Brownian Motion*

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This article explains the history and mathematics of Brownian motion.

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1. Introduction

Brownian Motion is a random process first observed by the Scottish botanist Robert Brown; later independently proposed as a model for stock price fluctuations by the French stock market analyst Louis Bachelier; a little later German-born Swiss/American Albert Einstein and Polish physicist Marian Smoluchowski arrived at the same process from molecular considerations; and still later Norbert Wiener created it mathematically. Once the last step was taken and matters were clarified, Andrei Kolmogorov (Russian) and Shizuo Kakutani (Japanese) related this to differential equations. Paul Levy (French) and a long list of mathematicians decorated this with several jewels with final crowns by Kiyoshi Ito (Japanese) in the form of Stochastic Calculus and by Paul Malliavin (French) with Stochastic Calculus of Variations. Once crowned, it started ruling both Newtonian world and Quantum world. We shall discuss some parts of this symphony.

2. Robert Brown

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Brownian motion, heat equation, Kolmogorov's formulation of probability, random walk, Wiener integral, stochastic differential equations, financial mathematics.

Robert Brown was a Scottish botanist famous for classification of plants. He noticed, around 1827, that pollen particles in water suspension displayed a very rapid, highly irregular zig-zag motion. He was persistent to find out causes of this motion: Are there any water currents? Is there any evaporation leading to this motion? Is there any attraction and repulsion of particles? Is the

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system in unstable equilibrium? Is there any capillary action? Are there air bubbles causing this?

He took pains to exclude the reasons he could think of. The motion still persisted and he even surmised that there may be life in the pollen particles. He experimented with materials other than pollen. He was so meticulous, he says: 'to give greater consistency to my statements, and to bring the subject as much as possible within the reach of general observation, I continued to employ throughout the whole of the inquiry the same lens with which it was commenced.' He honestly states that others too before him have seen this motion; 'obscurely seen by Needham, and distinctly by Gleichen'.

However, credit goes to Brown for insisting for an explanation and drawing wider attention to this phenomenon. He teaches us to be diligent and persist for explanation of any unexpected phenomenon that we notice.

3. Louis Bachelier

More than seventy years later in 1900, Louis Bachelier, a student of Henri Poincaré, in his doctoral thesis analyzed the stock market and built model for price fluctuations. His father (apart from other things) was a Wine merchant; mother was a Banker's daughter; a grand father was an important person in the financial business. His parents passed away soon after his graduation and he had to continue his father's business to take care of his sister and brother. Further, by 1850 the Paris Stock Market was already famous.

Let us say that X_t is price change (current price minus initial price) at time t. Due to the uncertainties in the market, X_t can not be deterministic and should be modelled as a random variable. What could be a good model for the distribution of this random variable? Let $p(t, x)dx$ be the probability that $X_t \in (x,\ x + dx)$. He argued that for the price change at time $(t_1 + t_2)$ to be near z, it should be near '*some x*' at time t_1 and then in the remaining time

t_2 there should be a further displacement of $z - x$. Thus

$$p(t_1 + t_2, z) = \int_{-\infty}^{\infty} p(t_1, x) p(t_2, z - x) dx \qquad (1)$$

With his expertise in physics he quickly recognized that the fundamental solution of heat equation fits the bill!

$$p(t, x) = \frac{1}{2\pi k \sqrt{t}} e^{-x^2/(4\pi k^2 t)}; \quad -\infty < x < \infty; \ t > 0.$$

Here $k > 0$ is a constant. This is of course normal density or Gaussian density, also commonly known as bell shaped curve. He leaves aside the issue whether there are other solutions of the above equation. Thus the upshot is $X_0 = 0$ and for $t > 0$, $a < b$ we have

$$P\{X_t \in (a, b)\} = \int_a^b p(t, x) dx.$$

Bachelier's thesis was unique in many ways. Mathematical Physics and Geometry were fashion of the day, not probability. Probability was not even well recognized, till Borel proved his Normal Number Law: in the decimal expansion of a typical number in $[0, 1]$ the ten digits occur with equal frequency. Continuous time processes were not considered till then. The concept of 'path' of process was considered for the first time. He discovered the equation (1), known now as the Chapman–Kolmogorov equation. He introduced the fundamental concept of arbitrage which is crucial in financial mathematics: expected gain at any instant is zero for any transaction at the Bourse. One moral to learn is this: have an open mind, ideas from other areas might be useful, you can innovate, not necessary to tread fashionable path.

