Classroom

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

A Visual Proof: $e \leq A \leq B \Rightarrow A^B > B^A$ *

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Fascination with the constants e and π has encouraged numerous visual proofs of the inequality $\pi^e < e^\pi$. Nakhli [4] used the fact that $\frac{1}{e}$ is a global maximum for $y = \frac{\ln(x)}{x}$ to conclude the relation, and Nelsen [5] used the fact that $y = e^{x/e}$ lies above the line $y = x$. More recently, Chakraborty [1] used Napier's inequality (see [6] for a general visual proof of this inequality), and then Chakraborty and Mukherjee together [3] utilized the fact that the line $y = x - 1$ lies above the curve $y = \ln(x)$ when $x > 1$.

Also I [7] have submitted an article in Intelligencer journal on $e^A > A^e$ and it is published. Gallant [2] provided the most general proof for which this inequality is a consequence, showing that when $e \leq A < B$, we have $A^B > B^A$; he used slopes of secant lines connecting the origin to points on the curve $y = \ln(x)$. We provide an alternate visual proof for this general inequality using an area argument.

Keywords
Visual proof, e, π , $\pi^e < e^\pi$.

Theorem. *For all real numbers A and B with $e \leq A < B$, $A^B > B^A$.*

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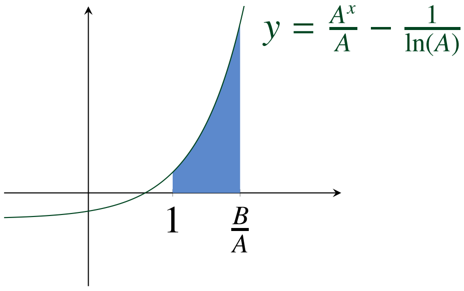
Proof.



$$0 < \int_1^{\frac{B}{A}} \left(\frac{A^x}{A} - \frac{1}{\ln(A)} \right) dx = \frac{A^{\frac{B}{A}}}{A \ln(A)} - \frac{B}{A \ln(A)}$$

$$\implies B < A^{\frac{B}{A}} \implies B^A < A^B. \qquad \square$$

Letting $A = e$ and $B = \pi$, the pictured equation becomes $y = e^{x-1} - 1$ and we conclude $\pi^e < e^\pi$.

Suggested Reading

[1] B Chakraborty, A visual proof that $\pi^e < e^\pi$, *Math. Intelligencer*, Vol.41, No.1, p.56, 2019.

[2] C Gallant, $A^B > B^A$ for $e \leq A \leq B$, *Math. Mag.*, Vol.64, No.1, p.31, 1991.

[3] A Mukherjee, B Chakraborty, Yet Another Visual Proof that $\pi^e < e^\pi$, *Math. Intelligencer*, Vol.41, No.2, p.60, 2019.

[4] Fouad Nakhli, $e^\pi > \pi^e$, *Math. Mag.*, Vol.60, No.3, p.165, 1987.

[5] R B Nelsen, Proof Without Words: Steiner's Problem on the Number e, *Math. Mag.*, Vol.82, No.2, p.102, 2009.

[6] R B Nelsen, Napier's Inequality (two proofs), College Math. J., Vol.24, No.2, p.165, 1993.

[7] Nazrul Haque, A visual proof that $e < A$ implies $e^A > A^e$, *Math. Intelligencer*, 2019 DOI: 10.1007/s00283-019-09964-x