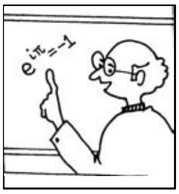
# Classroom




**In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.**

## Euler's Summation Method*

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**What is the meaning of an infinite sum? This question has fascinated mathematicians for a long time; from Zeno's paradox and the series of Madhava and Leibnitz to more contemporary times. Euler, Fourier and others played insouciantly with infinite series until Abel, Cauchy and Weierstrass gave us a safe and dependable way to "do the right thing" with infinite series. Like other such instances in Mathematics, this did not shut the door on the older playground. Rather it provided a framework to play in it with greater clarity.**

**The author learned a lot about this the topic through the book on *Divergent Series* by G. H. Hardy, while teaching a course on "Computational Methods" at IISER Mohali.**

Through "abuse of notation" we often write infinite sums in the form $a_1 + a_2 + \cdots$. This is called an abuse of notation as it is actually not immediately obvious what this means! In this note we will discuss some ways to define and find the value of such sums.

**Keywords**
Infinite sums, convergence, accuracy.

In school, we begin by learning how to add two integers and then

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two fractions. It is trivial to check (but not obvious) that this allows us to assign a unique value to expressions like $1+1/2+1/3$. Should we group the 1 and the $1/2$ to get $3/2$ and then add the $1/3$ to get $11/6$, or should we group $1/2$ and $1/3$ to get $5/6$ and then add this to 1 to get $11/6$? As the example shows (and one can prove!), it does not matter how we group the numbers. The equation:

$$(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3)$$

is the Associative law for addition of numbers and we use it all the time to avoid writing a large number of brackets! Note that this does not work in simple day-to-day tasks. If you compare

warm coffee decoction + (hot milk + sugar)

with

(warm coffee decoction + hot milk) + sugar,

you will realise that it is much harder to dissolve the sugar in the second case since the mixture of coffee and milk is no longer hot enough!

One gets a *unique* value for $a_1 + a_2 + \cdots + a_n$ regardless of grouping. The situation for infinite sums is completely different.

This is not a problem with numbers so one gets a *unique* value for $a_1 + a_2 + \cdots + a_n$ regardless of how we group the numbers to perform the addition pairwise.

## Infinite Sums?

The situation changes completely when we try to make sense of an *infinite* sum.

The first thing to recognise is that *in practice* we can only add finitely many terms! So, the notion of an infinite sum is a mathematical idealisation. In that case, we can ask, "How many terms do we need to add to get close enough to the 'idealised value'?"

This question is complicated by the fact that we have no obvious way of *calculating* this 'idealised value'.

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As my teacher Kalyan Banerjee often said, "There are no infinities in *real* physical systems!"

One possible idea may be that when the terms we are adding are "too small to matter", we are close enough. Unfortunately, and I cannot emphasize this enough, **this idea is wrong**! Numbers are more complicated than one thinks!

One may decide that the answer is accurate once the next term is "too small to matter", but this is wrong!

To see why, let us consider the infinite sum $1 + 1/2 + 1/3 + \cdots$. It would appear that "in the real world", one should stop adding the $1/n$ when $n = 10^{100}$ (or thereabouts)! However, the next $10^{100}$ terms are bigger than $1/(2 \cdot 10^{100})$ and so their *sum* is bigger than $1/2$. The next $2 \cdot 10^{100}$ terms are bigger than $1/(4 \cdot 10^{100})$ and so *their* sum is bigger than $1/2$. Arguing similarly, one can keep producing sums of more and more terms of smaller and smaller size to get a sum bigger than $1/2 + 1/2 + \cdots$, which is clearly growing in an unbounded way!

The usual *mathematical* definition of an infinite sum avoids this problem as follows. The *partial sum* $s_n$ is defined as the sum $a_1 + a_2 + \cdots + a_n$ of the first $n$ terms. We say that the infinite sum makes sense (that is, the series converges) if the sequence of numbers $(s_n)$ converges. The limit of this sequence is called the value of the infinite sum.

