots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$$

in n variables, where all a_{ij}'s are integers, $a_{ij} = a_{ji}$ for $i \neq j$ and which takes only positive values at all integers except at the point $(x_1, \cdots, x_n) = (0, \cdots, 0)$ where the value is evidently 0. They proved the remarkable *Conway–Schneeberger theorem*:

If f takes all the integer values from 1 to 15, it takes all integer values! Moreover 15 is the smallest such number (compare with Ramanujan's list with one error).

Conway–Schneeberger's proof was very involved, and Manjul Bhargava who came up with a simpler proof of this result was a graduate student in Princeton at that time, and generalized the result.

• Prior to the completion of the classification of finite simple groups—akin to the existence of elementary particles—someone would predict the existence of a group with certain properties, and much later, this would be proved with much more difficulty by someone else. A notorious such prediction was made in 1973 by Fischer and Griess, of a group now called 'the monster'. Conway was the one who gave the name 'Monster' to the Fischer–Griess finite simple group with the largest order among all the sporadic simple groups—the order is about 8×10^{53}.

Conway also named the 'baby monster' which is finite simple, of order something like 4×10^{33}. In 1980, Griess constructed the monster group as the automorphism group of a commutative, non-associative algebra of dimension 196883. Then, Mckay made the amazing observation that 196884 = 1 + 196883—alluding to the fact that the classical modular j-function has a Fourier expansion

$$j(q) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

The numbers 1 and 196883 are the dimensions of the smallest irreducible representations of the monster. Lest it be thought of as a coincidence, let us point out that

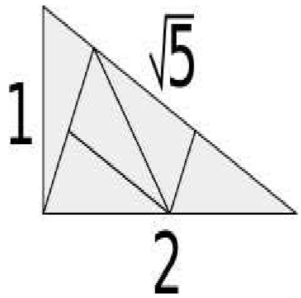
$$21493760 = 21296876 + 196883 + 1,$$

$$864299970 = 842609326 + 21296876 + 2 \times 196883 + 2 \times 1.$$

The left-hand sides are the Fourier coefficients of the j-function, and the right-hand sides give the dimensions of irreducible representations of the monster. Conway and Norton conjectured vast and bold generalizations of the existence of such hidden connections. It was Conway who introduced the nomenclature *'Monstrous Moonshine'*. The word 'moonshine' is slang for 'unreal' or for 'idle speculation', or 'an illusive shadow'. It is also said to be a reference to Shakespeare's *A Midsummer Night's Dream*, where this phrase was used to signify something crazy or unexpected. The 'monstrous moonshine' theory was an amazing and unexpected connection between two seemingly different areas of mathematics—modular functions and representations of the monster group, where the bridge came from physics—through conformal field theory. A whole hidden world started to emerge when Conway's graduate student Borcherds proved the moonshine conjectures; Borcherds won a Fields Medal for this work. Borcherds apparently said, "I was over the moon when I proved the moonshine conjecture."

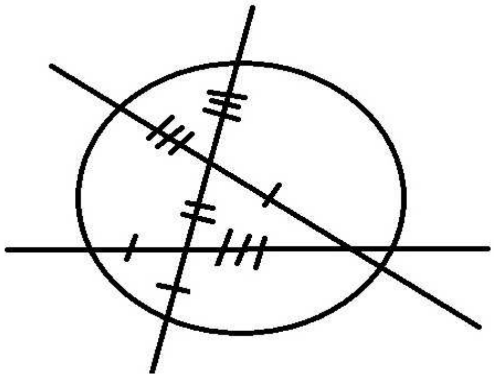
• Conway was a key contributor in the 'classification of finite simple groups' program as he himself created three sporadic groups! He was one of the authors of the *Atlas of Finite Groups*. The automorphism group of the Leech lattice is a group usually denoted by Co_0; its center is ± 1 and the quotient Co_1 is a simple group. Two other simple groups Co_2, Co_3 were also discovered by Conway as certain subgroups of Co_0; they are known as *Conway's simple groups*. Interestingly, originally Conway had named the groups Co_0, Co_1, Co_2, Co_3 respectively, as $.0, .1, .2, .3$. The notation .1 signified that this group was the stabilizer of the origin 0 and the notations $.2, 0.3$ alluded to the fact that they are stabilizers of vectors of type 2 and 3. The suggestion to look at the symmetry group of the Leech lattice is said to have come during ICM 1966 from McKay, then a graduate student.

