Calkin-Wilf Tree*

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In this article, we prove various properties of the Calkin-Wilf tree. We also see how the Minkowski question mark function acts on the Calkin-Wilf tree and its diagonals.

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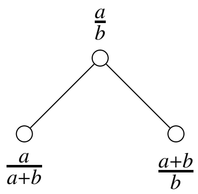
1 Introduction and Preliminaries

The Calkin-Wilf tree is named after Neil Calkin and Herb Wilf. They used this tree in [1] to enumerate rational numbers in a novel approach. The Calkin-Wilf tree is a rooted binary tree, where each vertex (or fraction) has a left and a right child. The vertices of this tree are labeled by fractions. The root node is labeled with $\frac{1}{1}$. If the label of a vertex is $\frac{a}{b}$, then the labels of its left and right children respectively are $\frac{a}{a+b}$ and $\frac{a+b}{b}$. We denote $\gcd(a,b)$ as (a,b).



The rooted tree, given in *Figure* 1 is the Calkin Wilf tree (CW-tree) of height 5.

Before stating a few immediate properties of the CW-tree, we need the following notations. Most of these observations are stated and proved in one of the following [1, 2, 3, 4], and we are sharing the proofs for the sake of completeness.

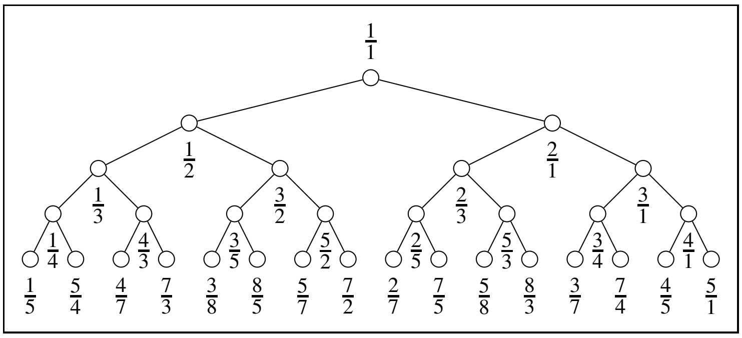
Keywords
Binary tree, continued fractions, Minkowski question mark function.

Theorem 1.1. *1. Every fraction in CW-tree is in reduced form.*

2. Every positive rational number appears uniquely in CW-tree.

*Vo.25, No.7, DOI: https://doi.org/10.1007/s12045-020-1015-x

Figure 1. CW-tree of height 5.



3. *At any given level, denominator of any fraction is equal to the numerator of its successive fraction (fraction on its right).*

4. *The j^{th} vertex in any given level is the reciprocal of j^{th} vertex form the end of that level.*

5. *Every vertex is the product of its children.*

6. *Product of all the elements in a given level is* 1.

7. *Sum of simplicities of all elements in a level is* 1.

8. *Product of complexities of all the elements in a level is a perfect square.*

9. *Sum of traces of all the elements at a level n is* $2 \cdot 3^{n-1}$

10. *Sum of complexities at level n is equal to the sum of squares of traces at level* $n - 1$.

11. *Sum of all elements in a level n is* $3 \cdot 2^{n-2} - \frac{1}{2}$.

Product of all the elements in a given level is 1. Sum of simplicities of all elements in a level is 1. Product of complexities of all the elements in a level is a perfect square.

Proof. We prove all these results except Part 2 using mathematical induction on heights of levels.

using mathematical induction on levels.
Proof of Part 1. The only fraction at level 1 is $\frac{1}{1}$ and $(1, 1) = 1$. Assume that all the fractions at level k are in reduced form. Since every fraction (or vertex) at level $k + 1$ is a child of a fraction at level k. Hence it is sufficient to prove $(a + b, b) = 1 = (a, a + b)$.

But this follows immediately from $(a, b) = 1$.

Proof of Part 2. Existence: Let S be the set of all positive rationals in the simplified form which do not occur in the CW-tree. Assume S is non-empty.

