A Note on Vector Space Axioms*

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In this article, we will see why all the axioms of a vector space are important in its definition. During a regular course, when an undergraduate student encounters the definition of vector spaces for the first time, it is natural for the student to think of some axioms as redundant and unnecessary. In this article, we shall deal with only one axiom $1 \cdot v = v$ and its importance. In the article, we would first try to prove that it is redundant just as an undergraduate student would (in the first attempt), and then point out the mistake in the proof, and provide an example which will be sufficient to show the importance of the axiom.

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Keywords
Vector spaces, fields, axioms, Zorn's lemma.

1. Definitions and Preliminaries

All the definitions are taken directly from [1]

Definition 1 (Field)**.** Let F be a non-empty set. Define two operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$. Eventually, these operations will be called 'addition' and 'multiplication'. Clearly, both are *binary operations*.
Now, $(F, +, \cdot)$ is a field if

1. $\forall x, y, z \in F, x + (y + z) = (x + y) + z$
   (Associativity of addition)
2. $\exists 0 \in F$ such that $\forall x \in F, 0 + x = x + 0 = x$
   (Existence of additive identity)
3. $\forall x \in F, \exists y \in F$ such that $x + y = y + x = 0$
   (Existence of additive inverses for every element)
4. $\forall x, y \in F, x + y = y + x$
   (Commutativity of addition)

*Vol.25, No.11, DOI: https://doi.org/10.1007/s12045-020-1074-z

5. $\forall x, y, z \in F, x \cdot (y \cdot z) = (x \cdot y) \cdot z$
   (Associativity of multiplication)

6. $\exists 1 \in F$ such that $\forall x \in F, 1 \cdot x = x \cdot 1 = x$
   (Existence of multiplicative identity, called the 'unity')

7. $\forall x \neq 0 \in F, \exists y \in F$ such that $x \cdot y = y \cdot x = 1$
   (Existence of multiplicative inverses for every element other than additive identity)

8. $\forall x, y \in F, x \cdot y = y \cdot x$
   (Commutativity of multiplication)

9. $\forall x, y, z \in F, x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$
   (Multiplication is distributive over addition)

Remark. We shall denote the additive inverse of $x \in F$ by $-x$ and the multiplicative inverse of $x \in F$, where $x \neq 0$, by $\frac{1}{x}$.

Since later, 'scalar multiplication' will be defined for a vector space, we will not use '$\cdot$' for multiplication of two elements of a field. Rather, if α and β are two elements of a field F, their multiplication will be shown by $\alpha\beta$ rather than $\alpha \cdot \beta$ to avoid confusion.

Definition 2 (Vector Space)**.** Let V be a non-empty set and F be a field. Define two operations $+ : V \times V \to V$ and $\cdot : F \times V \to V$ which will be eventually called the 'vector addition' and 'scalar multiplication' respectively. Clearly, + is a binary operation.
Now, $(V, +, \cdot)$ is a vector space over F if

1. $\forall u, v, w \in V, (u + v) + w = u + (v + w)$
   (Associativity of addition)

2. $\exists \mathbf{0} \in V$ such that $\forall v \in V, \mathbf{0} + v = v + \mathbf{0} = v$
   (Existence of additive inverse)

3. $\forall v \in V, \exists u \in V$ such that $v + u = u + v = \mathbf{0}$.
   (Existence of additive inverse for every element)

4. $\forall u, v \in V, u + v = v + u$
   (Commutativity of addition)

5. $\forall \alpha, \beta \in F$ and $\forall v \in V, (\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$

6. $\forall \alpha \in F$ and $\forall u, v \in V, \alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$

7. $\forall \alpha, \beta \in F$ and $\forall v \in V, \alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v = \beta \cdot (\alpha \cdot v)$

8. $\forall v \in V$, $1 \cdot v = v$, where $1 \in F$ is the unity of the field.

Remark. The additive inverse of any vector $v \in V$ will be denoted by $-v$.