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4. Einstein

A few years later, in 1905, Einstein was researching statistical mechanics and molecular kinetic theory of heat, not aware of the works of Brown and Bachelier. He says: 'It will be shown that according to the molecular kinetic theory of heat, bodies of microscopically-visible size suspended in a liquid will perform movements of such magnitude that they can be easily observed in a microscope ....'. He continues: 'it is possible that the movements discussed here are identical with the so called Brownian motion. However the information available to me is lacking in precision, I can form no judgement in the matter.'

If atomistic theory is correct, then the molecules of the fluid keep on hitting the suspended particles — the net displacement is indeed observable.

The idea is that if atomistic theory is correct, then the molecules of the fluid keep on hitting the suspended particles. Even though each individual hit results in a minute unobservable displacement, there are so many hits – a constant bombardment – that the net displacement is indeed observable.

He had a two part argument for the motion. In the first part he derives a differential equation. Let us discuss displacement in, say, X-direction. If $p(t, x)dx$ is the probability that the displacement at time t is near x, then for some constant D,

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}. \tag{2}$$

A word of clarification is in order here. Of course, he did not involve probability, he was talking about the proportion of particles that are displaced by an amount x at time t. The second part of the argument consists of relating D, coefficient of diffusion, to Avogadro number and coefficient of viscosity of the liquid. One knows that a solution of the above equation is the normal density,

$$p(t, x) = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/(4Dt)}; \quad -\infty < x < \infty, \ t > 0.$$

The equation (2) is known as the diffusion equation. The Polish physicist Smoluchowski apparently arrived at the Brownian motion prior to Einstein and was waiting to test his theoretical predictions; but decided to publish after he saw Einstein's paper.

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Einstein made one crucial assumption: 'we will introduce a time interval τ in our discussion which is to be very small compared with observed interval of time, but, nevertheless, of such magnitude that the movements executed by a particle in two consecutive intervals of time τ are to be considered as mutually independent'.

One important conclusion is that 'squared displacement is proportional to the duration of displacement.' You will see a precise version of these in the mathematical formulation later.

5. Wiener

Bertrand Rusell realized the importance of these researches of Einstein and encouraged his postdoctoral student Norbert Wiener to treat it rigorously. Several years later, in 1923 Wiener gave a rigorous mathematical argument to show the existence of such a process. What does this mean? He showed how to calculate expectations of functionals of the process. Wiener's calculations were enough to give a probability measure, according to a theory developed by Daniell and already available at that time.

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As J L Doob says 'He constructed this process rigorously more than a decade before probabilists made their subject respectable and he applied the process both inside and outside mathematics in many important problems'. Irving Segal says 'the novelty of Wiener's Brownian motion theory was such that it was not at all widely appreciated at that time and the few who did, such as H Cramér in Sweden and P Lévy in France were outside the United States.'

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– J L Doob

Wiener discovered several properties of this process, in particular showed that the particle has continuous paths.

6. Kolmogorov

Around 1933, Kolmogorov had given decisive measure theoretic formulation to the theory of probability. A model for chance experiment consists of three objects. Firstly, a set Ω, called sample space which collects all outcomes of the experiment. Secondly, a collection of subsets called events satisfying reasonable conditions. Thirdly, an assignment $P(A)$ for each event A in such a way that $P(\emptyset) = 0$; $P(\Omega) = 1$; and if $\{A_n, n \geq 1\}$ is a disjoint sequence of events, then $P(\cup A_n) = \sum P(A_n)$ (in particular the union must be an event). A random variable is a measurement performed on the outcome, thus it is a real valued function defined on Ω such that for every number c, the set $\{\omega \in \Omega : X(\omega) \leq c\}$ is an event. This last condition is to answer questions like: what are the chances that the value of measurement X is at most c? Distribution function of the random variable is the function $c \mapsto F(c) = P\{\omega \in \Omega : X(\omega) \leq c\}$. Expectation of X, denoted $E(X)$, is Integral $\int X dP$.