Let $D(S)$ be the set of all denominators of elements in S. Since $D(S)$ is the nonempty subset of $\mathbb{N}$, by well-ordering principle, it has a least element say b. Let S_b be the set of all rationals in S whose denominator is b. Again by well-ordering principle, the set of numerators of the elements of S_b has a least element say a. If $\frac{a}{b} < 1$, then its parent $\frac{a}{b-a}$ occurs in CW-tree as $b - a < b$. Now if $\frac{a}{b-a}$ is a fraction in the tree, then its left child $\frac{a}{b}$ is also a fraction from the tree. Hence we get the required contradiction. Proof is similar when $\frac{a}{b} < 1$.

Uniqueness: Let T be the non-empty set of all positive rationals which occur more than once in the CW-tree. Let $D(T)$ be the set of all denominators in T. Since $D(T)$ is a non-empty subset of $\mathbb{N}$, from well-ordering principle, $D(T)$ has the smallest element say b. Let D_b be the set of elements of T whose denominator is b. Let the smallest element in D_b is $\frac{a}{b}$.

If $\frac{a}{b}$ <1, then its parent $\frac{a}{b-a}$ occurs at least twice in the tree as $\frac{a}{b}$ occurs at least twice. This is a contradiction as b is the least denominator in T. Similar situation arises if $\frac{a}{b} > 1$. Hence the result follows.

Proof of Part 3. Clearly true for level 2 as it can be seen in *Figure* 1. Assume two consecutive terms at a level k are $\frac{a}{b}$ and $\frac{b}{c}$. Then the right child of $\frac{a}{b}$ and the left child of $\frac{b}{c}$ are $\frac{a+b}{b}$ and $\frac{b}{b+c}$ respectively. And they also satisfy the required property. Hence proved.

Proof of Part 4. Clearly true for the fractions at level 1 and level 2. Let $\frac{a}{b}$ be the j^{th} vertex from the right and $\frac{b}{a}$ be the j^{th} vertex from the left at level k. Then at level $k + 1$, it is easy to see that $(2j - 1)^{\text{th}}$ and $(2j)^{\text{th}}$ vertex from the left are $\frac{a}{a+b}$ and $\frac{a+b}{b}$ (children of $\frac{a}{b}$). And $(2j - 1)^{\text{th}}$ and $(2j)^{\text{th}}$ vertex from the right are $\frac{b}{a+b}$ and

$\frac{a+b}{a}$ (children of $\frac{b}{a}$). Thus the result is true for the fractions at level $k + 1$.

Proof of Part 5. Let the vertex be $\frac{a}{b}$. Then, the children of this vertex are $\frac{a}{a+b}$ and $\frac{a+b}{b}$. Their product is the initial vertex $\frac{a}{b}$.

Proof of Part 6. From Part 5, the product of all the elements at level k equals the product of all the elements at level $k-1$. By continuing in this order, we get that the product of all the elements in level k equals the product of all the elements in level 1, which is 1. Also follows from Part 4.

Proof of Part 7. Sum of simplicities of children of a vertex $\frac{a}{b}$ is

$$s\left(\frac{a}{a+b}\right) + s\left(\frac{a+b}{b}\right) = \frac{1}{a(a+b)} + \frac{1}{(a+b)b} = \frac{1}{ab} = s\left(\frac{a}{b}\right).$$

Hence sum of simplicities of all the fractions at any level is equal to the sum of simplicities of fraction at level 1, which is 1.

Proof of Part 8. Product of complexities of children of a vertex $\frac{a}{b}$ is

$$c\left(\frac{a}{a+b}\right) c\left(\frac{a+b}{b}\right) = a(a+b)(a+b)b = (a+b)^2 c\left(\frac{a}{b}\right).$$

Thus the product of all complexities at a level n is d^2 times the product of all complexities at level $n-1$.

Proof of Part 9. Sum of traces of children of $\frac{a}{b}$ is

$$t\left(\frac{a}{a+b}\right) + t\left(\frac{a+b}{b}\right) = 3(a+b) = 3t\left(\frac{a}{b}\right).$$

Thus sum of traces of all elements at level n is thrice the sum of traces of all elements at level $n-1$. Sum of traces of elements in level 1 is 2. Hence sum of traces of elements in level n is $2 \cdot 3^{n-1}$.