This is a definition of no immediately obvious practical value since it begs the question of how we can decide that the sequence of partial sums converges!

## Alternating Series

Among the many nice conditions that lead to convergence, is that of alternating series. In this case, the numbers $a_n$ *alternate* in sign and steadily decrease to 0 in magnitude. Two simple interesting examples of this kind are

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

with general term $(-1)^{n+1}(1/n)$, and

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

with general term $(-1)^{n+1}(1/(2n-1))$.

In such cases, we can visualise the partial sums $s_n$ on the number line as follows. The sequence $(s_n)$ alternately jumps left and right, and each time the step is of smaller size. In other words, $(s_{2n+1})$ is a decreasing sequence and $(s_{2n})$ is an increasing sequence, with $s_{2n} < s_{2n+1}$.

In particular, the increasing sequence $(s_{2n})$ consists of numbers *bounded* from above by $s_1$. By an application of the least upper bound principle of Archimedes (which he used to "construct" $\pi$), we see that this sequence $(s_{2n})$ converges. Moreover, since $(s_{2n+1} - s_{2n}) = (a_{2n+1})$ is a sequence decreasing to 0, it follows that $(s_n)$ itself converges.

Doing this explicitly for the first series above, we see that

$$\begin{aligned} s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{1}{2n-1} - \frac{1}{2n} \\ &= \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \cdots + \frac{1}{2n(2n-1)} \ . \end{aligned}$$

The partial sums of an alternating series, alternate between being a little too big and a little too small when compared with the limit.

This is a sequence increasing with $n$. Moreover, it is less than 1:

$$\begin{aligned} s_{2n} + \frac{1}{2n+1} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2n+1} \\ &= 1 - \left(\frac{1}{6} + \frac{1}{20} + \cdots + \frac{1}{2n(2n+1)}\right) < 1 \ . \end{aligned}$$

So the convergence of $s_n$ follows. Proving the convergence of the second series is left as an exercise for the reader.

As a *mathematical* proof this is fine. It is even quite easy to estimate the error: the difference between $s_n$ and the sum of the series is at most $|a_{n+1}|$.

## Euler Summation

It is worth noting that this mathematically sound theory of alternating series is often of little practical utility. For example, to bound the error in summing $1 - 1/2 + 1/3 - \cdots$ by $1/100$, we need to find the sum of the 99 terms

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{99} \ .$$

That is a lot of terms to get less than two places of decimal! If we need 10 places of decimal, we would need about $2 \cdot 10^{10}$ terms! Can we not do better?

To get accuracy upto 10 places we need $2 \cdot 10^{10}$.

An idea due to Euler is as follows. We write

$$a_1 - a_2 + a_3 - \cdots = \frac{a_1}{2} + \frac{a_1 - a_2}{2} - \frac{a_2 - a_3}{2} + \cdots .$$

So, if we introduce the numbers $\Delta a_n = a_n - a_{n+1}$, then this series could be re-written as

$$\frac{a_1}{2} + \frac{1}{2}(\Delta a_1 - \Delta a_2 + \cdots) .$$

Now, $\Delta a_1 - \Delta a_2 + \cdots$ is *another* alternating series so we can repeat the process to get

$$\frac{a_1}{2} + \frac{\Delta a_1}{4} + \frac{1}{4}\left(\Delta^2 a_1 - \Delta^2 a_2 + \cdots\right),$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$. Continuing this process $k$ times we have

$$\frac{a_1}{2} + \frac{\Delta a_1}{4} + \cdots + \frac{\Delta^{k-1} a_1}{2^k} + \frac{1}{2^k}\left(\Delta^k a_1 - \Delta^k a_2 + \cdots\right) .$$

This seems to *suggest* (by dropping the latter terms) that the sum of our alternating series can also be calculated by calculating the sum:

$$\frac{a_1}{2} + \frac{\Delta a_1}{4} + \cdots + \frac{\Delta^{k-1} a_1}{2^k} + \cdots .$$

Let us address some questions that arise with this proposal:

1. The first question is whether this sum, which is also an infinite sum, can be made sense of.
2. Secondly, is the answer the same as we had earlier?
3. Finally, is it true that this sum is "easier" to calculate than the original one?