• The right-angled triangle with side lengths in the ratio $1 : 2 : \sqrt{5}$ can be tiled by five mutually congruent right-angled triangles that are similar to the big triangle. This beautiful observation is also due to Conway.



• The *Angel Problem* is a game on an infinite chessboard that was proposed by Conway. The game is played by two players—the angel and the devil. The angel starts with a specific strength r, which is the maximum number of squares she can move if she moves like a king on the chessboard. Initially, the angel is at some square, and all other squares are empty. The devil blocks one square at each step, and the angel and the devil play alternatively. The angel can jump over a blocked square but is not allowed to land on one. The question was whether the angel could always keep on jumping without landing on a block or the devil could trap the angel. Conway offered a hundred dollar prize for anyone who could show that an angel with enough strength, to begin with, can keep escaping. He also offered a thousand dollars for proving that the devil could trap the angel no matter what her strength was. At the end of 2006, it was proved independently by several people that an angel with strength 2 can adopt a strategy to stay alive indefinitely. The proof by Mathé apparently uses a 'hands-on-the-wall' method that is used to come out of simple mazes.

• What has come to be known as *Conway's circle* is obtained by extending the sides of a triangle as shown in the figure here. The six points lie on a circle whose center is the incenter of the triangle.



Conway and Steve Sigur were writing an exhaustive book entitled *The Triangle Book* when tragically Sigur passed away before its completion. Sigur's website containing various interesting results with Conway can still be accessed at this webpage.

• The intermediate value theorem is one of the first theorems proved in calculus. It states that if a function f is continuous on a closed interval $[a, b]$ with $f(a) \neq f(b)$, then for any k between $f(a)$ and $f(b)$, there exists a $c \in (a, b)$ such that $f(c) = k$. It is natural to ask if the converse is true. That is, if, for every such k, there exists such a c, then is f continuous? Conway gave the ultimate counterexample to the converse by constructing an explicit function, now known as *Conway's base 13 function*, which takes on every real value in every interval $[a, b]$. This makes the function discontinuous everywhere!

To describe his function, we first fix some terminology. Given a real number x, first, write it in base 13. Apart from the usual decimal symbols commonly used, namely $0, 1, \ldots, 9$, we need three other symbols. Choose those three symbols to be $+, -$ and . (the decimal point). For example, one can check that the decimal number 54349589 is written as -34.128 and the decimal number 3629265 is $9 + 0 - -7$. Notice that the first makes sense as a decimal number, but the second does not.

Now, given a real number x, write it in base 13 in the notation describe above. If, after some position, the number makes sense as a decimal number (as in the first example), then that is the output. If no such position exists, the output is zero. For example, $19 + 0 - -7 + 3.1415$ outputs to $+3.1415$. One can show that in every interval $[a, b]$, this function takes on every possible real value.

• Conway worked on the so-called 'moving sofa' problem. This is the problem to determine the possible shape of a sofa that can be turned around a right-angled corner in a corridor. This has variations such as finding the optimal shape of a car that can turn at a T-junction. This came to be known as the *Conway car problem*.

• There are myriad other beautiful gems that we have not mentioned. For instance, *Conway's 150-method* allows us to quickly find the prime factors of any number having up to 4-digits. See the article by Arthur Benjamin [2].

Conway's Princeton colleague Simon Kochen said, "In mathematics and physics there are two kinds of geniuses. There are the ordinary geniuses – they are just like you and me but they are better at it; if we'd worked hard enough, maybe we could get some of the same results. But then there are the magical geniuses. Richard Feynman was a magical genius. And the same always struck me about John – he was a magical mathematician. He was a "magical genius" rather than an "ordinary genius."

Suggested Reading

[1] B Sury, Which positive integers are interesting, *Resonance*, Vol.20, No.8, pp.681–698, 2015.
[2] Arthur T Benjamin, Factoring numbers with Conway's 150 method, *The College Mathematics Journal*, Vol.49, pp.122–125, 2018.
[3] Genius At Play: The Curious Mind of John Horton Conway By Siobhan Roberts
[4] Videos of John Conway on Numberphile
[5] https://cameroncounts.wordpress.com/2020/04/12/john-conway/
[6] https://en.wikipedia.org/wiki/Surreal_number
[7] https://en.wikipedia.org/wiki/Conway_base_13_function

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