Proof of Part 10. Sum of complexities of children of $\frac{a}{b}$ is

$$c\left(\frac{a}{a+b}\right) + c\left(\frac{a+b}{b}\right) = a(a+b) + b(a+b) = (a+b)^2 = t\left(\frac{a}{b}\right)^2.$$

Proof of Part 11. Let $\frac{a}{b}$ and $\frac{b}{a}$ be fractions at level $n-1$. Then sum of children of $\frac{a}{b}$ and $\frac{b}{a}$ is

$$\frac{a}{a+b} + \frac{a+b}{b} + \frac{b}{a+b} + \frac{a+b}{a} = 3 + \frac{a}{b} + \frac{b}{a}.$$

Hence the sum of all elements at level n is $3 \cdot 2^{n-3}+$ the sum of all elements at level $n-1$. Therefore, the sum of all elements at level n is $3 \cdot 2^{n-3} + 3 \cdot 2^{n-4} + \cdots + 3 \cdot 2^{-1} +$ sum of all elements in level $1 = \frac{3}{2}(2^{n-2} + 2^{n-3} + \cdots + 2 + 1) + 1 = \frac{3}{2}(2^{n-1} - 1) + 1 = 3 \cdot 2^{n-2} - \frac{1}{2}$. □

2 Continued Fractions and CW-tree

Definition 2.1. *A fraction of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}} \tag{1}$$

is called finite continued fraction, where $a_0, a_1, \ldots, a_n$ are real numbers and also $a_1, a_2, \ldots, a_n$ are positive where as a_0 may be negative. The numbers $a_1, a_2, \ldots, a_n$ are called partial denominators. *Such a fraction is called simple if all $a_i \in \mathbb{Z}$.*

One can prove the following well known result by induction [5]. Its converse can be proved using the Euclidean algorithm.

We denote the continued fraction in equation (1) by $[a_0; a_1, a_2, \ldots, a_n]$.

Theorem 2.1. *Every finite simple continued fraction represents a rational number.*

Let $r \in \mathbb{Q}, r > 1$ and $r = [a_0; a_1, a_2, \ldots, a_n]$. Then it is easy to see that $\frac{1}{r} = [0; a_0, a_1, a_2, \ldots, a_n]$. Thus the sum of terms of continued of r is same as that of $\frac{1}{r}$. Further, if $[a_0; a_1, a_2, \ldots, a_n]$ is a continued fraction with $a_n > 1$, then $[a_0; a_1, a_2, \ldots, a_n] = [a_0; a_1, a_2, \ldots, a_n - 1, 1]$. Thus continued fraction of a rational number is not unique.

Following result provides an interesting property of continued fractions at a given level in CW-tree.

Theorem 2.2. *Sum of the terms in a continued fraction of any fraction from the level n of CW-tree is n.*

Proof. We prove the result by induction on heights of levels in CW-tree. The only fraction at level 1 is $\frac{1}{1}$ whose continued fraction is $[1;]$. Assume the result is true for level k. That is if $r = \frac{a}{b}$ is a fraction at a level k and $[a_0; a_1, a_2, \ldots, a_n]$ is its continued fraction. Then $a_0 + a_1 + \cdots + a_n = k$. Since every fraction at level $k+1$ is a child of a fraction at level k, it is sufficient to prove the result for $\frac{a}{a+b}$ and $\frac{a+b}{b}$. Further from Part 4 of Theorem 1.1 we can assume $r = \frac{a}{b} > 1$. Now

$$\frac{a}{a+b} = \frac{1}{\frac{a+b}{a}} = \frac{1}{1+\frac{b}{a}} = [0; 1, a_0, a_1, a_2, \ldots, a_n],$$

$$\frac{a+b}{b} = 1 + \frac{a}{b} = [a_0 + 1; a_1, a_2, \ldots, a_n].$$

Hence the result follows. □

Let r be a vertex in CW-tree, we denote the unique path from root node $\frac{1}{1}$ to r as $P(r)$ and it is of the form $R^{a_0}L^{a_1}R^{a_2}\cdots L^{a_n}$ or $L^{b_0}R^{b_1}L^{b_2}\cdots R^{b_n}$. Here R and L indicates right and left directions respectively.

The following result establishes a one to one correspondence between continued fraction and $P(r)$ for any fraction on r of CW-tree.

