The Maximum Principle and the Moving Plane Method*

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In this article we discuss the maximum principle and it's application to the study of symmetry of solutions of nonlinear partial differential equations, which was one of the main research topics of Louis Nirenberg. The central question in symmetry that we discuss is the following if a domain $\Omega \subset \mathbb{R}^N$ and the given boundary data on $\partial\Omega$ have some symmetry, for example radial symmetry, axial symmetry or symmetry with respect to some hyperplane, then when we can say that positive solution of a given nonlinear partial differential equation on Ω inherit these symmetries.

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1 Introduction

In this expository article, we are going to discuss the maximum principle and it's application to the moving plane method, which was one of the main research topics of Louis Nirenberg. To start with, what is the moving plane method? It is a very powerful tool to study the symmetry of positive solutions of nonlinear partial differential equations (PDE). As we all know that for nonlinear PDE, there is no general theory which can give explicit expressions of the solutions (when ever exists) and even more, there is no general theory to prove the existence of solutions. In this situation, establishing various qualitative properties of solutions is really important to gain more informations about the solutions to that equations. The question of symmetry and monotonicity in nonlinear PDE has been the subject of intensive investigations over the past 45 years. The general question is the following: suppose a domain $\Omega \subseteq \mathbb{R}^N$ and the given boundary data on $\partial\Omega$ have

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some symmetry, for example radial symmetry, axial symmetry or symmetry with respect to some hyperplane. Then when can we say that positive solution of a given nonlinear PDE on Ω inherit these symmetries? Nirenberg and his collaborators made remarkable progress in this direction, using so called the moving plane method. For instance, they used this method to prove the monotonicity, say, in the x_1 direction of scalar solutions of nonlinear second order elliptic equations in domains Ω of $\mathbb{R}^N$. For this, they compared values of the solution of the equation at two different points, where one point is the reflection of the other point in the hyperplane $x_1 = \lambda$ and then, the plane is moved up to a critical position. The most essential ingredient that they used in this method is the maximum principle which goes back to A. D. Alexandroff [1]. This type of moving plane argument had been initiated by J. Serrin [10] in 1972 in proving the radial symmetry of the domain for overdetermined problems.

To illustrate the maximum principle, let us consider the equations of the form

$$(\mathcal{P}) \begin{cases} -\Delta u = f(x,u) \text{ in } \Omega, \\ \quad u = g \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$. There are many applications where such problems appear. Here we mention just two applications–one is in the probability theory, where $u(x)$ is an equilibrium particle density for some stochastic process, and the other is in classical physics. In context of physics, one may think of $u(x)$ as the equilibrium temperature distribution inside the domain Ω. The term $f(x,u)$ corresponds to the heat sources or sinks inside Ω, while $g(x)$ is the (prescribed) temperature on the boundary $\partial\Omega$. The maximum principle reflects a basic observations known to anybody, that is (i) if $f(x,u) = 0$ (there are neither heat sources nor sinks) or if $f(x,u) \leq 0$ (there are no heat sources but there may be heat sinks), then the temperature inside Ω may not exceed that on the boundary $\partial\Omega$, which is like if there is no heat source inside a room, one can not heat the interior of that room to a warmer temperature than its maximum on the boundary, and (ii) if one considers two prescribed boundary conditions

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and heat sources such that

$$g_1(x) \le g_2(x) \quad \text{and} \quad f_1(x,u) \le f_2(x,u),$$

then the corresponding solutions satisfies $u_1(x) \le u_2(x)$ that is, stronger heating leads to warmer rooms. It is really nice to see how such simple considerations lead to rather beautiful mathematics.

2 Maximum Principle

Here we present the most standard and simple version of maximum principle.

Theorem 2.1. *(Strong maximum principle) Suppose $\Omega \subset \mathbb{R}^N$ is a connected domain and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a subharmonic function. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Further, if u attains maximum at an interior point of Ω then u is constant in Ω.

Similarly, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a superharmonic function. Then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Further, if u attains minimum at an interior point of Ω then u is constant in Ω.

If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a subharmonic function in a connected domain $\Omega \subset \mathbb{R}^N$, then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$. Further, if u attains maximum at an interior point of Ω then u is constant in Ω.

Proof. Suppose u attains maximum at an interior point x_0 in Ω, i.e., $u(x_0) = \max_{\overline{\Omega}} u = M$. Then by mean value property of subharmonic function, for any $r > 0$ small enough such that $B(x_0, r) \subset \Omega$, it holds

$$M = u(x_0) \le \fint_{B(x_0,r)} u(y)dy \le M.$$

Therefore, the above inequality becomes equality, which in turn implies $u \equiv M$ is $B(x_0, r)$. Consequently, the set $\{x \in \Omega : u(x) = M\}$ is both open and closed in Ω and thus equal to Ω, since Ω is connected. This proves $u(x) \equiv M$ for all $x \in \Omega$. □

Proof of Theorem 3.2 in next chapter uses Stampacchia's version of the maximum principle which we state below.

