S S Shrikhande: The Euler Spoiler*

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After retirement from Mumbai University in 2010, Sharad Sane was at IIT Bombay until he joined the Chennai Mathematical Institute in 2019. Besides being a distinguished alumnus of the IIT Bombay, he was also a National Coordinator of the Mathematical Olympiad activity. He specialises in combinatorics.

1. Introduction and Early Life

I am humbled by being asked to write on S S Shrikhande, not only because of his mathematical stature but also because of his greatness as a human being. However, being fairly close to him, I hope to prove somewhat equal to the task, at least in terms of some of his mathematical work as also him as a person. My apologies right at the beginning for some indulgence in personal reminiscences, though I think they are worth the narration.

Professor Sharadchandra Shankar Shrikhande was born at Sagar in Madhya Pradesh on 17 October 1917 into a middle-class Marathi family of 9 children including himself. The huge economic slowdown in the shadows of the Second World War had made the situation difficult for the middle-class family of his. He landed at the Indian Statistical Institute in Kolkata after reading a small advertisement for the post of statistical assistant. Professor R C Bose had joined the Institute around the same time. It was also around this time that Bose published his seminal paper on the construction of block designs [1]. In the mid-1940s, Bose migrated to the U.S.A. to take up an academic position at the University of North Carolina, and Shrikhande obtained a scholarship to work for a PhD degree at the University of North Carolina. He did his research work mainly on the design existence questions with applications from arithmetic number theory under Bose's supervision, thus becoming his first PhD student. Shrikhande's thesis obtained some interesting results on the use of Hasse–Minkowski invariant for proving the non-existence of designs with certain pa-

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rameters. Shrikhande returned to Nagpur to take up a position at the Science College. The association he established with Bose (as his first PhD student) continued throughout his life,n and he continued writing path-breaking papers in collaboration with Bose for a very long time. Shrikhande visited the U.S. again in 1959 when, among other things, both Bose and Shrikhande planned to collaborate on a disproof of the long-standing (170 years) Euler's conjecture. They both had the key ideas on how they can proceed to tackle this problem. Among many results that Shrikhande proved, the Bose–Shrikhande–Parker theorem, acquired a special historical status. It proved that even outstanding mathematicians of the calibre of Euler can sometimes make errors in judgment. News of the disproof of Euler conjecture found a place as a front-page photograph and a fairly long report in a Sunday issue of *New York Times*. I will postpone the discussion of Euler spoiler result to Section 2.

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Shrikhande returned to India in the early 1960s and initially joined the Banaras Hindu University. He took up the position of the Head of the Department and Director of the Centre of Advanced Studies in Mathematics at Bombay University in 1963 and retired from that position in 1978. Shrikhande was then offered the directorship of Mehta Research Institute (now called Harishchandra Research Institute), Allahabad. He worked as the Director there for some years before returning to Nagpur in the early 1990s.

Shrikhande's selected papers [2] were published by the Charles Babbage Research Centre at Winnipeg, Canada. This book begins with a foreword by eminent mathematicians R G Stanton, K A Bush and J Srivastava. A quote from J Srivastava: "In any given combinatorial setting, he has the knack of discovering questions that are deep, significant and elegant". This is the hallmark of Professor Shrikhande's research work. The selected papers collection is divided into five sections and each section begins with a commentary from Professor Shrikhande himself that makes his insight available to the reader. I will briefly go over these sections.

The existence question of symmetric designs is a hard question and the non-existence question is even harder! Reasonably sophisticated number theory is employed for this purpose and it is a fact that among this difficult mass of research papers in this area, the ones authored by him are the easiest in terms of understandability. This section contains a joint work with D Raghavarao on affine resolvable designs and this paper is among his highly cited papers even today. Raghavarao also authored the first book [3] on combinatorial designs. This book has a large number of chapters devoted to various aspects of the statistical theory of block designs. Shrikhande truly was a mentor for Raghavarao like he was to many others. Since most of Shrikhande's research work is in the area of combinatorial designs, it would be apt to conclude this section with the following quote from Gianco Rota [4]:

"Block Designs are generally acknowledged to be the most complex structures that can be defined from scratch in a few lines. Progress in understanding and classification has been slow and has proceeded by leaps and bounds, one ray of sunlight being followed by years of darkness. $\cdots$ the subject has been made even more mysterious, a battleground of number theory, projective geometry and plain cleverness. This is probably the most difficult combinatorics going on today ..."