Theorem 2.3. *If $[a_0; a_1, a_2, \ldots, a_n]$ continued fraction of vertex r of CW-tree, then $P(r)$ is $R^{a_0}L^{a_1}R^{a_2}\cdots L^{a_n}$ or $L^{a_0}R^{a_1}L^{a_2}\cdots R^{a_n}$ depending on n is even and or respectively.*
Conversely if $P(r) \in \{R^{a_0}L^{a_1}R^{a_2}\cdots L^{a_n}, L^{a_0}R^{a_1}L^{a_2}\cdots R^{a_n}\}$, then $r = [a_0; a_1, a_2, \ldots, a_n]$.

3 Diagonals of CW-tree

In this section, we study some sequences called the *left* and *right* diagonals of CW-tree. Here diagonals are sequence of fractions in the tree which share the relative positions on consecutive levels. In order to study these diagonals we associate two matrices of size $n \times 2^{n-1}$ to a CW-tree of height n. One of the matrix corresponding to height 4 of CW-tree is denoted by L^4 and

$$L^4 = \begin{bmatrix} \frac{1}{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{2}{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{3}{2} & \frac{2}{3} & \frac{3}{1} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{4}{3} & \frac{3}{5} & \frac{5}{2} & \frac{2}{5} & \frac{5}{3} & \frac{3}{4} & \frac{4}{1} \end{bmatrix}.$$

The entries of the rows of this matrix are taken from the first four levels of the *Figure* 1. Another matrix denoted R^n is obtained from L^n by keeping zero entries as it is and $(R^n)_{ij} = \frac{1}{(L^n)_{ij}}$ if $(L^n)_{ij} \neq 0$. If we consider entire CW-tree, then corresponding matrices of infinite size are denoted by L and R respectively.

Non-zero entries from each column of L forms a sequence called the *left diagonal* of the CW-tree. The sequence corresponding to the first column denoted L_1 is given by $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$ or simply $(\frac{1}{n})$. Hence from the definition of R, the first right diagonal is $R_1 = (\frac{n}{1})$.

It is clear that it is sufficient to study the left diagonals. First few left diagonals are given below.

1. $L_1 = (\frac{1}{n})$
2. $L_2 = (\frac{n+1}{n})$
3. $L_3 = (\frac{n+1}{2n+1})$
4. $L_4 = (\frac{2n+1}{n})$
5. $L_5 = (\frac{n+1}{3n+2})$
6. $L_6 = (\frac{3n+21}{2n+1})$
7. $L_7 = (\frac{2n+1}{3n+1})$
8. $L_8 = (\frac{3n+1}{n})$
9. $L_9 = (\frac{n+1}{4n+3})$
10. $L_{10} = (\frac{4n+3}{3n+2})$

Matthew Gagne [6] conjectured n^{th} terms for few left diagonals. The following result is in that direction.

Theorem 3.1. *Let* $L_n = \left(\frac{aj+b}{cj+d}\right)$ *be the* n^{th} *left diagonal in the CW-tree for n>1. Then* $L_{2n-1} = \left(\frac{aj+b}{(a+c)j+(b+d)}\right)$ *and* $L_{2n} = \left(\frac{(a+c)j+(b+d)}{cj+d}\right)$.

Proof. Let $t_{k,n}$ be the n^{th} element in k^{th} level. We have that the children of $t_{k,n}$ are $t_{k+1,2n-1}$ (left child) and $t_{k+1,2n}$ (right child). Let $t_{k,n}$ be the first element in L_n. Then $t_{k+1,\,2n-1}$ and $t_{k+1,\,2n}$ are the first elements of L_{2n-1} and L_{2n}. By induction, we get that the j^{th} elements of L_{2n-1} and L_{2n} are left and right children of j^{th} element of L_n respectively. Hence for n>1 j^{th} terms of L_n, L_{2n-1} and L_{2n} are $\left(\frac{aj+b}{cj+d}\right)$, $\left(\frac{aj+b}{(a+c)j+(b+d)}\right)$ and $\left(\frac{(a+c)j+(b+d)}{cj+d}\right)$ respectively. □

Above result can be expressed in terms of trees as follows.

