A cube is sliced in half by a plane cut orthogonal to a diagonal. We want to label the ten vertices of the half-cube with digits 0 through 9 so that the sum of the labels around each triangle is a constant, and the sum of the labels around each pentagon is another constant. We count all such labels.

Introduction

We pose a mathematical puzzle that can be explained to a high school student. We describe the genesis of the puzzle in Section 1. We give a formal statement of the puzzle in Section 2, and urge readers to solve the puzzle on their own before reading the rest of the article. We solve the main problem in Section 3 and other related problems in Section 4. We state some open questions in Section 5, which concludes the article.

1. Genesis of the Puzzle

My collaborators and I study random walks on different graphs. After completing our study of walks on the vertices of a cube, we looked for other interesting shapes.

We sliced a unit cube into two identical halves using a plane cut through the center and orthogonal to the main diagonal of the cube. See Figure 1(a), which is taken from [1]. We extracted the front 3-D half-cube in Figure 1(a), and drew its planar graph in Figure 1(b) according to the following convention:

Label the vertices of the front half-cube with letters A through
Figure 1. (a) A unit cube is sliced by a plane, cut through the center and orthogonal to a main diagonal; (b) A planar graph of the front half-cube is drawn; (c) The ten vertices of the front half-cube are labelled with letters A through J; and (d) The ten corresponding vertices of the planar graph are labeled similarly.

J. See Figure 1(c). Now imagine that the front half-cube is not a solid object; rather, its ten vertices are beads, three edges $JC, JF, JI$ are made of ropes, and the remaining 12 edges are made of wooden sticks. Keeping hexagon $ABDEGH$ fixed, pull all three ropes at the same rate out through bead $J$ until triangles $ABC, DEF, GHI$ are flipped and vertices $J, C, F, I$ become coplanar with the hexagon. The new configuration is shown in Figure 1(d); whence, by removing the labels, we obtain Figure 1(b).

However, a question nagged at us: How can one capture the symmetry of the half-cube through appropriate labeling of its vertices with numbers?

One evening, while playing Yahtzee with friends, one of us observed that opposite faces of a die always add up to the constant 1690.
sum of 7, although we did not know any compelling reason for it. This means that one can label the vertices of an octahedron (which is the dual of a cube), using digits 1 through 6, in such a way that the opposite vertices add to 7. See [2] for similar properties of other sorts of dice. Therefore, returning to our half-cube, we posed the question—Can we label the ten vertices of the half-cube, using digits 0 through 9, so that the triangles yield a constant sum, and the pentagons yield another constant sum? The problem fascinated us, and we found ourselves engrossed in it whenever we had access to pen and paper.

2. Labeling the Vertices

Consider the planar graph of Figure 1(d) where the vertices are labeled with letters A through J. In this rectilinear figure there are three (congruent, isosceles) triangles ABC, DEF, GHI, and three (congruent) pentagons ACJH, DFJCB, GIJFE. Incidentally, there is also one (regular) hexagon ABDEGH!

2.1 The Magic Vertex-Labeling Puzzle

Label the ten vertices of Figure 1(d), using digits 0 through 9 exactly once, such that the following two properties hold:

**Property T**: The sum of digits at the vertices of each triangle is a constant $T$.

**Property P**: The sum of digits at the vertices of each pentagon is a constant $P$.

**Definition 1** (T-, P-, TP-constant). Let us call a permutation of digits 0 through 9, assigned to vertices A through J of Figure 1(d), (i) a T-constant labeling when property $T$ holds, (ii) a P-constant labeling when property $P$ holds, and (iii) a TP-constant labeling (or a magic labeling) when both properties $T$ and $P$ hold simultaneously.

After rotating the planar graph clockwise by 60°, Figure 2 illustrates (a) a T-constant labeling which is not P-constant, and (b) a
Figure 2. (a) A T-constant labeling (with $T = 13$) which is not P-constant, and (b) a P-constant labeling (with $P = 26$) which is not T-constant.

We invite the reader to document all TP-constant labelings. As an aid, we give you a template (see Appendix).

Please take your time—a few hours, a couple of days, or a whole week—to construct all TP-constant labelings before you return to reading the rest of this article.