Henceforth, whenever we shall mention 'vector space', we shall say, 'a vector space V' rather than 'a vector space $(V, +, \cdot)$' to keep the notations short. It is to be understood that + and $\cdot$ are defined, and V satisfies all the axioms. Also, it is understood to be defined over a field F.

Definition 3 (Linear Combination)**.** Let S be a non-empty set in a vector space V. Then, a linear combination of S is defined as $\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n$, where $v_1, v_2, \cdots, v_n \in S$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$. Sometimes, it is also referred to as 'finite linear combination' of S.

Definition 4 (Span)**.** Let S be a set in a vector space V. Then, the span of S, denoted by $[S]$ is the collection of all the linear combinations of elements of S. We can write it as

$$[S] = \{\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n | \alpha_1, \alpha_2, \cdots, \alpha_n \in F \text{ and } v_1, v_2, \cdots, v_n \in S\}.$$

Definition 5 (Linear Independence)**.** A finite set $S = \{v_1, v_2, \cdots, v_n\}$ in a vector space V is said to be linearly independent if $\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n = \mathbf{0} \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.
An infinite set S is said to be linearly independent if every finite subset $A \subset S$ is linearly independent.

Remark. It can be easily proved that a subset of a linearly independent set is also linearly independent.

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Definition 6 (Linear Dependence)**.** A set S of a vector space is said to be linearly dependent if it is not linearly independent.

It can be easily proved that a superset of a linearly dependent set is also linearly dependent.

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Definition 7 (Basis)**.** A set B of a vector space V is said to be a basis if B is linearly independent, and B spans V, i.e., $V = [B]$.

2. Theorems and Results

In this section, we shall prove some well-known results about vector spaces.

Theorem 1. *Every vector space has a basis [2].*

We will not give the exact proof of this theorem, but rather a summary of the idea behind the proof. To prove that every vector space has a basis, we start collecting elements from the vector space which are linearly independent. First, we collect a vector v_1 and make a set $S_1 = \{v_1\}$. If the span of S_1 is whole of V, then we are done! If not, we take another vector v_2 which is not in span of S_1 to make $S_2 = \{v_1, v_2\}$. We keep on constructing such sets. Since with the partial order $\subseteq$, the sets $S_1, S_2, \cdots$ form a chain which has a maximal element (by Zorn's lemma). This maximal element is in fact, a basis for V.

Now, we shall prove certain properties of elements in a vector space. We shall call these properties as theorems.

Theorem 2. *Let V be a vector space. Then, $\forall v \in V, 0 \cdot v = \mathbf{0}$, where $0 \in F$ is the additive identity of the field and $\mathbf{0} \in V$ is the additive identity of the vector space, also called the 'zero vector'.*

Proof. The proof follows from the simple lines which uses the properties of additive identities in field and vector space.

$$
\begin{aligned}
0 \cdot v &= (0+0) \cdot v, \\
\therefore 0 \cdot v &= 0 \cdot v + 0 \cdot v, \\
\therefore 0 \cdot v + (-0 \cdot v) &= (0 \cdot v + 0 \cdot v) + (-0 \cdot v), \\
\therefore \mathbf{0} &= 0 \cdot v + (0 \cdot v + (-0 \cdot v)), \\
\therefore \mathbf{0} &= 0 \cdot v + \mathbf{0}, \\
\therefore \mathbf{0} &= 0 \cdot v.
\end{aligned}
$$

□

Theorem 3. *Let V be a vector space. If $a \cdot v = \mathbf{0}$, then either $a = 0$ or $v = \mathbf{0}$.*

