Estimating a Genius – Louis Nirenberg*

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Louis Nirenberg (1925–2020) made monumental contributions to partial differential equations and related areas of mathematics. This article attempts to present aspects of his life as well as his works to mathematically mature laymen without doing justice to either.

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The study of differential equations, whether ordinary (ODE) or partial (PDE) is akin to theology. The central questions are often about existence and uniqueness[1]. It is impossible to find an area of real life that does not involve differential equations, whether it be the current rage – epidemiology or a past glory like rocket science. Even in "pure" mathematics, differential equations play a role in seemingly unrelated subjects like topology and geometry (Poincaré conjecture), algebraic geometry (deformation invariance of plurigenera), and number theory (modular forms). This vast subject relies crucially on establishing deviously clever estimates of relevant quantities, i.e., inequalities are prized over equalities (despite the final answer being an equality!). One of its best practitioners, Louis Nirenberg, passed away on 26 January 2020. There are several articles on the life and work of Nirenberg, all of them excellent [2, 3, 4, 5, 6]. While this note is drawn heavily from them, I shall make a brave attempt to describe a tiny portion of his highly technical work to non-mathematicians. I am sincerely grateful to Joel Spruck (of the Monge-Ampère type

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1 While most high school students may think that one can always find explicit formulae that solve differential equations, there are innocent looking ones for which one can prove that there are no solutions at all! Leave alone an explicit formula! In this vein, for many differential equations, even when solutions exist, one cannot write explicit formulae involving elementary functions. There are theorems to this effect (forming an edifice known as differential Galois theory).

equations fame) who was kind enough to share some thoughts about Louis Nirenberg. I shall sprinkle these snippets throughout the article[2].

[2] For instance, apparently, Louis Nirenberg hated it when people (including the famous geometer/analyst Fred Almgren) pronounced his first name rhyming with "Lewis" [7].

Louis Nirenberg was born in 1925 in Canada. His schooling happened during the time of the great depression. As a consequence, the only teachers who had jobs were excellent [3] and he was torn between studying mathematics and physics. He studied these subjects in McGill university and then went to do a summer job at the National Research Council. There, his acquaintance with a son of Courant (of the Courant–Hilbert fame) led him to join New York University (NYU). He pursued a Master's degree in mathematics and then went to be a PhD student, along with other prodigies such as the Abel prize winner Peter Lax (of the Lax-Milgram fame) and Cathleen Morawetz (she of the Morawetz estimates). Apparently, Nirenberg turned into a sophisticated New Yorker who loved food, travel, music, dancing, and wry humour[3] very quickly [7].

[3] He went to an Italian pastry shop with Joel Spruck and told him "Joel, I am very cheap and am not going to treat you." [7]

Nirenberg's PhD adviser was Jim Stoker. However, he considered Kurt Friedrichs (of the Friedrichs' extension) his mathematical mentor (as he put it, his Sensei [3, 4]). It appears that Friedrichs imparted a love for inequalities in him. He continued his love affair with inequalities as a faculty in NYU and produced revolutionary work in analysis and PDE. For his monumental contributions to the field of PDE, he was awarded the Abel Prize in 2015 along with another giant, the Nobel laureate John Nash (of the "Beautiful mind" fame). For the remainder of this article, I shall describe his work in miserly detail. Since his work is highly technical, the reader must forgive me for omissions as well as glossing over important points.

1. Differential geometry : Differential geometry is the study of distances and angles on curved surfaces (and higher dimensional objects) like the surface of the earth. The "curvature" is not a single number in high dimensions, but for surfaces, the Gaussian curvature is a number that varies from point to point and measures how much the surface curves intrinsically (as opposed to say, creating a cylinder out of flat paper). One interesting ques-

tion is: Given a metric (a way to measure to distances and angles) on a sphere with "positive curvature" (meaning it does not look like a horse saddle anywhere), is there a way to "embed" it without changing distances and angles into 3-D space (the isometric embedding problem of Weyl). This problem was studied by the mathematician, physicist, and philosopher Hermann Weyl (of the Weyl fermion fame) as a PDE but was missing some crucial estimates. Impressively enough, Nirenberg proved those estimates and finished the problem for his PhD thesis[4]. He generalised the question to "Which functions can be the curvature of some metric on the sphere?" (the Nirenberg problem). It was studied by many including Kazdan, Warner, Moser, Yang, Chang, and Struwe. Its generalisations to higher dimensions is still an active subject of research [6].