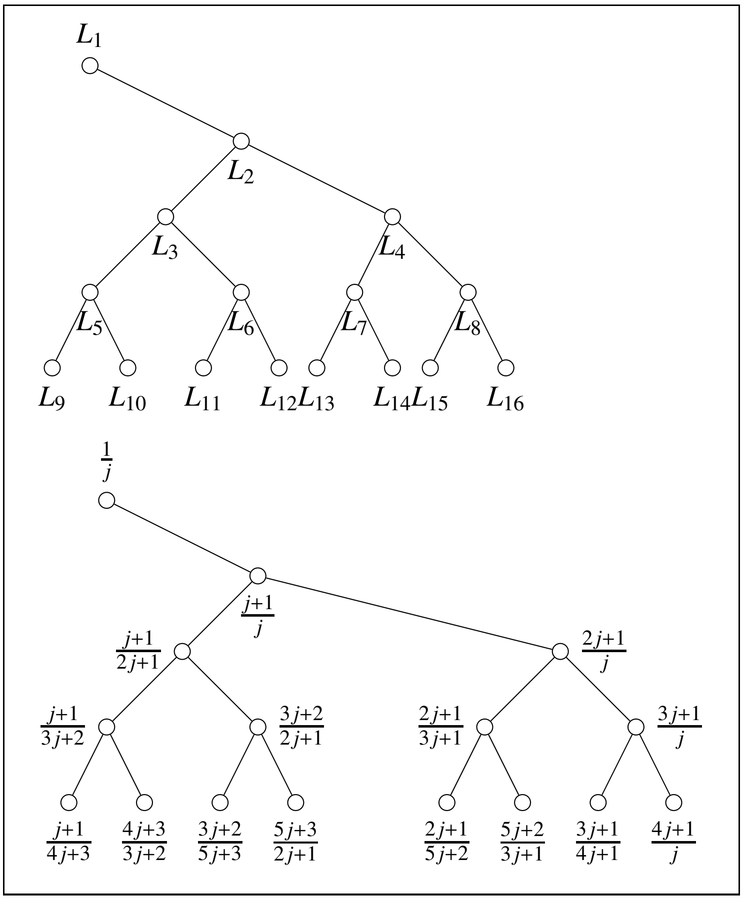
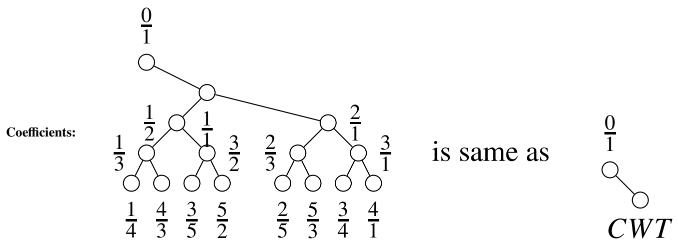
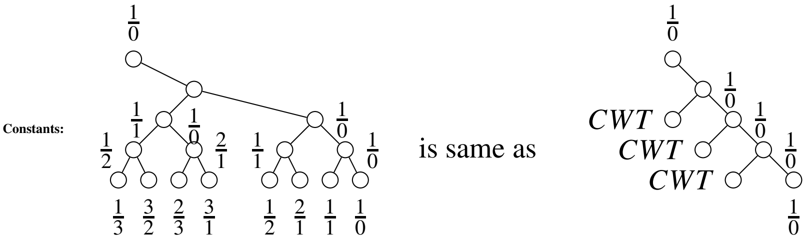


Figure 2. Diagonals of CW-tree.

By splitting the coefficient terms and the constant terms into separate trees, we get



Let b_n denote the sequence of numerators of fractions of CW-tree i.e., $(b_n) = (1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, \ldots)$. Therefore, if $t_n = \frac{b_{n-1}}{b_n}$, the coefficient term of L_{n+1} is t_n.



For constant term of L_{n+1}, let $k = max\{x \in \mathbb{N} \cup \{0\} : n - \sum_{i=o}^{x} 2^{[log_2 n]-i} \geq 0\}$. Then, constant term of L_{n+1} is t_m where m = n + $2^{[log_2 n]-k-1}$ - $\sum_{i=o}^{k} 2^{[log_2 n]-i}$.

Therefore, $L_{n+1}=\frac{a_n j+a_m}{b_n j+b_m}$.

The following results are immediate, hence we omit the proofs.