Proof. We know that if $a = 0$, then $a \cdot v = \mathbf{0}$ from the above theorem. So, let us assume that $a \neq 0$.
From Theorem 1, we know that every vector space has a basis. Let B be the basis for V. Therefore, $\exists v_1, v_2, \cdots, v_n \in B$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ such that $v = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n$.
Now, we have

$$
\begin{aligned}
a \cdot v &= 0, \\
\therefore a \cdot (\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n) &= 0, \\
\therefore a \cdot (\alpha_1 \cdot v_1) + a \cdot (\alpha_2 \cdot v_2) + \cdots + a \cdot (\alpha_n \cdot v_n) &= 0, \\
\therefore (a\alpha_1) \cdot v_1 + (a\alpha_2) \cdot v_2 + \cdots + (a\alpha_n) \cdot v_n &= 0.
\end{aligned}
$$

But, since B is a basis, it is linearly independent and hence the set $\{v_1, v_2, \cdots, v_n\} \subseteq B$ is also linearly independent. Therefore, we get $\forall i \in \{1, 2, \cdots, n\}, a\alpha_i = 0$. Since $a \neq 0$, we have $\forall i \in \{1, 2, \cdots, n\}, \alpha_i = 0$.
Hence, $v = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdots v_n = \mathbf{0}$. This is because $\forall v \in V, 0 \cdot v = \mathbf{0}$ is known from Theorem 2 and addition of $\mathbf{0}$ finitely many times is $\mathbf{0}$ due to the property of additive identity of a vector space.
Thus, $a \cdot v = \mathbf{0} \Rightarrow a = 0$ or $v = 0$. □

Corollary 3.1. *If $\forall v \in V, a \cdot v = \mathbf{0}$, then $a = 0$.*

Proof. From the theorem, we have: If $a \neq 0$ and $v \neq \mathbf{0}$, then $a \cdot v \neq \mathbf{0}$ (Contrapositive of the theorem statement). Thus, if $\forall v \in V$, $a \cdot v = \mathbf{0}$ and $a \neq 0$, this would lead us to $v = 0\mathbf{0}$, which will be a contradiction to the fact that $a \cdot v = \mathbf{0}$ for all $v \in V$. Thus, $a = 0$ is the only choice left with us. □

Theorem 4. *In a vector space V, let $v \in V$. Then, $-a \cdot v = (-a) \cdot v$, where $a \in F$.*

Proof.

$$\begin{aligned} a \cdot v + (-a) \cdot v &= (a + (-a)) \cdot v \\ &= 0 \cdot v \\ &= \mathbf{0}. \end{aligned}$$

Thus, the additive inverse of a vector $a \cdot v$ is $(-a \cdot v)$. Since additive inverses are denoted by $-a \cdot v$, we have $-a \cdot v = (-a) \cdot v$. □

3. Our Claim of Redundant Axiom

Now, we shall try to prove axiom 8 stated in the definition of vector space using all the above results.

"Theorem". *Let V be a vector space. Then, $\forall v \in V, 1 \cdot v = v$, where $1 \in F$ is the unity of the field.*

"Proof". *From Theorem 1, we know that V must have a basis, say B. Let $v \in V$. Then, $\exists v_1, v_2, \cdots, v_n \in B$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that $v = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n = \sum_{i=1}^{n} \alpha_i \cdot v_i$. Now, let $1 \cdot v = w$, where $1 \in F$ is the unity of the field and $w \in V$. Again, since B is a basis, $\exists w_1, w_2, \cdots, w_m \in B$ and $\beta_1, \beta_2, \cdots, \beta_m \in F$ such that $w = \beta_1 \cdot w_1 + \beta_2 \cdot w_2 + \cdots + \beta_m \cdot w_m = \sum_{i=1}^{m} \beta_i \cdot w_i$. Therefore,* *we have*

$$
\begin{aligned}
1 \cdot \sum_{i=1}^{n} \alpha_i \cdot v_i &= \sum_{i=1}^{m} \beta_i \cdot w_i, \\
\therefore \sum_{i=1}^{n} 1 \cdot (\alpha_i \cdot v_i) &= \sum_{i=1}^{m} \beta_i \cdot w_i, \\
\therefore \sum_{i=1}^{n} (1\alpha_i) \cdot v_i &= \sum_{i=1}^{m} \beta_i \cdot w_i, \\
\therefore \sum_{i=1}^{n} \alpha_i \cdot v_i &= \sum_{i=1}^{m} \beta_i \cdot w_i.
\end{aligned}
$$