7. The Model

Let us see how to mathematically model the motion we are talking about. To simplify matters we consider one dimension. Imagine a particle performing motion in one dimension, subject to random bombardments, no force acting. Let X_t be the position of the particle at time t, for $t \geq 0$. The first thing to notice is that each X_t is a random variable. Thus we have a probability space. As mentioned above this means a set Ω with a nice collection of subsets called events and Probability $P(A)$ defined for each event A. For each $t \geq 0$, X_t is a real valued function defined on the space Ω. Points in Ω are denoted by ω. Thus for each $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ gives one possible path of the particle, or one possible scenario of the motion. Since we would like to be not too technical we shall follow the maxim: do not scratch unless it itches.

Initially we assume that the particle is at position zero. If you do not like it, imagine that we are modelling displacement, so that X_t denotes the displacement from initial position during the time interval $[0, t]$. Thus we want:

(i) $X_0(\omega) = 0$ for all ω.

Next, the particle performs continuous motion, it does not jump. In other words, each scenario is a continuous path. Thus we want:

(ii) For every ω, the path $t \mapsto X_t(\omega)$ is a continuous function of t.

There is no force acting on the particle, the motion is only due to bombardment of the particle by the molecules of the fluid. These pushes being random from all directions we expect that the net displacement is zero, on the average. That is, there is no bias to move the particle in a particular direction. Thus we want:

(iii) $E(X_t) = 0$ for every t.

The displacement during a time interval depends on the molecular hits during that interval and so depends on the duration of the interval; in other words, on the length of that interval. Thus we want:

(iv) If $s < t$ and $u < v$ such that $t - s = v - u$, then $X_t - X_s$ and $X_v - X_u$ have the same distribution.

Note that X_t being displacement upto time t, the quantity $(X_t - X_s)$ is indeed displacement during the time interval (s, t). In particular $X_t - X_0 = X_t$ has the same distribution as $X_{t+s} - X_s$, whatever be $s \geq 0$ and $t \geq 0$.

Finally we assume that displacements during non-overlapping time intervals are independent. This reflects a fact that Einstein assumed. Thus we want:

(v) If $0 = t_0 < t_1 < \ldots < t_k < \infty$ then $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$ are independent.

Is there such a process? Yes. It is also unique in the sense of distribution. Thus Brownian Motion exists as a concrete mathematical object. A word of clarification is in order. When you model chance phenomena, it is enough if the demands are met on an event of probability one. One does not really worry if the demands are not met on an event of chance zero. This is because main concern in such an analysis is to answer chances of something interesting happening or not happening. Thus it is customary to demand that requirements (i) and (ii) hold on an event of probability one.

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8. Distributions

Our first concern is the fact that there was no mention of the distribution of the random variable X_t in the above formulation. We saw Bachelier, Einstein and Smoluchowski arrived at the bell shaped curve for the distribution of X_t.

If you are familiar with Central Limit Theorem of probability, you would guess that X_t must be a normal random variable. $X_t = X_{t/2} + (X_t - X_{t/2})$, sum of two random variables which are independent [assumption (v)] and have the same distribution [assumption (iv)]. Of course X_t is sum of four random variables too if you consider $X_{t/4}$, $(X_{t/2} - X_{t/4})$, $(X_{3t/4} - X_{t/2})$ and $(X_t - X_{3t/4})$. In fact given any n, X_t is sum of n independent random variables, each of the n having the same distribution (depending on n). Since $s \mapsto X_s$ is a continuous function [assumption (ii)], you also expect these differences, whose sum makes up X_t, are also very small. In other words X_t is sum of a large number of very small random variables which are independent and identically distributed. Hence it must be Gaussian – this is the spirit of Central Limit Theorem. Of course to show this precisely, you need to argue carefully.