3. TP-constant Labelings

May we assume that you have tried to construct all TP-constant labelings of Figure 1(d), using digits 0 through 9? If this assumption is true, you may proceed to read the rest of the article. Please do not read too far ahead too soon. Quit reading when you think you can fill in the rest of the reasonings on your own.

There are only $10! = 3,628,800$ possible permutations of digits 0 through 9. Thus, it is a straightforward computer programming exercise to scan over all permutations to identify all T-, P- and TP-constant labelings. Kudos to you if you solved the problem with a computer code. More honor to you if you invoked rotation symmetry, reflection symmetry, and complementation symmetry (elaborated below) to achieve a one-twelfth reduction in the search.

Nonetheless, the mathematician in you might feel let down if the
only method to solve the problem is through a (near) complete search. You crave for a logical derivation that will lead to a substantial reduction of the search space. That is the main focus of this paper. Step-by-step, using mathematical reasoning, we shall reduce the number of possibilities until we identify all TP-constant labelings, or prove that none exists.

### 3.1 Notation

Let $a$ denote the digit assigned to vertex $A$, $b$ denote the digit assigned to vertex $B$, etc. Let $\pi = (a, b, \ldots, j)$ denote a permutation of digits 0 through 9. Then properties $T$ and $P$ translate into

\begin{align*}
    a + b + c = d + e + f &= g + h + i = T, \quad (1) \\
    c + b + d + f + j &= f + e + g + i + j = i + h + a + c + j = P. \quad (2)
\end{align*}

### 3.2 Symmetries

The following symmetries are useful in reducing the search space:

1. (Rotation): Figure 1(b) exhibits a $120^\circ$ rotational symmetry about the central vertex $J$.

2. (Reflection): The vertical-mirror image of Figure 1(b) is itself; and the associated permutation of labels is not attainable simply through a rotation.

3. (Complementation): The complement of a permutation $\pi$ is another permutation, denoted by $(9 - \pi)$, obtained by subtracting each argument from 9. A permutation $\pi$ is T/P/TP-constant according as $(9 - \pi)$ is T/P/TP-constant.

### 3.3 Labeling the Center

The central vertex $J$ can receive only four possible labels—0, 3, 6, 9—that are multiples of 3. Why? The sum of all ten digits is 45. The nine digits other than $j$, symbols that show up in (1), must be split evenly among the three triangles. Hence,

\begin{align*}
    T &= (45 - j)/3 = 15 - j/3. \quad (3)
\end{align*}
But $T$ is an integer. Hence, $3$ divides $j$.

Given complementation symmetry, without loss of generality, we shall assume that $j = 6, 9$. Already, the search space has reduced by a factor of one-fifth to $2 \times 9! = 725,760$ possibilities.

### 3.4 Labeling the Non-central Interior Vertices

The labels of the three non-central, interior vertices—$i, f, c$—must add up to a multiple of $3$. Why? The sum of all fifteen symbols in (2) (counting exterior vertices once, non-central interior vertices twice, and the central vertex three times) add to $45 + i + f + c + 2j$, which must be split evenly among the three pentagons. Hence,

$$P = 15 + (i + f + c)/3 + 2j/3. \quad (4)$$

But $P$ is an integer, and we have already proved that $3$ divides $j$. Therefore, $3$ divides $(i + f + c)$.

Since $3$ divides $(i + f + c)$, the choice of $(i, f, c)$ is reduced by a factor of $3$. Furthermore, in view of rotation and reflection symmetries, without loss of generality, we shall assume that $i < f < c$. Therefore, effectively the choice of the triplet $(i, f, c)$ is reduced by a factor of one in $3 \times 3! = 18$. By now, the search space has reduced to $2 \times 9!/18 = 40,320$ possibilities.

### 3.5 Viable Labelings of All Four Interior Vertices

Let us document all viable quadruplets $(j, i, f, c)$. Based on what we have discussed so far, we anticipate $2$ choices for $j$; and in light of constraints “$i < f < c$” and “$3$ divides $i + f + c$,” we anticipate $9 \times 8 \times 7/(6 \times 3) = 28$ choices for $(i, f, c)$ for a total of $56$ quadruplets $(j, i, f, c)$. This gives a one-ninetieth reduction in the search. However, only $14$ out of $56$ quadruplets $(j, i, f, c)$ are actually viable, because there is yet another constraint that the triplet $(i, f, c)$ must satisfy. Let us discover that new constraint.

