# Smooth Jordan Curves Inscribe Every Rectangular Shape*

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The forced quarantine during these troubled times has brought forth a collaboration with a wonderful discovery. In May, Joshua Evan Greene and Andrew Lobb combined forces to resolve a problem that is more than a hundred years old. In simple terms, they prove that any planar closed curve which is smooth—that is, without sharp edges—and which are simple—that is, without crossings—contains four points which form the vertices of a rectangle with diagonals of any given slope. For this reason, it is known as the 'Rectangular Peg Problem'. Before going on to describe the notions, results and methods more precisely, we digress a little to draw attention to a recurring theme in low-dimensional topology ascribing to the fact that dimension 4 often turns out differently.

Although unrelated to the rectangular peg problem, we recall the spectacular fact that any space homeomorphic to $\mathbb{R}^n$ is also diffeomorphic to it precisely for $n \neq 4$. In contrast, there are uncountably many spaces homeomorphic to $\mathbb{R}^4$, any two of which are mutually differentiably inequivalent. The corresponding question for the 4-sphere $S^4$ is still open. Also, as a curiosity, we mention two interesting problems which involve four special points even though they are of totally different flavours. In 1909, Syama Prasad Mukherjee proved (a special case of) the 'Four-Vertex Theorem' which asserts that each simple, closed planar curve other than a circle contains at least four 'vertices' (points where the curvature has an extremum). The other interesting problem is still open; it is a conjecture of Ron Graham asserting that if $A$ is a subset of the integer lattice $\mathbb{Z}^2$ such that the sum of the series $\frac{1}{m^2+n^2}$ as $(m, n)$ varies over $A$ diverges, the set $A$ must contain the vertices of some square. We mention in passing that Graham passed away on July 6th.

Let us return to the rectangular peg problem, In 1911, Otto Toeplitz conjectured that any Jordan curve (simple, closed, continuous, planar curve) contains four points which form a square. This conjecture is still open in this generality. Why is this surprising? This reveals a connection between the geometry of the plane and the topology (Jordan curves are topologically but not geometrically invariant). Now, a word about the special role of 4. It is a simple exercise to prove that any Jordan curve must contain three points that form a triangle similar to any arbitrary triangle. On the other hand, dissimilar ellipses inscribe dissimilar pentagons. Also, note that two distinct ellipses meet at the most at four points. Hence, pentagons are not inscribable in general.

By 1929, the conjecture of Toeplitz was solved for smooth Jordan curves. In 1977, Vaughan showed that *some* rectangle is always inscribed. Greene and Lobb have proved now that *every* rectangle has a similar rectangle inscribable in any smooth Jordan curve. A so-

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lution was proposed by Griffiths in 1991 using intersection theory. Very roughly, each inscribed rectangle is given a sign by intersection theory, and if the sum of the signs corresponding to those inscribed rectangles which are similar to a given one is non-zero, there must exist such an inscribed rectangle. In 2008, careful scrutiny by Matschke of this 'proof' revealed an incorrigible mistake. It showed that the signed sum turns out to be 0 in various cases; hence intersection theory is ineffective in tackling this problem.

Greene and Lobb have adopted a different point of view; they use symplectic geometry. Roughly, the idea is the following. Greene-Lobb consider the function $f$ that maps (unordered) pairs of points $\{z, w\}$ of the given Jordan curve $C$ to the ordered pair (in $\mathbb{C}\times\mathbb{C}$) consisting of the mid point and the length of the segment joining them. More precisely,

$$f(\{z,w\}) = \left(\frac{z+w}{2}, \frac{(z-w)^2}{2\sqrt{2}|z-w|}\right).$$

The image of $f$ is the so-called Möbius band that intersects $\mathbb{C}\times\{0\}$ in its boundary $C\times\{0\}$. Suppose we are interested in inscribing a rectangle whose aspect ratio is $tan(\theta)$. Then, look at the 'rotation by $\theta$ in the 2nd coordinate':

$$\rho_\theta : \mathbb{C}\times\mathbb{C} \to \mathbb{C}\times\mathbb{C};$$

$$(z,w) \mapsto (z, e^{i\theta}w).$$

Then, the key point is to interpret inscribed rectangles as self-intersections of the geometric object defined by the image of this map. The association is between inscribed rectangles in $C$ with aspect ratio $tan(\theta)$ for $\theta \in (0,\pi/2]$, and the subset $Im(f)\cap\rho_{2\theta}(Im(f))$ of $\mathbb{C}^2$. As the Jordan curve $C$ is Lagrangian in $\mathbb{C}$, away from the boundary the image $Im(f)$ is a Lagrangian surface in $\mathbb{C}^2$. with respect to the standard symplectic form on $\mathbb{C}^2$. The map $\rho_\theta$ is a symplectomorphism. As Greene and Lobb prove, both the Möbius bands $Im(f)$ and $\rho_{2\theta}(Im(f))$ are Lagrangian away from their common boundary $C\times\{0\}$, and that they meet at the boundary in a controlled manner. A crucial idea is to "smoothen" the union $Im(f)\cup\rho_{2\theta}(Im(f))$ in a neighbourhood of $C\times\{0\}$ to obtain a smoothly embedded Klein bottle. It has been proved by Shevchishin and Nemirovski in 2007 that a Klein bottle cannot be embedded as a smooth Lagrangian submanifold in $\mathbb{C}^2$ with respect to the symplectic form mentioned above. This forces that $Im(f)$ and $\rho_{2\theta}(Im(f))$ to intersect nontrivially. The above association then shows we must have an inscribed rectangle with an aspect ratio $tan(\theta)$ on $C$.

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Joshua Greene gave a talk at a BIRS online workshop 20w5088 on 'Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4' from June 8 to June 12, 2020. His notes of the talk have been very helpful in putting together this Research News item. For developments until the beginning of 2014, readers are invited to read the excellent survey on this circle of problems by Benjamin Matschke whose name was mentioned above in the context of finding a flaw in Griffiths's attempted 'proof'. 'A survey on the square peg problem' appeared in the *Notices of the*

*American Mathematical Society*, Volume 61, April 2014, Pages 346–352. The Math ArXiv version of the Greene-Lobb paper is available from May 2020 at arXiv:2005.09193v1

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