Now that we believe each X_t is gaussian, we only need to know its mean and variance. But we know mean is zero [assumption (i)]. We only need to know its variance. Let us denote it by $v(t)$. Clearly $v(0) = 0$.

$$v(t+s) = var(X_{t+s}) = var[X_t + (X_{t+s} - X_t)] = v(t) + v(s)$$

where the last equality is from assumptions (iv) and (v): $(X_{t+s} - X_t)$ has same distribution as X_s and is independent of X_t. Assumption (ii) makes us believe that $v(t)$ should be continuous, which can be proved. As a result the equation above tells us that $v(t) = ct$ for some number c. The case $c < 0$ can not hold because variance is non-negative. The case $c = 0$ is uninteresting, because then X_t is a constant random variable for each t and mean zero tells $P(X_t = 0) = 1$ for all t. Continuity of paths tells $P(X_t = 0 \text{ for all } t) = 1$. In other words there is no motion. Thus $c > 0$. By choosing suitable scaling we can assume $c = 1$. So $v(X_t) = t$, and by assumption (iii) we see $v(X_t - X_s) = v(X_{t-s}) = t - s$.

These considerations lead us to another definition, essentially equivalent to the previous one.

A process $\{X_t, t \geq 0\}$ is called standard Wiener process or standard Brownian motion if (i) it starts at zero, that is $X_0 \equiv 0$; (ii) it has continuous paths, that is $t \mapsto X_t(\omega)$ is a continuous function of t for each sample point ω; (iii) $X_t - X_s \sim N(0, t-s)$ for $0 \leq s < t < \infty$; and (iv) has independent increments, that is for $0 = t_0 < t_1 < \ldots < t_k < \infty$ the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$ are independent.

Let $x \in R$. A process $\{X_t, t \geq 0\}$ is called Wiener process starting at x or Brownian motion starting at x if the process $\{X_t - x : t \geq 0\}$ is standard BM.

As the name suggests, for BM starting at x, we have $X_0 \equiv x$ and for each t, $E(X_t) = x$. Clearly, standard BM is just BM starting at zero.

But where did the motivation disappear: Are we not supposed to model 'motion of a particle under constant bombardment by molecules'? Well, it did not disappear.

9. Continuous Update of Interest

It is always tricky to make meaning of 'continuous' something. Most of our understanding of continuous phenomena is via 'limit of discrete phenomena'. Let us first discuss an elementary problem where this was already encountered in connection with simple interest, compound interest and continuous interest payments.

It is always tricky to make meaning of 'continuous' something. Most of our understanding of continuous phenomena is via ' limit of discrete phenomena'.

Suppose you put one rupee in a bank. Assume the interest is calculated yearly and the rate is r Rs. per rupee per year. Thus if the interest rate, in the customary sense, is 6% per annum, then $r = 0.06$. So at the end of the year you get $(1 + r)$ Rs. Suppose the interest is calculated half-yearly and the rate is $r/2$ Rs. per half year per rupee; this is called pro-rata (at the same rate). Then after a half year you have Rs. $(1 + \frac{r}{2})$ and at the end of the year you get Rs. $(1 + \frac{r}{2})^2$. More generally, if the interest is calculated $(1/n)$-yearly and the rate is r/n Rs. per rupee per $(1/n)$-year, then at the end of the year you get $(1 + \frac{r}{n})^n$ Rs.

If the bank says it updates interest continuously, then what should it mean? A reasonable meaning is that you take limit of these above numbers, which, luckily exists giving e^r rupees at the end of one year. As you would notice, we did not say that the interest is Rs. r per year and *hence* it is r/n per $1/n$-year. We outright said it is r/n Rs. per $1/n$-year, as if prorata is God-given. The subtle point is that the bank has the option of announcing a possibly different interest rate, but in reality it has no choice!