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–Gianco Rota

2. Set of MOLS and Euler's Conjecture

Suppose we have n^2 army officers coming from n different ranks. The number of officers of each rank also equals n, and we have to arrange them in an $n \times n$ grid so that every row and every column should have no two officers of the same rank. Then, we are merely asking the question of constructing an $n \times n$ array of numbers from 1 through n (the ranks) such that each row and each column is a permutation of the set $[n] = \{1, 2, \cdots, n\}$. Such an arrangement (or grid) can be easily constructed (for example, just cyclically shift the rows (modulo n) taking the first row to consist of numbers from 1 to n in the natural order) and is called a 'Latin Square'. Now consider a more difficult problem in which (in addition to the previous stipulation), the n^2 officers come from n different ranks (with n officers of each rank), and also n different regiments (with n officers of each regiment). This then requires the construction of two Latin Squares (LSs for brief) $L = [\ell_{i,j}]$ and $M = [m_{i,j}]$ of the same order n, which, when superposed, have the property that all the n^2 ordered pairs $(\ell_{i,j}, m_{i,j})$ so obtained are different (and hence obtains the Cartesian product, the set of ordered pairs is $[n] \times [n]$). Such a pair of LSs is called an orthogonal pair, and in the historical terminology, we have a pair $\{L, M\}$ of 'Mutually Orthogonal Latin Squares' (MOLS for short). Interchange of rows (or columns) of an LS and an application of a permutation to the symbols clearly renders the properties invariant and we record it in the following elementary assertion.

Lemma: *Both the properties "being a LS" and "being a pair of MOLS" are invariant under any of the following.*

(i) (Simultaneous for two LSs) Interchange of rows/columns.

(ii) Application of the same permutation to all the entries the LSs under consideration.

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Orthogonality of LSs is used in the statistical design of experiments to remove 'two-way heterogeneity', though these terms were obviously not at all in vogue at the time of Euler. If two LSs L and L' are orthogonal, then L' is an *orthogonal mate* of L (also the other way round). Here is a pair of orthogonal LSs.

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{bmatrix}$$

Unfortunately, 'orthogonality' is not a transitive property. Thus, the fact that L is orthogonal to both M and M' does not make M and M' orthogonal. Thanks to the Lemma, we may assume that the MOLS are in a form where the first row of all the squares is the identity permutation $[1, 2, \cdots, n]$. Superposition of two squares then obtains all the pairs (i, i) in the first row itself and hence the entries at the cell (position) $(2, 1)$ must all be distinct numbers other than 1. Thus the largest number of MOLS of order n can be no larger than $n-1$. Such a set of $n-1$ MOLS of order n is called 'a complete set of MOLS'. It was probably known to Euler that if n is a prime power, then there is a complete set of MOLS of order n. The converse, which is still open, is a big long-standing question in the theory of combinatorial configurations. Restricting to the question of finding just a pair of MOLS of order n, Euler could also derive a conclusion that a pair of MOLS exists for all $n \geq 3$ if n is not 'oddly even' (that is, n is not an even number of the form $2 \ (mod\ 4)$). This result uses prime power factorization of n and composition techniques of constructing larger order MOLS from those of smaller orders and was also known to Euler. Based on his findings, Euler went on to make the following (bold) conjecture:

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Euler's Conjecture: *Let $n \equiv 2 \ (mod\ 4)$. Then there does not exist (even) a pair of MOLS of order n.*

It is easily seen that we do not have a pair of MOLS of order 2 (the number is too small), and Euler checked (though it is not clear as to how) that there is no pair of MOLS of order 6. An exhaustive proof of this was worked out by Tarry in 1900; a much shorter proof that there is no pair of MOLS of order 6 was given by Stinson [5].