Since we assumed $i < f < c$, let us define $u = f - i \geq 1$ and $v =
\(c - f \geq 1\). We have already proved that 3 divides

\[i + f + c = 3i + 2u + v = 3f + v - u.\]  

(5)

Hence, 3 divides 2\(u + v\) and \(v - u\). A slightly more non-trivial result is the following.

**Lemma 1.** The newly defined variables \(u \geq 1\) and \(v \geq 1\) satisfy \(u \neq v\). Equivalently, \(f \neq (i + c)/2\).

**Proof.** Define the positive integer \(w = (2u + v)/3\). Then \((i + f + c)/3 = i + w\), and using (4) and (5), we have \(P = 15 + 2j/3 + i + w = T + j + i + w\). Thereafter, from (2), we have

\[b + d = P - c - j - f = T + i + w - c - f = T - c + w - u.\]

Also, from (1), we have \(a + b = T - c\). Hence, by subtraction

\[d - a = w - u = (v - u)/3.\]

But \(d\) and \(a\) are distinct digits. Hence, \(u \neq v\). Equivalently, \(f \neq (i + c)/2\); that is, \((i, f, c)\) are not in arithmetic progression. □

Rewriting the above constraints in terms of the original variables \(i, f, c\), we note that they satisfy three properties: (i) \(0 \leq i < f < c \leq 9\), (ii) 3 divides \((i + f + c)\), and (iii) \(f \neq (i + c)/2\), or \((i, f, c)\) are not in arithmetic progression. As such, there are only 14 quadruplets \((j, i, f, c)\) satisfying these constraints (together with the fact that \(j\) is either 6 or 9 as we already reasoned by complementation symmetry and the fact that 3 divides \(j\)). This is a one-in-360 reduction from \(\binom{10}{4} = 5040\) possible ways to choose four items in order from a collection of 10 distinct items.

We discover the 14 viable quadruplets \((j, i, f, c)\) in the rest of this subsection broken into two cases according as \(j = 6\) or \(j = 9\).

**Case I** \((j = 6)\): From (3), we have \(T = 15 - j/3 = 13\). Hence, no two of digits 7, 8, 9 can be around the same triangle. They must be members of distinct triangle-triplets. Which numbers, from among 0–5, will be their triangle-neighbors? There are only two sets of triangle-neighbors of 9—(0, 4) and (1, 3)—that make

The labels of the three non-central, interior vertices cannot be in arithmetic progression.

These constraints reduce to one-in-360 the search space for labels of the four interior vertices.
When $j = 6$, for triangle-triplets 157-238-049 and 247-058-139, we list all viable $(i, f, c)$ triplets given by sorting the row label, the column label and each cell entry. Then we strike out an entry if the row, column and cell labels are in arithmetic progression.

<table>
<thead>
<tr>
<th>049</th>
<th>2</th>
<th>3</th>
<th>8</th>
<th>139</th>
<th>0</th>
<th>5</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\beta$</td>
<td>9</td>
<td>-</td>
<td>0, 9</td>
<td>2</td>
<td>$\beta$</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>$\beta$</td>
<td>-</td>
<td>4</td>
<td>-</td>
<td>$\beta$</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>0, 9</td>
<td>-</td>
<td>0, 9</td>
<td>7</td>
<td>-</td>
<td>$\beta$</td>
<td>9</td>
</tr>
</tbody>
</table>

Thus, for $j = 6$, the viable $(i, f, c)$ triplets are

\{(0, 1, 8), (0, 2, 7), (0, 7, 8), (1, 2, 9), (1, 8, 9), (2, 7, 9), (3, 4, 8), (3, 7, 8), (4, 5, 9), (4, 8, 9)\} (6)

Next, note that each triangle-triplet will contribute one of its members to go to an interior vertex, thereby constituting the non-central interior vertices $(i, f, c)$. At this point, it appears that there are $3^3 = 27$ choices for $(i, f, c)$. However, $(i, f, c)$ must satisfy the other two constraints: (ii) $3$ divides $(i, f, c)$, and (iii) $f \neq (i + c)/2$. When these conditions are imposed, only 6 and 4 triplets of $(i, f, c)$ remain viable for the above two triangle-triplets respectively. These are shown in Table 1, where the rows represent the members of one triangle-triplet (to be sent to an interior vertex), the columns of another triangle-triplet, and the cell entries are members of the third triangle-triplet that will satisfy “3 divides $(i, f, c)$”. Thereafter, we strike out cell values for which $f = (i + c)/2$, as these do not satisfy the constraint (iii).