With continuous updating, can the bank afford to announce rate of Rs. $r/\sqrt{n}$ per rupee per $1/n$ year and then say interest accrues continuously? No. because then at the end of the year they should pay you Rs. $\lim(1+\frac{r}{\sqrt{n}})^n$ which is ∞! Indeed

$$\left(1+\frac{r}{\sqrt{n}}\right)^n \geq 1+n.\frac{r}{\sqrt{n}} > r\sqrt{n} \to \infty. \tag{3}$$

On the other hand, suppose the bank announces a rate of Rs. r/n^2 per rupee per $1/n$-year and then say interest accrues continuously; will you accept? No. Because then you will receive no interest at all; $\lim(1+\frac{r}{n^2})^n = 1$. Indeed, given any $\epsilon > 0$, for all sufficiently large n, we have $\frac{r}{n} < \epsilon$ so that

$$\limsup\left(1+\frac{r}{n^2}\right)^n \leq \limsup\left(1+\frac{\epsilon}{n}\right)^n = e^{\epsilon}. \tag{4}$$

This being true for every $\epsilon > 0$ we conclude that the lim sup is at most one. But limsup is at least one because each term of our sequence is so. Thus the limit exists and equals one.

The square root is unimportant, bank can not afford to have any power of $1/n$ which is smaller than one. This is because the proof of (3) shows that,

$$\lim\left(1+\frac{r}{n^{\alpha}}\right)^n = \infty \quad \text{for any } \alpha < 1.$$

Similarly any power of $1/n$ larger than one is not acceptable to you. The proof of (4) shows

$$\lim\left(1+\frac{r}{n^{\alpha}}\right)^n = 1 \quad \text{for any } \alpha > 1.$$

Thus there must be a match between the time units and the rates in order to take meaningful limits. Since we already started with an

r, the above discussion may appear like a tautology. You should carefully think about what we did: in practice, there are only discrete time models to start with and we need to give a meaning to continuous time model.

10. Molecular Bombardments

Let us see how we can view Brownian Motion as model for continuous bombardment of the particle by surrounding molecules. This would be interpreted as limit of discrete motions. Just to make matters simple let us consider only time duration $[0, 1]$.

Fix an integer $n > 1$. Consider the particle position at time points $\{k/n : k = 0, 1, 2, \ldots, n\}$. Let us pretend that the displacement between successive time points equals in magnitude $1/\sqrt{n}$. Remember Einstein's conclusion: squared displacement during an interval is proportional to the duration of that interval. We took the constant of proportionality to be one. But since the average displacement is zero the actual displacement is $\pm 1/\sqrt{n}$, each with probability 1/2. Combine with the fact that displacements during disjoint intervals are independent. The upshot is the following. There are 2^n possible scenarios for this discrete motion. If $\{\epsilon_i; 1 \leq i \leq n\}$ are independent random variables each taking value $\pm 1/\sqrt{n}$ with equal probabilities; then $X_0^n = 0$ and $X_{k/n}^n = \sum_{i=1}^{k} \epsilon_i$. The superscript indicates the discrete motion, not to be confused with the Brownian motion (X_t). If you are familiar with random walk, you recognize this as symmetric random walk.

The question now is whether the above discrete motions have any limit. There are two problems. First is the following. The discrete motion $\{X^n\}$ is defined for certain time points and this set of time points changes with n. This is easy to rectify. Define $\{X_t^n\}$ for all t in $[0, 1]$ by saying that between two successive time points k/n and $(k + 1)/n$ we do not move, stay at that position. Or you can say move linearly to the next position from the current position. In the first case the particle jumps at time points k/n; in the second case it moves continuously deterministically to the next designated position. Either way we have now for each n, a process $(X_t^n, 0 \leq t \leq 1)$. The second problem is: limit in what sense? This can be made precise, but unfortunately becomes very technical, so we will not do so. The limit exists and is indeed the BM described above.