3. Disproof of Euler's Conjecture

Given LS $L = [\ell_{i,j}]$, when does L have an orthogonal mate? Just to see that the problem is non-trivial, consider the following example of an LS (constructed using the cyclic group $\mathbb{Z}_4$):

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

As an exercise, we ask the reader to prove that L has no orthogonal mate. Motivated by this idea, define 'a transversal' $T = T(\sigma)$

(where σ is permutation of $[n]$) to be the set

$$T = T(\sigma) = \{(i, \sigma(i), \ell_{i,\sigma(i)}) : i \in [n]\}.$$

Call the first two coordinates $\{(i, \sigma(i)) : i \in [n]\}$ the support of T. Thus the support of the transversal T is the set of n cells, one in each row and one in each column (corresponding to the permutation σ) of L. If L has an orthogonal mate $M = [m_{i,j}]$, then the set of all the cells (i, j) with $m_{i,j} = 1$ must constitute a transversal of L. The following assertion is then obvious.

L has an orthogonal mate if and only if L has n transversals T_k (with $k \in [n]$) such that the supports of any two transversals are disjoint.

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Theorem 1. *[6]: Let $L = [\ell_{i,j}]$ be the Latin Square of order n constructed using the cyclic group $\mathbb{Z}_n$ (of integers modulo n): $\ell_{i,j} = i + j - 1$. Suppose n is even. Then L has no transversal.*

Proof. For a proof, define $\Delta(x, y, z) := z - x - y$ where all the computations are in the cyclic group $\mathbb{Z}_n$. Suppose $T = T(\sigma)$ is a transversal of L. Define

$$\Delta(T) = \sum \Delta(i, \sigma(i), \ell_{i,\sigma(i)}),$$

and make a two way counting of $\Delta(T)$ to obtain the required contradiction if n is an even integer. □

Thus, the obvious (natural) candidate L (LS over $\mathbb{Z}_n$) for which one would like to explore the possibility of finding an orthogonal mate does not work. It fails miserably as it does not even possess a single transversal! In contrast, the Bose–Shrikhande–Parker theorem essentially disproves Euler's conjecture for all the values $n \equiv 2$ (*mod* 4):

Theorem 2. *Bose–Shrikhande–Parker (BSP) theorem [7]: Let $n \equiv 2$ (mod 4) with $n \geq 10$. Then there is a pair of MOLS of order n.*

Shrikhande visited the U.S. again in 1959, in his second stint, when, among other things, both Bose and Shrikhande planned (they had even exchanged ideas on how to disprove the conjecture since they both thought that it was false) the ambitious feat of disproving the conjecture. A brief introduction to the ideas used in the disproof of Euler's conjecture follows, along with some developments on that topic. A pairwise balanced design (PBD) is a collection of subsets (lines) of a v-set each with at least two elements (points) such that every point pair is contained in a unique line. This is also called a (combinatorial) linear space. The de Bruijn-Erdös theorem (or more generally, the Fisher inequality) tells us that the number of lines must be at least v, the number of points. Exploiting a technique similar to the one used by MacNeish where he constructed larger sized MOLS from the ones with smaller sizes, Bose–Shrikhande–Parker used what they called the PBD-closure technique to recursively construct MOLS of larger and larger sizes. What is most important here is that the PBD-closure technique was used by R M Wilson to prove a very strong existence result in design theory, a seminal work of a very high standing in the area that was published in the mid-1970s.