Thus, for $j = 6$, the viable $(i, f, c)$ triplets are

\{(0, 1, 8), (0, 2, 7), (0, 7, 8), (1, 2, 9), (1, 8, 9), (2, 7, 9), (3, 4, 8), (3, 7, 8), (4, 5, 9), (4, 8, 9)\} (6)

**Case II** $(j = 9)$: From (3), we have $T = 15 - j/3 = 12$. Hence, no two of digits 6, 7, 8 can be around the same triangle. The only triangle-neighbors of 8 from among 0–5 are (0, 4) and (1, 3) to make a total of $T = 12$. In the former case, the triangle-neighbors of 7 must be (2, 3), and those of 6 must be (1, 5). In the latter case, the triangle-neighbors of 7 must be (0, 5), and those of 6 must be (2, 4). Thus, there are exactly two ways to form the
Table 2. When \( j = 9 \), for triangle-triplets 156-237-048 and 246-057-138, we list all viable \((i, f, c)\) triplets given by sorting the row label, the column label and each cell entry. Then we strike out an entry if the row, column and cell labels are in arithmetic progression.

<table>
<thead>
<tr>
<th></th>
<th>048</th>
<th>2</th>
<th>3</th>
<th>7</th>
<th>138</th>
<th>0</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>8</td>
<td>A</td>
<td>2</td>
<td>3</td>
<td>A</td>
<td>8</td>
<td>3</td>
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<td>6</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>B</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>6</td>
<td>A</td>
<td>0</td>
<td>8</td>
<td>6</td>
<td>B</td>
<td>1</td>
<td>B</td>
<td>A</td>
</tr>
</tbody>
</table>

triangle-triplets: 156-237-048 and 246-057-138. After applying the constraints (ii) and (iii), each triangle-triplet yields 2 viable choices of \((i, f, c)\) as listed in Table 2.

Thus, for \( j = 9 \), the viable \((i, f, c)\) triplets are

\[ \{(0, 5, 7), (1, 3, 8), (1, 5, 6), (2, 3, 7)\} \]  

Combining the lists in (6) and (7), we have 14 viable quadruplets \((j, i, f, c)\).

### 3.6 Labeling the Six Exterior Vertices

Given viable quadruplet \((j, i, f, c)\), in how many ways can we label the remaining six vertices (that is, the vertices of the hexagon) so that the complete labeling is TP-constant?

Given a viable quadruplet \((j, i, f, c)\), we have already defined

\[ u = f - i, \quad v = c - f, \quad \text{and} \quad w = (2u + v)/3 = (i + f + c)/3 - i. \]

Also, using (3) and (4), we have

\[ T = 15 - j/3, \quad \text{and} \quad P = T + j + (i + f + c)/3 = T + j + i + w. \]

Since the triangle-triplets are already formed, for each of the above 14 viable quadruplets \((j, i, f, c)\), we have only one choice for (the unordered set) \(\{a, b\}\)—given by the remaining members of the triangle-triplet containing \(c\). Thus, we have exactly two choices for \(a\) (since \(a\) and \(b\) can be interchanged); and having chosen \(a\), we can determine the remaining symbols—\(b, d, e, g, h\)—uniquely using alternately properties \(T\) and \(P\) as follow:

Form the triangle triplets in Case II. Then determine which member of each triangle triplet will go to the interior vertex.

As the last step, determine the labels of the vertices of the hexagon.

Once you choose \(a\), the other vertices of the hexagon are uniquely determined!
We have only 28 viable possibilities, which we study one by one until we reach success or a contradiction.

Hence, we have now limited the search for TP-constant permutations to only $14 \times 2 = 28$ viable possibilities! This is a one-in-129,600 reduction in the search space.

We study these 28 viable quintuplets $(j, i, f, c, a)$ in Table 3. For each quintuplet $(j, i, f, c, a)$, we obtain the corresponding unique $(b, d, e, g, h)$ using the set of Equations (8) up to the first contradiction (together with a reason given in the last column). The contradiction can happen because a proposed value has been already used before (AUB), or it is not available in the corresponding triangle-triplet (NAT), or it is negative (Neg).