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2. Fluid mechanics : The motion of many fluids (technically speaking, Newtonian fluids) is described by the famous Navier–Stokes equations[5]. These equations for an incompressible fluid are as follows.

$$\nabla.\vec{v} = 0$$
$$\frac{d\vec{v}}{dt} + (\vec{v}.\nabla)\vec{v} = -\nabla p + \mu\Delta\vec{v}, \qquad (0..1)$$

where $\vec{v}$ is the velocity of the fluid, p is the pressure, and μ is the viscosity coefficient. Taking divergence on both sides of the second equation and using the first equation, one can "solve" (as an integral operator) for p and thus reduce the number of equations and unknowns. The resulting system is remarkably complex. Due to the presence of the viscosity term, it behaves like diffusion but because of the rest of the terms, it behaves like waves. Leray proved that this equation has a "weak solution", i.e., a solution in some sense but not in the usual calculus sense. For two spatial dimensions, it has been known since the 1960s that it has smooth solutions for smooth initial data for all time. A (literally !) million-dollar question [1] is whether it has smooth solutions in three spatial dimensions for all time for all smooth initial data or not. One of the best results for this problem was due to Caffarelli–Kohn–Nirenberg who proved that there are suitable weak solutions for all time in $\mathbb{R}^3$ such that they become non-smooth ("blow up") on a very small set of points (which for in-

[4]Then again, Simon Donaldson won the fields medal for his PhD thesis!

[5]Unlike most other things mentioned in this article, these equations are genuinely famous. Read Chetan Bhagat's "Five point someone" for details.

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stance, cannot contain a curve). The Navier–Stokes equations are very different from the usual equations Nirenberg is associated with. It is remarkable that he contributed to one of the best results in a field that he was not originally trained in! The Navier–Stokes problem is still open. Nirenberg felt [3] that it could go either way. Indeed, years later, Terence Tao proved that [8] for a related "averaged" version of these equations, blow-up does happen for some initial data.

3. PDE theory : Most of Nirenberg's work was in the theory of PDE. One extremely important contribution is the Gagliardo–Sobolev–Nirenberg inequality (and the related interpolation inequality). Let u be a smooth function in $\mathbb{R}^n$ with compact support (simply means that the function vanishes outside a large enough ball) and let $1 \leq p < n$ be an integer. Then,

$$\left(\int_{\mathbb{R}^n} u^q dx_1 dx_2 \ldots\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |\nabla u|^p dx_1 \ldots\right)^{1/p}, \qquad (0..2)$$

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where $q = \frac{np}{n-p}$. The importance of these kinds of inequalities lies in the fact that whilst studying PDE, one often proves that a "weak" solution exists and then proves that this solution is actually quite well-behaved and is a solution in the usual calculus sense. The latter step (and for nonlinear PDE, even the former step) involves proving inequalities that control how fast the solution changes. In conjunction with inequalities like the one above, one can then proved that the weak solution is much better behaved than it appears at first blush. For those who are familiar with Fourier analysis, this philosophy is closely related to proving that the high frequency components carry very little energy (the less shrill the music, the more melodious it sounds[6]) Nirenberg's work in this regard was fundamental. In addition to inequalities like the above, he, along with Douglis and Agmon generalised the polish mathematician Schauder's work on second order elliptic PDE to systems of PDE.

[6] At least if, like Aamir Khan in "Dil Chahta Hai", you do not enjoy opera-style music.