This is satisfying. BM does model motion of particle subject to random bombardments. One question remains, did we take $1/\sqrt{n}$ displacement for the n-th discrete motion only because of Einstein's conclusions? Not really, it can be shown that $1/n^\alpha$ would not work for $\alpha > 1/2$ or $\alpha < 1/2$. In the first case the limit exists and corresponds to 'no motion'; in the second case the 'fluctuations are so huge' there is no limit. This is similar to the simple phenomenon we discussed earlier.

11. A Closer Look

Does our mathematical model confirm the observed 'zig-zag highly irregular' motion? Yes, one can show that a typical path has several, mathematically non-smooth, properties.

Does our mathematical model confirm the observed 'zig-zag highly irregular' motion? Yes, one can show that a typical path has several, mathematically non-smooth, properties. For instance, a typical path $t \mapsto X_t(\omega)$ is a nowhere differentiable function. A typical path is non-monotone on any interval you take. Typical path passes through each real number infinitely many times, that is, given any real number a, the set $\{t : X_t(\omega) = a\}$ is an infinite set. However if you start BM from x, then for any $y \neq x$, the expected time taken to hit y is infinite.

You start BM from zero and dozed off; at time 5 you suddenly saw that the particle is at $x = 23$ and wonder how the future motion proceeds. Well, the future motion proceeds as if it now started at 23. The way it reached from zero to 23 during time duration $[0, 5]$ is irrelevant. In other words, the future motion depends only on the present and not on the past. This is known as Markov property. Something more is true. You start BM from zero. Fix a position $a \in R$. You set an alarm to ring as soon as the particle first reaches a. It may be at time $t = 5$ or $t = 30$ or $t = 1000$. Obviously, when the alarm rings, the particle is at a. You want to know how does it proceed from now on? The answer is that it proceeds as if started at a now. This is called the strong Markov property.

There are many interesting questions you can ask and can answer many of them. For example, start BM from zero and follow the particle until time $t = 1$. Consider the proportion of time the particle spends on the positive side. Naturally it depends on the scenario, that is, on the path it takes. So it is a random variable. How is it distributed?

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12. Beginnings of Calculus

We shall conclude with a very interesting idea of Wiener. Suppose you have two finite dimensional vector spaces with inner product; say V and W with inner products $\langle .,. \rangle_V$ and $\langle .,. \rangle_W$. To make life simple consider only real vector spaces. Suppose that you have a subset $S \subset V$ which spans V as a vector space. Suppose we have a map $I : S \to W$ which is inner product preserving, that is $\langle x, y \rangle_V = \langle I(x), I(y) \rangle_W$. Then I can be extended as an inner product preserving linear map from V to W. This is a nice simple exercise you should try. Note that, this in particular means that if $x, y, x + y$ are in S then the hypothesis already implies $I(x+y) = I(x)+I(y)$, though we did not say this explicitly.

Let $(X_t, t \geq 0)$ be standard BM defined on a probability space (Ω, P). Let us now extend our imagination and replace V and W as follows.

Let $V = L^2[0, \infty)$, with usual Lebesgue integral. Thus it consists of all square integrable functions on $[0, \infty)$ and $\langle f, g \rangle_V = \int f(x)g(x)dx$. Recall Ω is the space on which our BM is defined. Take $W = L^2(\Omega)$. Thus it consists of all random variables which are square integrable and $\langle X, Y \rangle_W = E(XY) = \int XY \, dP$. For each $t > 0$ let e_t be the indicator function of $[0, t]$. More precisely, e_t is defined on $[0, \infty)$ and $e_t(x) = 1$ if $x \leq t$ and $e_t(x) = 0$ if $x > t$.

Then $e_t \in V$. Define $I(e_t) = X_t$, remember $(X_t : t \geq 0)$ is our standard BM. Thus $I(e_t) \in W$.