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I will close this discussion by discussing a more ambitious question. BSP result proves that for all values n other than $1, 2$ and 6 there is a pair of MOLS of order n. If n happens to be a prime power, then a finite field structure can be exploited to construct not just 2 but a complete set of (i.e. $n - 1$) MOLS of order n. It is a long-standing conjecture that a complete set of MOLS of order n exists if and only if n is a prime power (this is the famous projective plane conjecture) and this conjecture is still unsolved. For a given n, one may define $N(n)$ to be the largest number of MOLS of order n that can be gotten, and we already know that for almost all n, this function is bounded below by 2, thanks to the BSP theorem and by $n - 1$ from above. How does this function grow? Using the PBD techniques developed by BSP and a considerable amount of number theory, Chowla, Erdös and Strauss proved that $N(n)$ is asymptotic with n (so goes to infinity with n). They actually proved that [8] for sufficiently large n, the function

$N(n) \geq n^{\frac{1}{91}}$. This was later improved to a lower bound $N(n) \geq n^{\frac{1}{13}}$ for sufficiently large n by Wilson [9].

I close this section with the information that the cyclic LS of order 4 alluded to earlier in this section, has connections with Shrikhande's other famous work in what is now called the Shrikhande graph, a reference for which is my earlier expository article in Resonance [10].

4. Personal Reminiscences

Shrikhande retired from Mumbai University's Mathematics Department in 1978, and the department was then headed by Professor M G Nadkarni who continued Shrikhande's traditions. I joined the department in 1982 and needless to say, thanks to the best practices evolved, we were among the best postgraduate teaching departments in India. I first visited the Indian Statistical Institute, Kolkata in December 1982 for the Shrikhande 65 Conference. Shrikhande subsequently became the Director of the Mehta Research Institute (now the Harishchandra Research Institute), Allahabad, and I made two visits to Allahabad during the time he was the Director. After his term in Allahabad, Shrikhande moved to Nagpur. Later, Shrikhande spent his time between the U.S. and Nagpur or Sagar where he lived for a few years. On his way to the U.S. almost every year, he would stay at our quarters in Kalina and would take off from there. Despite having lived in the U.S. for so many years, Shrikhande remained a very simple person in his habits. On one visit to the U.S. when he got his first great-grandchild, he showed us what all things he was carrying for the great-grandchild. After the year 2000, Shrikhande spent a lot of his time staying with his youngest son, Anil's family in New Delhi. Anil had two different stints separated by a few years of stay in the same bungalow on Kautilya Marg in Delhi.

It was at the end of Anil's stay in Delhi that Shrikhande had to undergo a major orthopaedic surgery in Delhi. Then he shifted to Chinmaya Ashram, Vijayawada where he lived the last nine years of his life. His 100th birthday celebration on 19 October 2017 at the Ashram was a simple but spectacular affair. I distinctly remember that about a dozen couples had assembled there and each and every one of them said that they had never seen such a calm and saintly person in their lives. It so happened that during the last calendar year (2019), I visited Vijayawada thrice, twice to the Vijnana University, Guntur, and the last visit was on 7 December 2019. He had already completed 102 years of his life and was also exhausted and tired, though very well looked after by his attendant, Bhupendra. He passed away in April 2020.

His life is inspiring to all who came in contact with him and certainly to me. Though a fairly reserved person, Shrikhande was a thorough rationalist gentleman with a keen and constructive desire to support and encourage excellence in myriad facets of life as has been witnessed by a very large number of people who came in contact with him. Since brooding over the past was never his wont, it will be in the fitness of things to end this article by a light anecdote narrated to me by Rajeeva Karandikar. The faculty selection committee at Indore University selected Professor M G Nadkarni (who succeeded Shrikhande as the Head at Mumbai University later) as a professor in that department. Professor Shrikhande was on that committee. A citizen of Indore appealed to the Chancellor challenging the selection saying that "Professor Shrikhande is an eminent statistician and he had no problems there, but how could he select a professor for the Mathematics Department?" The appeal was, of course, turned down, but the point was well made. Shrikhande only cared for quality and looked far beyond bureaucratic exercises!

The Epilogue

A few close academic acquaintances of Shrikhande have told me that they think they are quite like him. I replied that it is true to a large extent, and the acquaintances also have some of the fine qualities of Shrikhande. But there is a difference. Shrikhande, in many ways, was very unique all through his life like the unique graph [10] that now bears his name!

Suggested Reading

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