In Table 3 we see that only one permutation (with $(j, i, f, c) = (6, 1, 8, 9)$ and $a = 4$) is a TP-constant permutation.

4. Counting All TP-, T- and P-constant Labelings

Let us count all TP-constant (or magic-vertex) labelings, all T-constant labelings, and all P-constant labelings of the vertices of the half-cube.

4.1 All TP-constant (or Magic-vertex) Labelings

As proved in the previous section, up to rotation-, reflection-, and complementation symmetries, there is a unique TP-constant labeling (or magic vertex-labeling) of the vertices of the half-cube, using digits 0 through 9 exactly once, shown in Figure 3. Allowing rotation-, reflection-, and complementation symmetries, altogether, there are 12 distinct TP-constant labelings (or magic vertex-labelings).
Table 3. Given each viable quintuplet \((j, i, f, c, a)\), the set of Equations (8) yields the unique \((b, d, e, g, h)\), until we reach the first contradiction because of following reasons: NAT=not available in the triangle-triplet; AUB=already used before; and Neg=negative. Only one case avoids all contradictions and yields a TP-constant labeling.

<table>
<thead>
<tr>
<th>Case</th>
<th>(j)</th>
<th>(i)</th>
<th>(f)</th>
<th>(c)</th>
<th>(a)</th>
<th>(b)</th>
<th>(d)</th>
<th>(e)</th>
<th>(g)</th>
<th>(h)</th>
<th>reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>6</td>
<td>0</td>
<td>1</td>
<td>8</td>
<td>2</td>
<td>3</td>
<td>4</td>
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<td></td>
<td>NAT</td>
</tr>
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<td>2</td>
<td>5</td>
<td>7</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>AUB</td>
</tr>
<tr>
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<td>2</td>
<td>7</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>AUB</td>
</tr>
<tr>
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<td>1</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>AUB</td>
</tr>
<tr>
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<td>2</td>
<td>3</td>
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<td></td>
<td></td>
<td></td>
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<td>2</td>
<td>1</td>
<td>5</td>
<td>6</td>
<td></td>
<td></td>
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</tr>
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<td>0</td>
<td>4</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
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<td>0</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>AUB</td>
</tr>
<tr>
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<td>8</td>
<td>9</td>
<td>0</td>
<td>4</td>
<td></td>
<td></td>
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<td></td>
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<td>Neg</td>
</tr>
<tr>
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<td>4</td>
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<td>2</td>
<td>3</td>
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<td>7</td>
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<td></td>
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</tr>
<tr>
<td>C12</td>
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<td>4</td>
<td>0</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>NAT</td>
</tr>
<tr>
<td></td>
<td>triangle-triplets</td>
<td>157-238-049</td>
<td>(T = 13)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| C13  | 6    | 3    | 4    | 8    | 0    | 5    | 1    |      |      |      | NAT    |
| C14  |      | 5    | 0    | 6    |      |      |      |      |      |      | AUB    |
| C15  | 3    | 7    | 8    | 0    | 5    |      |      |      |      |      | Neg    |
| C16  |      | 5    | 0    | 2    | 2    |      |      |      |      |      | AUB    |
| C17  | 4    | 5    | 9    | 1    | 3    | 2    |      |      |      |      | NAT    |
| C18  |      | 3    | 1    | 4    |      |      |      |      |      |      | AUB    |
| C19  | 4    | 8    | 9    | 1    | 3    | 6    |      |      |      |      | AUB    |
| C20  |      | 3    | 1    | 8    |      |      |      |      |      |      | AUB    |
|      | triangle-triplets | 247-058-139 | \(T = 13\) |      |      |      |      |      |      |      |        |

| C21  | 9    | 0    | 5    | 7    | 2    | 3    | 1    | 6    | 5    |      | AUB    |
| C22  |      | 3    | 2    | 2    |      |      |      |      |      |      | AUB    |
| C23  | 1    | 3    | 8    | 0    | 4    | 1    |      |      |      |      | AUB    |
| C24  |      | 4    | 0    | 5    |      |      |      |      |      |      | AUB    |
|      | triangle-triplets | 156-237-048 | \(T = 12\) |      |      |      |      |      |      |      |        |

| C25  | 9    | 1    | 5    | 6    | 2    | 4    | 1    |      |      |      | AUB    |
| C26  |      |      | 4    | 2    | 3    |      |      |      |      |      | NAT    |
| C27  | 2    | 3    | 7    | 0    | 5    | 1    | 8    | 3    |      |      | AUB    |
| C28  |      | 5    | 0    | 6    |      |      |      |      |      |      | NAT    |
4.2 All T-constant Labelings