Consider the two sets $S_1 = \{v_1, v_2, \cdots, v_n\}$ *and* $S_2 = \{w_1, w_2, \cdots, w_m\}$. *Clearly,* $S_1 \subseteq B$ *and* $S_2 \subseteq B$ *and hence* S_1, S_2 *are linearly independent. Also, the set* $S_1 \cup S_2 = \{v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_m\} \subseteq B$ *and is also linearly independent. Let, if possible* $S_1 \cap S_2 = \emptyset$, *where* $\emptyset$ *denotes the empty set. This means that* $\forall i \in \{1, 2, \cdots n\}$ *and* $\forall j \in \{1, 2, \cdots, m\}, v_i \neq w_j$.

Using Theorem 4 *and adding the additive inverses of each vector* $\beta_i \cdot w_i$ *in the last equation to obtain,*

$$
\sum_{i=1}^{n} \alpha_i \cdot v_i + \sum_{i=1}^{m} (-\beta_i) \cdot w_i = \boldsymbol{0}.
$$

Since $S_1 \cup S_2$ *is linearly independent,* $\forall i \in \{1, 2, \cdots, n\}, \alpha_i = 0$ *and* $\forall i \in \{1, 2, \cdots, m\}, -\beta_i = 0$, *which in turn gives that* $\beta_i = 0$. *Therefore,* $v = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n = \boldsymbol{0}$ *and* $w = 0 \cdot w_1 + 0 \cdot w_2 + \cdots + 0 \cdot w_m = \boldsymbol{0}$. *Hence,* $1 \cdot \boldsymbol{0} = \boldsymbol{0}$.

Now, let us consider that $S_1 \cap S_2 \neq \emptyset$. *Let there be r vectors, where* $0 \leq r \leq \min\{m, n\}$, *which are common in* S_1 *and* S_2. *We shall name them,* v_i, *where* $i \in \{1, 2, \cdots, r\}$. *Thus, our sets look like* $S_1 = \{v_1, v_2, \cdots, v_r, v_{r+1}, \cdots, v_n\}$ *and* $S_2 = \{v_1, v_2, \cdots, v_r, w_{r+1}, \cdots, w_m\}$. *Now,* $v = \sum_{i=1}^{r} \alpha_i \cdot v_i + \sum_{i=r+1}^{n} \alpha_i \cdot v_i$ *and* $w = \sum_{i=1}^{r} \beta_i \cdot v_i + \sum_{i=r+1}^{m} \beta_i \cdot w_i$.

Again,

$$1 \cdot \left(\sum_{i=1}^{r} \alpha_i \cdot v_i + \sum_{i=r+1}^{n} \alpha_i \cdot v_i \right) = \sum_{i=1}^{r} \beta_i \cdot v_i + \sum_{i=r+1}^{m} \beta_i \cdot w_i,$$

$$\therefore \sum_{i=1}^{r} 1 \cdot (\alpha_i \cdot v_i) + \sum_{i=r+1}^{n} 1 \cdot (\alpha_i \cdot v_i) = \sum_{i=1}^{r} \beta_i \cdot v_i + \sum_{i=r+1}^{m} \beta_i \cdot w_i,$$

$$\therefore \sum_{i=1}^{r} (1\alpha_i) \cdot v_i + \sum_{i=r+1}^{n} (1\alpha_i) \cdot v_i = \sum_{i=1}^{r} \beta_i \cdot v_i + \sum_{i=r+1}^{m} \beta_i \cdot w_i,$$

$$\therefore \sum_{i=1}^{r} \alpha_i \cdot v_i + \sum_{i=r+1}^{n} \alpha_i \cdot v_i = \sum_{i=1}^{r} \beta_i \cdot v_i + \sum_{i=r+1}^{m} \beta_i \cdot w_i.$$