Nirenberg, along with Gidas and Ni proved that certain PDE like

$$\Delta u = f(u)$$

on a symmetric domain (like a ball) with symmetric boundary conditions have symmetric solutions (and hence there is some hope of finding an explicit formula for them). Along with Kohn, Nirenberg developed the theory of pseudodifferential operators (which despite the obscure sounding name, is extremely important in the study of PDE). Nirenberg wrote a paper with the German-born mathematician Fritz John[7] which initiated the theory of Bounded Mean Oscillation (BMO) functions. Nirenberg wrote a series of landmark papers with Caffarelli and Spruck (one of them was with Kohn) on the theory of fully nonlinear elliptic equations, in particular, the Monge-Ampère equation. As promised earlier, despite PDE theory being Nirenberg's "core" area so to say, we shall not delve deeper into these highly technical but important contributions.

[7]Nirenberg himself counts [4] Fritz John as one of his most important colleagues.

4. Complex analysis : Among differential geometers, especially those who work in complex differential geometry (basically, a marriage of convenience between complex numbers and the theory of curved objects developed to serve the needs of algebraic geometry in addition to being interesting in its own right) Nirenberg is most famous for the Newlander–Nirenberg theorem. Nirenberg proved it along with his brilliant student August Newlander[8] in 1957 when another giant André Weil (of the Weil conjectures fame) suggested to him the problem of integrability of almost complex structures [3]. Unfortunately, the most famous of Nirenberg's works is also one of the most technical to even state. At a high level, sometimes PDE may need some "compatibility conditions" to be met before even the hope of a solution is born. Here is a particular simple example : Suppose we want to find a smooth function $u(x, y)$ such that $\frac{\partial u}{\partial x} = F_1$ and $\frac{\partial u}{\partial y} = F_2$ where F_1, F_2 are given smooth funcions. Then notice that

[8]Unfortunately it appears that Newlander did not pursue mathematics after his PhD due to personal reasons.

$$\frac{\partial F_2}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial F_1}{\partial y}.$$

In fact, in many cases (but not all !) this compatibility condition (such conditions are sometimes called "integrability conditions") is enough to guarantee the existence of such a u.

Akin to the example above, the Newlander–Nirenberg theorem states that a particular PDE related to complex analysis can be solved if a certain integrability condition is met. (For the experts, an almost complex structure defines a complex structure if and only if the Nijenhuis tensor vanishes.)

The Newlander–Nirenberg theorem states that a particular PDE related to complex analysis can be solved if a certain integrability condition is met.

The versatility of Nirenberg's work is rare in current mathematics, which is becoming increasingly specialised[9]. He was quite pleasant and liked to collaborate with many mathematicians, especially with young ones[10]. Even confined to a wheelchair in his later years, he continued to travel and talk about mathematics extensively. Alas! all good things must come to an end, including the popular and beloved Louis Nirenberg, and this article.

[9] He wrote a couple papers even in Economics [4].

[10] When the then young Spruck gave a lecture in the Courant analysis seminar, he said "that's good enough to steal" [7].

Suggested Reading

[1] **Official statement of the problem, Clay Mathematics Institute : https://www.claymath.org/sites/default/files/navierstokes.pdf**

[2] **K Chang, "Louis Nirenberg, 'One of the Great Mathematicians,' Dies at 94", *The New York Times*, Jan 31 2020.**

[3] **A Jackson, "Interview with Louis Nirenberg", *Notices of the AMS*, Vol.49, No.4, pp.441–449, 2002.**

[4] **M Raussen and C Skau, "Interview with Louis Nirenberg", *Notices of the AMS*, Vol.63, No.2, pp.137–140, 2016.**

[5] **B Shields and M J Barany, "Louis Nirenberg (1925–2020)", *Nature*, Vol.578, No.359, pp.359–359, 2020.**

[6] **C Sormani and X C Vilagut, "Recent applications of Nirenberg's classical ideas", *Notices of the AMS*, Vol.63, No.2, pp.126–134, 2016.**

[7] **J Spruck, *Private Communication*.**

[8] **T Tao, "Finite time blowup for an averaged three-dimensional NavierStokes equation", *Journal of the American Mathematical Society*, Vol.29, No.3, pp.601–674, 2016.**

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