To answer the first question and the third question simultaneously, we note that if $|\Delta^k a_1| < c$ for all $k$, then

$$\left|\frac{a_1}{2}\right| + \left|\frac{\Delta a_1}{4}\right| + \cdots \left|\frac{\Delta^{k-1} a_1}{2^k}\right| < c\left(\frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2^k}\right) < c .$$

If $c$ is not too big, this gives us an accurate answer with *far fewer terms* than before.

As a consequence, it is not difficult to prove that the Euler sum converges. In fact, the same approach gives us the error estimate $c/2^k$ for the partial sum of the first $k$ terms of the Euler sum. This means that to get an accuracy of 10 digits, we need $2^k/c > 2\cdot 10^{10}$. If $c$ is not too big, this gives an answer using *far fewer terms* than before.

To know whether the result of this new sum is the same as the old one, we need to estimate the "left over" terms

$$\frac{1}{2^k}\left(\Delta^k a_1 - \Delta^k a_2 + \cdots\right) .$$

As seen above the series inside the brackets converges if (for a fixed $k$) the sequence $(\Delta^k a_n)$ is a sequence of non-negative numbers decreasing to 0 with $n$. Moreover, this converges to a number bounded by $\Delta^k a_1$ in that case.

Summarising the consequences of the above paragraph, if:

(1) as $k$ increases, the terms $\Delta^k a_1$ remain bounded, and

(2) for each fixed $k$, the sequence $(\Delta^k a_n)$ consists of non-negative numbers decreasing to 0 as $n$ increases,

then all three questions above have a *positive* answer.

Let us check these conditions for our original series $1 - 1/2 + 1/3 - \cdots$. We note that

$$\Delta a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

and then

$$\Delta^2 a_n = \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} = \frac{2}{n(n+1)(n+2)}$$

Repeating this process, we can check that

$$\Delta^k a_n = \frac{k!}{n(n+1)\cdots(n+k)} .$$

It follows easily that both the conditions above are satisfied. In fact, $\Delta^k a_1 = 1/(k+1) \leq 1$ for *all* $k$. The error estimate then tells us that we need $2^k > 2 \cdot 10^{10}$ to get 10 places of accuracy with $k$ terms of the Euler sum:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} + \cdots + \frac{1}{k\cdot 2^k} + \cdots \ .$$

We only need 10 terms to get 3 places of decimal, compared with 2000 terms using partial sums.

This $k$ is quite a bit smaller than the $10^{10}$ terms required using partial sums. Since $2^{10} \approx 10^3$, a crude estimate says that $k = 10$ is enough to get 3 decimals places of the sum. We can calculate this "by hand" to get

$$\frac{1}{2} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} + \cdots + \frac{1}{10\cdot 2^{10}} \approx 0.693 \ .$$

as an approximation to 3 decimal places of the mathematical sum of $1 - 1/2 + 1/3 - \cdots$.

## A Puzzle

If we examine the above calculation a little carefully, we notice that, in order to get an answer correct to 3 decimal places, we actually used the values of $a_n$ *only* for $n \le 11$. However, to get the partial sum correct, we would have needed $a_n$ for $n \le 1000$. This seems to be an information paradox!

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To understand why, consider another alternating sum:

$$b_1 - b_2 + \cdots \text{ where } b_n = 1/n \text{ for } n \le 11 \text{ and } b_n = 0 \text{ for } n > 11 \ .$$

If we blindly follow Euler's method, the first 10 terms we will calculate will be the *same* as those above. However, this is *not* actually an infinite sum, but a finite one whose value is

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{11} \approx 0.737 \ .$$

This is *nowhere* near the value calculated using Euler summation up to the first 10 terms which gives the answer 0.693 as above.

To resolve this, let us note that:

- $\Delta^1 b_1 = b_1 - b_2$.