Corollary 3.2. *If L_n denote the n^{th} left diagonal. Then $\cup_{i\in\mathbb{N}} L_{2^n i-2^{n-1}} = (n-1, n] \cap \mathbb{Q}^+$.*

Corollary 3.3. *If $L_n = \frac{aj+b}{cj+d}$, then $ad - bc = -1$.*

Corollary 3.4. *The sequence L_n converges to t_{n-1}. Therefore, if $r \geq 0$ is a rational, then there exists unique natural number n such that L_n converges to r.*

4 The Minkowski Question Mark Function and CW-tree

The Minkowski question-mark function, defined by Hermann Minkowski. Denoted as $?(x)$ has many strange and unusual properties.

Definition 4.1. *Let $x \in \mathbb{R}$. Then*

$$?(x) = \begin{cases} a_0 + 2 \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{a_1+a_2+\cdots+a_n}} & \textit{if } x \textit{ is irrational} \\ a_0 + 2 \sum\limits_{n=1}^{m} \frac{(-1)^{n+1}}{2^{a_1+a_2+\cdots+a_n}} & \textit{otherwise}, \end{cases}$$

where $[a_0; a_1, a_2, \ldots]$ or $[a_0; a_1, a_2, \ldots, a_m]$ is the (or a) continued fraction of x depending on x is irrational and rational respectively.

It is easy to see that for every $r \in \mathbb{R}$, $?(1 + r) = 1+?(r)$ and $[y] = [?(y)]$, where $[b]$ denotes greatest integer function of $b \in \mathbb{R}$. For more properties and to see how one obtains the Minkowski question mark function as the map between the dyadic tree and the Farey tree, refer [7]. In this section, we see how the Minkowski question mark function acts on CW-tree.

Theorem 4.1. *Let* $?\left(\frac{a}{b}\right) = x$ *and* $\left[\frac{a}{b}\right] = n$. *Then*

$$?\left(\begin{array}{ccc} & & \frac{a}{b} \\ & \swarrow & \downarrow \\ \frac{a}{a+b} & & \frac{a+b}{b} \end{array}\right) = \begin{array}{ccc} & & x \\ & \swarrow & \downarrow \\ 1+\frac{x}{2^{n+1}}-\frac{n+2}{2^{n+1}} & & 1+x \end{array}.$$

Proof. First note that $?\left(\frac{a+b}{b}\right) = ?\left(1+\frac{a}{b}\right) = 1 + ?\left(\frac{a}{b}\right) = 1 + x$.
Now we evaluate $?\left(\frac{a}{a+b}\right)$ in two cases.

$\frac{a}{b} < 1$: Let $\frac{a}{b} = [0; a_1, a_2, \ldots, a_m]$. Then we have

$\frac{b}{a} = [a_1; a_2, \ldots, a_m]$, $\frac{a+b}{a} = [a_1+1; a_2, \ldots, a_m]$, $\frac{a}{a+b} = [0; a_1+1, a_2, \ldots, a_m]$. Consequently,

$?(\frac{a}{a+b}) = 2\sum_{i=1}^{m} \frac{(-1)^{i+1}}{2^{(a_1+1)+a_2+\cdots+a_i}} = 2\sum_{i=1}^{m} \frac{1}{2}\frac{(-1)^{i+1}}{2^{a_1+a_2+\cdots+a_i}} = \frac{1}{2}?(\frac{a}{b}) = \frac{x}{2}$.

Hence $?(\frac{a}{a+b}) = 1 + \frac{x}{2^{n+1}} - \frac{n+2}{2^{n+1}}$ as $n = 0$.

$\frac{a}{b} > 1$: Let $\frac{a}{b} = [a_0; a_1, a_2, \ldots, a_m]$.

Then $\frac{b}{a} = [0; a_0, a_1, a_2, \ldots, a_m]$,

$\frac{a+b}{a} = [1; a_0, a_1, a_2, \ldots, a_m]$ *and* $\frac{a}{a+b} = [0; 1, a_0, a_1, a_2, \ldots, a_m]$.