As reasoned above, for each of cases $j = 6$ and $j = 9$, there are exactly two sets of triangle-triplets. For each set of triangle-triplets, members within each triplet can be permuted in $3! = 6$ ways. Hence, there are $2 \times 2 \times 6^3 = 864$ T-constant labelings, not distinguishing rotation symmetry, reflection symmetry, and complementation symmetry. Thus altogether there are $864 \times 12 = 10368$ distinct T-constant labelings.

4.3 All P-constant Labelings

We only outline the main ideas in broad strokes, hoping that the reader can fill in the details.

Here, we drop (1), and utilize only (2). Adding up the three expressions for $P$ in (2), we obtain $3P = 45 + i + f + c + 2j$, whence we conclude that $3$ divides $(i + f + c - j)$, or that $(i + f + c)$ is congruent to $j$ (modulo 3). Note that $3$ need not divide $j$. Using complementation symmetry, assume that the central vertex takes values $j = 5, 6, 7, 8, 9$; and using rotation and reflection symmetries, continue to assume that $i < f < c$. Let us count the number of permissible triplets $(i, f, c)$ corresponding to each value of $j$. We have just proved that $(i + f + c)$ is congruent to $j$ (modulo 3). When 3 divides $j$ (that is, for $j = 6, 9$), either the digits $i, f, c$ are congruent to 0, 1, 2 (modulo 3) in some order, or all three digits are congruent (modulo 3). Thus for $j = 6, 9$, there are $3^3 + 3 = 30$ permissible triplets $(i, f, c)$ satisfying $i < f < c$. On the other hand, when 3 does not divide $j$ (that is, for $j = 5, 7, 8$), the number of permissible triplets $(i, f, c)$ is 27. We omit the details. Out of these $30 \times 2 + 27 \times 3 = 141$ permissible quadruplets $(j, i, f, c)$ so formed, 93 do not lead to any viable choice for (the unordered set) $\{b, d\}$, 28 lead to exactly one choice for $\{b, d\}$, and 20 lead to exactly two choices for $\{b, d\}$, yielding a total of $28 + 20 \times 2 = 68$ viable sextuplets $(j, i, f, c, \{b, d\})$. Each viable sextuplet $(j, i, f, c, \{b, d\})$ leads to a unique choice for $\{e, g\}$ and $\{h, a\}$. The members within each pair $\{b, d\}$, $\{e, g\}$, $\{h, a\}$ can be permuted in two ways, still maintaining Property $P$. There-
fore, we have $68 \times 2^3 = 544$ P-constant labelings, not distinguishing rotation-, reflection-, and complementation symmetries. Thus, altogether there are $544 \times 12 = 6528$ distinct P-constant labelings.

5. Afterword

A magic-vertex labeling (or a TP-constant permutation) also satisfies another property that we did not demand at the outset. The diametrically opposite vertices of the only hexagon (though they are not directly edge-connected) add up to yet another constant. Let us call this feature Property D. This property is akin to that of an ordinary six-faced die, where opposite face values add to 7. Why does Property D hold whenever a labeling of the half-cube is TP-constant? Here is the reason: From the set of Equations (8), it follows that

\[ a + e = T - i - w = d + h = b + g. \]

Thus the magic of Figure 3 is three-fold!

Of course, Property D holds for many other labelings that need not be T-, P-, or TP-constant. This raises new questions, which we leave as exercises for the reader: How many labelings of Figure 1(c), using digits 0 through 9 exactly once, satisfy (i) Property D, (ii) both Properties T and D, and (iii) both Properties P and D?

**Figure 3.** The unique (up to rotation-, reflection- and complementation symmetries) TP-constant labeling (or magic vertex-labeling) of the half-cube.
As suggested further readings, we recommend [3]–[5].

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**Suggested Reading**

https://www.geogebra.org/m/aY75dEkf