Take $S = \{e_t : t > 0\}$. A simple calculation with Gaussian distribution shows that this map I is inner product preserving; S spans V and I can be extended to all of V. Of course, in the finite dimensional case span is simply the linear span; but in the present case span means closed linear span. For $f \in L^2[0, \infty)$, the random variable $I(f)$ is called Wiener integral and usually denoted by $\int_0^\infty f(t)dX_t$. One can define indefinite integral too. Thanks to the theory of martingales, developed by J L Doob, this indefinite integral is a nice process. A simple, yet profound, extension of the above integral with far-reaching consequences is the Ito integral. Here the integrand f can depend not only on t but also on ω. Thus f itself is a process. Thus you can integrate reasonable processes with BM.

Differential equations in the present set up are called Stochastic Differential Equations (SDE). Paul Malliavin developed powerful techniques to elicit information about existence and smoothness of densities of random variables that arise from solutions of SDE. This body of knowledge goes by the name of Malliavin calculus.

Once you have integration you can discuss differential equations. After all, the ordinary differential equation $x'(t) = x(t); x(0) = 1$ is simply same as saying $x(t) = 1 + \int_0^t x(s)ds$. Differential equations in the present set up are called Stochastic Differential Equations (SDE). Paul Malliavin developed powerful techniques to elicit information about existence and smoothness of densities of random variables that arise from solutions of SDE. This body of knowledge goes by the name of Malliavin calculus. This theory sheds light on properties of solutions of certain partial differential equations and also has applications to stability problems in mathematical finance.

We must mention here an important and tragic historical event – related to Wolfgang Doeblin.

We must mention here an important and tragic historical event. In the thirties, the German-born French probabilist Wolfgang Doeblin, during his researches on Chapman–Kolmogorov equation, already solved some SDE! He did not develop stochastic integral, instead he considered 'differential version' as explained below. As found by Marc Yor and others, his work has ideas of martingales, Ito formula and random time change.

To get the right perspective, keep in mind that solving the ordinary differential equation $x'(t) = x(t)$ is 'same as' the following. Find a function x with the data: if $x(t) = a$ then in a small interval $(t, t + dt)$ the displacement $[x(t + dt) - x(t)]$ equals $a\,dt$. In the present context we want to model random motion of a particle. For $t \geq 0$, let Z_t be the position of the particle at time t. Here is the data: if $Z_t = x$, then in a small interval $(t, t + dt)$ the particle has a deterministic displacement $a(t, x)dt$ and a random displacement which is Gaussian with mean zero and standard deviation $\sigma(t, x)\sqrt{dt}$. Here a and σ are given functions. In the present day terminology, this amounts to solving the SDE: $dZ_t = a(t, Z_t)dt + \sigma(t, Z_t)dX_t$ where (X_t) is the standard BM.

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From the war front he sent his working papers, in a sealed envelope, to the French Academy for safe custody. His plan was to collect the papers after the war. Alas, it was not to be – surrounded by the Nazi army, he shot himself rather than surrendering. This was in 1940 when he was just twenty five years old. The sealed envelope of Doeblin was opened in May, 2000. Paul Lévy compared him to Galois and Abel.

Returning to SDE, you get the feeling that matters are getting too technical, more and more mathematical. Then how come this theory found applications in several diverse areas, be it physics or mathematical finance or signal processing or biology? The reason is simple. Once you can talk about SDE and are able to solve them, you can construct more and more processes. Once you succeed in this, you can model more and more phenomena. Once you can model a phenomenon, you can understand it better.

As the statisticians J Neyman and E L Scott say: 'Each attempt to treat mathematically a complicated category of phenomena must rely on idealizations of certain factors deemed of predominant importance and must ignore innumerable other factors. Whether, as a whole the model is close enough to the actual phenomenon to be useful for practical purposes, for example, prediction, can be established only through comparison with data. However, even if a proposed model proves totally inadequate, it is hoped that the mere process of establishing the model's inadequacy will contribute to a better understanding of the fascinating phenomenon

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which is being modelled'.

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Suggested Reading

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