Notation: We denote $f^+(x) := \max\{f(x), 0\}$ and $f^-(x) = -\min\{0, f(x)\}$. Therefore, $f = f^+ - f^-$.

Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $-\Delta u + c(x)u \leq 0$ in Ω and $u \leq 0$ on $\partial\Omega$, where $\|c^-\|_{L^{\frac{N}{2}}(\Omega)} < S_N$ and S_N is the best Sobolev constant in $\mathbb{R}^N$. Then $u \leq 0$ in Ω.

Theorem 2.2. *Suppose* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfies*

$$-\Delta u + c(x)u \leq 0 \quad in \quad \Omega, \quad u \leq 0 \quad on \quad \partial\Omega,$$

where $\|c^-\|_{L^{\frac{N}{2}}(\Omega)} < S_N$ *and* S_N *is the best Sobolev constant in* $\mathbb{R}^N$. *Then* $u \leq 0$ *in* Ω.

In particular the assumption on c always holds in small domains provided the sup-norm of c^- is bounded.

3 The Moving Plane Method

In this section we are going to discuss the moving plane method in order to study the symmetry of solutions. The following result was first proved in 1979 by B. Gidas, W. M. Ni and L. Nirenberg in their celebrated paper [7].

Suppose $u \in C^2(\Omega)$ is any positive solution of $-\Delta u = f(u)$ in Ω and $u = 0$ on $\partial\Omega$, where Ω is the open unit ball in $\mathbb{R}^N$ and $f \in C^1$. Then u is radially symmetric and the radial derivative $u'(r) < 0$ for $0 < r < 1$.

Theorem 3.1. *[7] Let* Ω *be the open unit ball in* $\mathbb{R}^N$ *and* $u \in C^2(\Omega)$ *be a positive solution of* $(\mathcal{P})$*, where* $f \in C^1$ *and* $f(x,t) = f(t)$ *for all* $x \in \Omega$. *Further assume,* $g = 0$. *Then* u *is radially symmetric and the radial derivative* $u'(r)$ *is negative for* $0 < r < 1$.

The paper of Gidas, Ni and Nirenberg [7] has become hugely popular over the years for many reasons:

(i) It established radial symmetry of solutions for a large class of problems and also became an incentive for studying symmetry in many other situations.

(ii) The method is very flexible.

In fact, it has been successfully adapted to address large class of questions arising in concrete problems. For example, Amick and Fraenkel [2] used it in connection with vortex rings. Craig and Sternberg [6] have used it to solve an open problem on water waves.

Later in 1991, Berestycki and Nirenberg generalized the above theorem for more general domains in $\mathbb{R}^N$.

Theorem 3.2. *[4] Let Ω be a bounded convex set in $\mathbb{R}^N$ which is symmetric with respect to some hyperplane, say $x_1 = 0$. Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $(\mathcal{P})$ with $g = 0$, $f(x,t) = f(t)$ for all $x \in \Omega$ and f is locally Lipschitz. Then u is symmetric with respect to x_1 and $\frac{\partial u}{\partial x_1} < 0$ for $x_1 > 0$ in Ω.*

Remark 3.1. *Theorem 3.2 was originally proved in [7] but under an additional assumption that Ω is C^2. Therefore the domains with Lipschitz boundary (for e.g. cube) could not be handled. The assumption Ω is convex can not be relaxed. For e.g, in [5], Brezis and Nirenberg have constructed positive nonradial solution of*

$$\begin{cases} -\Delta u = u^p + \lambda u \ \ in \ \ \Omega = annulus, \\ \quad u = 0 \ \ on \ \ \partial\Omega, \end{cases}$$

Radial symmetry of solutions of $(\mathcal{P})$ in the entire space $\mathbb{R}^N$ (that is when $\Omega = \mathbb{R}^N$) with $u(x) \to 0$ as $|x| \to \infty$ has been originally proved in [8]. The radial symmetry was used in order to get a complete description of all positive solutions of $-\Delta u = u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$. This classification plays an important role in the study of the Yamabe problem, in particular in the work of R. Schoen [9]. When Ω is any half space for example $\Omega = \{x \in \mathbb{R}^N \ : \ x_n > 0\}$ in $(\mathcal{P})$ with $g = 0$ and $f(x,t) = t$, Berestycki, Caffarelli and Nirenberg [3] have established symmetry ($u = u(x_n)$) and monotonicity property of u provided u is bounded and $f(\sup u) \leq 0$.

Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is any positive solution of $-\Delta u = f(u)$ in Ω and $u = 0$ on $\partial\Omega$, where f is locally Lipschitz and Ω is a bounded convex set in $\mathbb{R}^N$ which is symmetric with respect to some hyperplane, say $x_1 = 0$. Then u is symmetric with respect to x_1 and $\frac{\partial u}{\partial x_1} < 0$ for $x_1 > 0$ in Ω. The assumption Ω is convex can not be relaxed. For e.g, Brezis and Nirenberg [5] have constructed positive nonradial solution of the above equation in $\Omega = annulus$ with $f(u) = u^p + \lambda u$.