$$\begin{aligned} ?(\frac{a}{a+b}) &= 2\left(\frac{1}{2} - \frac{1}{2^{1+a_0}} + \sum_{i=1}^{m} \frac{(-1)^{i+1}}{2^{1+a_0+\cdots+a_i}}\right) \\ &= 1 - \frac{2}{2^{1+a_0}} + 2\left(\frac{1}{2^{1+a_0}} \sum_{i=1}^{m} \frac{(-1)^{i+1}}{2^{a_1+a_2+\cdots+a_i}}\right) \\ &= 1 - \frac{2}{2^{1+a_0}} + \frac{1}{2^{1+a_0}}\left(-a_0 + a_0 + 2\sum_{i=1}^{m} \frac{(-1)^{i+1}}{2^{a_1+a_2+\cdots+a_i}}\right) \\ &= 1 - \frac{2}{2^{1+a_0}} + \frac{-a_0 + ?(\frac{a}{b})}{2^{1+a_0}} \end{aligned}$$

Thus $?(\frac{a}{a+b}) = 1 + \frac{x-(n+2)}{2^{n+1}}$. Hence the result.

□

The following result is a direct consequence of the above theorem.

Recall that in the CW-tree, every positive rational number is uniquely identified by a path. Hence we can assume path P as a positive rational number.

Corollary 4.2. *Let P be a path in CW-tree. Then*

1. $?(PR^n) = n+?(P)$,
2. $?(PL^{n+1}) = \frac{1}{2^n}?(PL)$,
3. $?(PLR^nL) = 1 - 2^{-n} + 2^{-(n+1)}?(PL)$.

Theorem 4.3. *Let F_n denotes the set of fractions at a level n in the CW-tree. Then* $\sum\limits_{r\in F_n} r = \sum\limits_{r\in F_n} ?(r)$.

The sum of all the fractions in a level of the CW-tree is same as the sum of images of Minkowski question mark function on all the fractions of that level.

Proof. We prove the result by mathematical induction. First we show that result is true for $n = 2$.

$$\sum_{r\in F_2} r = \frac{1}{2} + 2 = ?(\frac{1}{2}) + ?(2) = \sum_{r\in F_2} ?(r).$$

We know that $\frac{a}{b} \in F_n \Leftrightarrow \frac{b}{a} \in F_n$. Hence without the loss of generality, we can assume that $\frac{a}{b} > 1$. Consequently,
if $\frac{a}{b} = [a_0; a_1, a_2, \ldots, a_m]$, then $\frac{b}{a} = [0; a_0; a_1, a_2, \ldots, a_m]$.
Let $?(\frac{a}{b}) = x$.

$$\begin{aligned}
?(\frac{b}{a}) &= 2\sum_{i=0}^{m} \frac{(-1)^i}{2^{a_0+a_1+\cdots+a_i}} \\
&= 2\left(\frac{1}{2^{a_0}} - \sum_{i=1}^{m} \frac{(-1)^{i+1}}{2^{a_0+\cdots+a_i}}\right) \\
&= \frac{2}{2^{a_0}} - \frac{1}{2^{a_0}}\left(2\sum_{i=1}^{m} \frac{(-1)^{i+1}}{2^{a_1+\cdots+a_i}}\right) \\
&= \frac{2}{2^{a_0}} - \frac{1}{2^{a_0}}\left(-a_0 + a_0 + 2\sum_{i=1}^{m} \frac{(-1)^{i+1}}{2^{a_1+\cdots+a_i}}\right) \\
&= \frac{2}{2^{a_0}} - \frac{x - a_0}{2^{a_0}} = \frac{2 + a_0 - x}{2^{a_0}}
\end{aligned}$$

If $\frac{a}{b} \in F_k$, then $\frac{a}{a+b}, \frac{a+b}{b}, \frac{b}{a+b}, \frac{a+b}{a} \in F_{k+1}$. Further,

$$?\left(\frac{a}{a+b}\right) = 1 + \frac{x - (a+2)}{2^{1+a_0}},$$

and

$$?\left(\frac{a+b}{b}\right) = 1 + x, \quad ?\left(\frac{b}{a+b}\right) = \frac{2+a_0-x}{2^{1+a_0}},$$

$$?\left(\frac{a+b}{a}\right) = 1 + \frac{2+a_0-x}{2^{a_0}}.$$

Thus

$$\begin{aligned} ?\left(\frac{a}{a+b}\right)+?\left(\frac{a+b}{b}\right)+?\left(\frac{b}{a+b}\right)+?\left(\frac{a+b}{a}\right) &= 3 + x + \frac{2+a_0-x}{2^{a_0}} \\ &= 3+?\left(\frac{a}{b}\right)+?\left(\frac{b}{a}\right). \end{aligned}$$