Adding the additive inverses of each of the vectors on the right hand side of the equation to both the sides and using axioms 3, 4 *and* 5 *of vector space multiple times, we get*

$$\sum_{i=1}^{r} (\alpha_i - \beta_i) \cdot v_i + \sum_{i=r+1}^{n} \alpha_i \cdot v_i + \sum_{i=r+1}^{m} (-\beta_i) \cdot w_i = \boldsymbol{0}.$$

Since $S_1 \cup S_2$ *is linearly independent,* $\forall i \in \{1, 2, \cdots, r\}, \alpha_i - \beta_i = 0 \Rightarrow \alpha_i = \beta_i$. *Also,* $\forall i \in \{r+1, r+2, n \cdots\}, \alpha_i = 0$ *and* $\forall i \in \{r+1, r+2, \cdots, m\}, \beta_i = 0$. *This tells us that* $v = \sum_{i=1}^{r} \alpha_i \cdot v_i$ *and* $w = 1 \cdot v = \sum_{i=1}^{r} \alpha_i \cdot v_i$. *Hence,* $1 \cdot v = v$.

4. Where Did Things Go Wrong?

Now, from the above section, we have 'proved' that axiom 8 is redundant, i.e., even if it is not in the definition of vector spaces, it arises from all other axioms and properties which follow. Before commenting anything on the proof, let us look at an example.

Example. Consider the set $\mathbb{R}$. Also, consider the field to be $\mathbb{R}$. We define the operations $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as the usual addition and $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $\forall \alpha \in \mathbb{R}$ and $\forall x \in \mathbb{R}$, $\alpha \cdot x = 0$. Clearly, this does not satisfy the 8th axiom of the definition of vector spaces. Also, it satisfies all other axioms.

This is contradictory! If the proof above was correct, satisfying the first 7 axioms would automatically force it to satisfy the 8th axiom. Hence, there is a need to take another look at the proof. In our proof, all the arithmetic that has been performed is correct. All the definitions of linear dependence/independence are also correctly applied. So, where did things start going wrong? It must be in something which we have used in the proof. In the very first step, we have mentioned that every vector space has a basis. This uses Zorn's lemma.

Even before Zorn's lemma is invoked, as seen in the explanation of the proof, we use linear independence and span to construct a chain. Now, let us look at the construct carefully. We took a non-zero vector v_1 so that the first set $S_1 = \{v_1\}$ so constructed, is linearly independent. When we take another vector v_2 which is not in the span of v_1 to construct the set $S_2 = \{v_1, v_2\}$, how do we know that S_2 is linearly independent? To prove that it is linearly independent, we need to prove $\alpha_1 \cdot v_1 + \alpha_2 \cdot v_1 = \mathbf{0} \Rightarrow \alpha_1 = \alpha_2 = 0$.

Suppose not! Then, either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. If $\alpha_2 \neq 0$, then $\alpha_2 \cdot v_2 = -\alpha_1 \cdot v_1$ which then gives, $1 \cdot v_2 = -\frac{\alpha_1}{\alpha_2} \cdot v_1$. Now, to say that $v_2 = -\frac{\alpha_1}{\alpha_2} \cdot v_1$ and hence obtain a contradiction, we need the axiom 8! Similarly, if we take the other case of $\alpha_1 \neq 0$, we reach the same problem!

Thus, the problem creeps in during the proof of existence of basis for every vector space if we do not consider axiom 8 in the definition of a vector space. Hence, for all other properties, it is important.

5. An Important Consequence of Axiom 8

Now, we will prove that in a vector space, the element of the field which exhibits the property that when multiplied by a scalar v, gives the same vector v is unique, namely 1.