Below we present the proof of Theorem 3.2.

Proof of Theorem 3.2: We denote a point $x \in \mathbb{R}^N$ as $x = (x_1, y)$, where $y = (x_2, \cdots, x_n)$ and set $a = \max\{x_1 : (x_1, y) \in \Omega\}$. We will prove that

$$u(x_1, y) < u(x_1', y) \quad \text{for all } x = (x_1, y) \in \Omega \text{ with } x_1 > 0, \text{ and every } x_1' \text{ with } |x_1'| < x_1. \tag{3.1}$$

Then letting $x_1' \to -x_1$, we get $u(x_1, y) \leq u(-x_1, y)$. Moreover, by defining $\tilde{u}(x) = u(-x_1, y)$, which is also a solution of $(\mathcal{P})$, we can find that $u(x_1, y) = u(-x_1, y)$, i.e., u is symmetric w.r.t x_1. Furthermore, it's easy to see that the assertion $\frac{\partial u}{\partial x_1} < 0$ for $x_1 > 0$ in Ω follows from (3.1).

For $0 < \lambda < a$, we define

$$\Sigma_\lambda := \{x \in \Omega : x_1 > \lambda\}, \quad T_\lambda := \{x_1 = \lambda\}.$$

For $x = (x_1, y)$, we define $x_\lambda = (2\lambda - x_1, y)$. Next we set,

$$w_\lambda := u(x) - u(x_\lambda), \quad \text{for} \quad x \in \Sigma_\lambda.$$

Note that w_λ is well defined on Σ_λ since, Ω is convex and symmetric about the hyperplane $x_1 = 0$. Then w_λ satisfies

$$\begin{cases} -\Delta w_\lambda = c_\lambda(x) w_\lambda \text{ in } \Sigma_\lambda, \\ \quad w_\lambda \leq 0 \text{ on } \partial\Sigma_\lambda, \quad w_\lambda \not\equiv 0, \end{cases}$$

where

$$c_\lambda(x) := \begin{cases} \frac{f(u(x)) - f(u_\lambda(x))}{w_\lambda} & \text{if } w_\lambda(x) \neq 0 \\ 0 & \text{if } w_\lambda(x) = 0. \end{cases}$$

Clearly, $\|c_\lambda\|_{L^\infty(\Omega)} \leq L$, where L is the Lipschitz constant of f in $[-\|u\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)}]$. We need to show $w_\lambda < 0$ for any $\lambda \in (0, a)$. Note that, for λ close to a, Σ_λ has small measure and therefore, by Theorem 2.2, $w_\lambda \leq 0$ in Σ_λ. Then, applying strong maximum principle, we obtain $w_\lambda < 0$ in Σ_λ when λ is close to a. Define,

$$\lambda_0 := \inf\{\lambda \in (0, a) : w_\lambda < 0 \text{ in } \Sigma_\lambda\}.$$

We claim $\lambda_0 = 0$. If not, then suppose $\lambda_0 > 0$. Clearly, by continuity $w_{\lambda_0} \leq 0$ in Σ_{λ_0} and $w_{\lambda_0} \not\equiv 0$ in $\partial\Sigma_{\lambda_0}$. Therefore, again by strong maximum principle $w_{\lambda_0} < 0$ in Σ_{λ_0}. Fix, $\delta > 0$ (to be determined later). Let K be any (smooth) compact set in Σ_{λ_0} such that $|\Sigma_{\lambda_0} \setminus K| < \delta/2$. Since, $w_{\lambda_0} < 0$ in Σ_{λ_0}, we have $w_{\lambda_0} \leq -\eta < 0$ in K. By continuity $w_{\lambda_0 - \varepsilon} < 0$ on K when $\varepsilon > 0$ is small enough. Also for $\varepsilon > 0$ small enough $|\Sigma_{\lambda_0 - \varepsilon} \setminus K| < \delta$. We choose $\delta > 0$ sufficiently small so that Theorem 2.2 can be applied to $w_{\lambda_0 - \varepsilon}$ in $\Sigma_{\lambda_0 - \varepsilon} \setminus K$ with $\|c_{\lambda_0 - \varepsilon}\|_{L^\infty} \leq L$ (the Lipschitz constant of f).

Hence, $w_{\lambda_0-\varepsilon} \leq 0$ in $\Sigma_{\lambda_0-\varepsilon} \setminus K$. Again applying strong maximum principle, we have $w_{\lambda_0-\varepsilon} < 0$ in $\Sigma_{\lambda_0-\varepsilon} \setminus K$. Hence, for small $\varepsilon > 0$

$$w_{\lambda_0-\varepsilon} < 0 \quad \text{in} \quad \Sigma_{\lambda_0-\varepsilon}.$$

This contradicts the definition of λ_0. Hence the theorem follows. Q.E.D

Suggested Reading

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