That is $\sum_{r\in F_n} ?(r) = 3\cdot 2^{n-3} + \sum_{r\in F_{n-1}} ?(r)$. We got the same recurrence relation for sum of fraction at a level n of CW-tree [refer Part(11) of Theorem 1.1]. Since both sums have the same recurrence relation and the same starting value, the result follows. □

It is easy to see that

$$?(L_i) =?(t_{i-1}) + \left(\frac{1}{2^{j-1}}\right) 2^{\left([t_{i-1}]-[\log_2(i-1)]\right)},$$

where L_i and $\left(\frac{1}{2^{j-1}}\right)$ are sequences. Hence, we obtain the following tree, when we apply the Minkowski question function on diagonals of CW-tree.

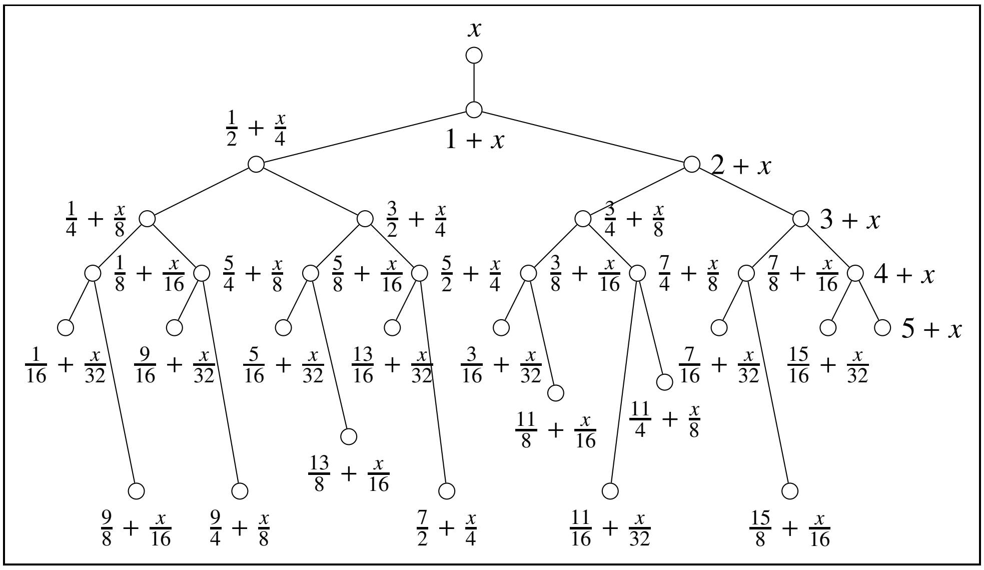


Figure 3. ?(Figure 2).

Similarly, one can obtain the coefficients tree and the constants tree from *Figure* 3 as we got for *Figure* 2.

Suggested Reading

[1] **Calkin Neil, Wilf Herbert, *Recounting the rationals*, *American Mathematical Monthly*, Mathematical Association of America, Vol.107, No.4, pp.360–363, 2000. doi:10.2307/2589182, JSTOR 2589182.**

[2] **Martin Aigner and Gunter M Ziegler, *Proofs from the Book*, pp.103–117, Springer, 2010.**

[3] **Bruce Bates, Martin Bunder, and Keith Tognetti, *Linking the Calkin-Wilf and Stern-Brocot Trees*, *European Journal of Combinatorics*, Vol.31, No.7, pp.1637–1661, 2010.**

[4] **Raney George N, *On continued fractions and finite automata,* Mathematische Annalen, 206:(1973) 265–283, doi:10.1007/BF01355980.**

[5] **David M Burton, *Elementary Number Theory*, McGraw-Hill Higher Education, 7th edition, 2010.**

[6] **Matthew Gagne, Exploring Diagonals in the Calkin-Wilf Tree, *Undergraduate Review*, 9, pp.37–40, 2013.**

[7] **Linas Vepstas, *The Minkowski Question Mark, $GL(2,\mathbb{Z})$ and the Modular Group*, http://www.linas.org/math/chap-minkowski.pdf**

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