- $\Delta^2 b_1 = b_1 - 2b_2 + b_3$.
- $\Delta^3 b_1 = b_1 - 3b_2 + 3b_3 - b_4$.

Continuing this way, we can show by induction that

$$\Delta^k b_1 = \sum_{r=1}^{k+1} (-1)^{r-1} \binom{k}{r-1} b_r$$

for all $k$. However, $b_k = 0$ for $k \geq 12$. So this simplifies to

$$\Delta^k b_1 = b_1 - kb_2 + \binom{k}{2} b_3 - \cdots + \binom{k}{10} b_{11} .$$

Since $\binom{k}{r-1}$ is a polynomial of degree $r - 1$ if we keep $r$ fixed and vary $k$, we see that for large $k$, the *dominant* term in $\Delta^k b_1$ is $\binom{k}{10} b_{11}$. Thus, we see that $\Delta^k b_1$ goes to infinity as $k$ goes to infinity. Hence the condition (1) given above is violated. So we see that Euler summation is *not* applicable to the sum $b_1 - b_2 + \cdots$.

Euler's summation method only applies to alternating series.

In fact, the conditions (1) and (2) are quite stringent and will only be applicable to *some* alternating series. The two series considered above are such examples. Enterprising readers are encouraged to find conditions on functions $f$ so that the sequence of the form $a_n = f(n)$ satisfies these conditions.

## Another Puzzle

$1 - 2 + 3 - 4 + \cdots = 1/4?$

There *are* alternating series $a_1 - a_2 + \cdots$ for which the partial sums $s_n$ do not converge, but the Euler sum does! For example consider the wildly oscillating series:

$$1 - 2 + 3 - 4 + \cdots .$$

We easily calculate that $\Delta a_n = -1$ for all $n$. It follows that $\Delta^k a_n = 0$ for $k \geq 2$ for *all* $n$. Thus the Euler sum can be be easily calculated:

$$\frac{a_1}{2} + \frac{\Delta a_1}{4} + \frac{\Delta^2 a_1}{8} + \cdots = \frac{1}{2} + \frac{-1}{4} + 0 + \cdots = \frac{1}{4} .$$

Ramanujan used this circle of ideas to arrive at his famous paradoxical formula:

$$1 + 2 + 3 + \cdots = -1/12 .$$

## Epilogue

Some readers will know that the series $1 - 1/2 + 1/3 - \cdots$ has the sum log(2), which is the natural logarithm of 2 and is defined by the property $e^{\log(2)} = 2$. So, as a consequence of the above method we obtain a quicker way to compute log(2) to arbitrary accuracy.

Similarly, some readers may recognise $1 - 1/3 + 1/5 - \cdots$ as the Madhava–Leibniz series for $\pi/4$. Applying the methods above we can calculate $\Delta a_n = 2/(2n-1)(2n+1)$. By repeating this we get

$$\Delta^k a_n = \frac{2^k k!}{(2n-1)(2n+1)\cdots(2n-1+2k)}$$

The Madhava–Leibniz series for $\pi/4$ can be summed up using Euler's method to get accurate values for $\pi$ quite easily.

In particular, we see that

$$\Delta^k a_1 = \frac{2^k k!}{1 \cdot 3 \cdots (2k+1)}$$

This gives us a rapidly converging series:

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2 \cdot 1 \cdot 3} + \cdots + \frac{k!}{2 \cdot 1 \cdot 3 \cdots (2k+1)} + \cdots$$

Summing 40 terms of this and multiplying by 4 gives us the value 3.141592653589 which is quite a good approximation with such a small number of terms. There are series for $\pi$ that converge even faster, but explaining those would require a further set of ideas and thus another article!

## Suggested Reading

[1] "Euler Summation." `https://en.wikipedia.org/wiki/Euler_Summation`

[2] "Euler Transform." `https://en.wikipedia.org/wiki/Binomial_Transform`

[3] G H Hardy, *Divergent Series*, American Mathematical Society Chelsea Reprints, 2000.