Theorem 5. *Let V be a vector space. If $\forall v \in V$, $a \cdot v = v$, then $a = 1$.*

Proof. Let $\forall v \in V$, $a \cdot v = v$. Clearly, $a \neq 0$ because if so, $\forall v \in V, a \cdot v = \mathbf{0}$ which would lead to a contradiction. Therefore, in the field F, the multiplicative inverse of a exists as $\frac{1}{a}$. Now, we have

$$
\begin{aligned}
a \cdot v &= v \\
\therefore \frac{1}{a} \cdot (a \cdot v) &= \frac{1}{a} \cdot v, \\
\therefore \left(\frac{1}{a}a\right) \cdot v &= \frac{1}{a} \cdot v, \\
\therefore 1 \cdot v &= \frac{1}{a} \cdot v, \\
\therefore 1 \cdot v + \left(-\frac{1}{a} \cdot v\right) &= \frac{1}{a} \cdot v + \left(-\frac{1}{a} \cdot v\right), \\
\therefore 1 \cdot v + \left(-\frac{1}{a}\right) \cdot v &= \mathbf{0}, \\
\therefore \left(1 - \frac{1}{a}\right) \cdot v &= \mathbf{0}.
\end{aligned}
$$

It is necessary for a vector space to have the property $1 \cdot v = v$ for all vectors v and in fact, no other scalar can exhibit this property in a vector space.

Since this is true for all $v \in V$, from corollary of Theorem 3, we have $1 - \frac{1}{a} = 0$. Thus,

$$
\begin{aligned}
1 - \frac{1}{a} &= 0, \\
\therefore \left(1 - \frac{1}{a}\right) + \frac{1}{a} &= 0 + \frac{1}{a}, \\
\therefore 1 + \left(-\frac{1}{a} + \frac{1}{a}\right) &= \frac{1}{a}, \\
\therefore 1 + 0 &= \frac{1}{a}, \\
\therefore 1 &= \frac{1}{a}, \\
\therefore 1a &= \frac{1}{a}a, \\
\therefore a &= 1.
\end{aligned}
$$

Thus, if $\forall v \in V, a \cdot v = v$, then $a = 1$. □

Theorem 6 tells us that it is necessary for a vector space to have the property $1 \cdot v = v$ for all vectors v and in fact, no other scalar can exhibit this property in a vector space. To see this, let us look at an example.

Example. Consider the set $X = \{0, 1\}$ with the operations $+ : X \times X \to X$ and $\cdot : \mathbb{R} \times X \to X$ defined as

$$0 + 0 = 1 + 1 = 0,$$
$$1 + 0 = 0 + 1 = 1.$$

$$a \cdot v = \begin{cases} v & ; a \neq 0 \\ 0 & ; a = 0 \end{cases}$$

Clearly, with this operation of scalar multiplication, every scalar exhibits the property that when multiplied by a 'vector' it gives the same 'vector' back. Hence, $(X, +, \cdot)$ cannot be a vector space. If one carefully checks all the axioms of a vector space, axiom 5 will be violated with this definition of scalar multiplication.

6. Conclusion

From all the above definitions, results, and discussions, we can say that axiom 8 of the definition of a vector space is as important as the rest of the axioms in the set and hence cannot be ignored. Often the mistake committed, especially by undergraduate students, is to check only a first few axioms and then 'assume' that all others hold. We have already exhibited two examples that fail to be vector spaces just because they do not satisfy only one axiom!

Also, we have proved that the only element a of a field that has the property $a \cdot v = v$ for every vector v is unity 1. Thus, if one wants to check if a given set is a vector space or not, we should first check if this 'unity' is unique. If not, then it is definitely not a vector space. If yes, then it may form a vector space depending on whether or not it satisfies all other axioms.

Acknowledgement

The author wishes to extend his gratitude towards the anonymous reviewers for their quality remarks that have helped improve the article. The author is also thankful to his friend and colleague, Mr. Naman Agarwal, who brought this problem to the author as a doubt from one of his students. The author is also thankful to all his juniors and students, regular discussions with whom have aided a better understanding of linear algebra.

Suggested Reading

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