Classroom

In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Group Laws Satisfying $1+1=11$ and $2+2=22$

In a recent election campaign, Prime Minister Modi claimed that if one works hard in a correct political climate, one can make $1+1=11$. In a similar speech, Mr Sitaram Yachuri mentioned that if we all work together then our combined strength will make $2+2=22$. Here, we address the question: Is there a field $k$ with a polynomially defined binary law of composition $+$ such that the two equations $1+1=11$ and $2+2=22$ are both valid in $k$. In this note, we show that there are infinitely many such fields and characterize them all.

1. Introduction

While browsing through funny math vides, I came across a YouTube short film comedy, Alternative Math, produced by Ideaman Studios (see [1]). It is a hilarious exaggeration of a math teacher who is dragged through the mud for teaching that $2+2=4$ and not $22$ as Dany, a young student insisted. In a recent election campaign in India, the Indian Prime Minister Narendra Modi declared, ‘will make $1+1=11$’, perhaps by way of hinting the

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### CLASSROOM

<table>
<thead>
<tr>
<th>Formula for $x \oplus y$</th>
<th>Associative?</th>
<th>Commutative?</th>
<th>$1 \oplus 1 = 11$?</th>
<th>$2 \oplus 2 = 22$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>usual plus $x + y$</td>
<td>Yes</td>
<td>Yes</td>
<td>No, $1 + 1 = 2$</td>
<td>No, $2 + 2 = 4$</td>
</tr>
<tr>
<td>Concatenation</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$x \oplus y = x + y + 9$</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$x \oplus y = x + y + 18$</td>
<td>Yes</td>
<td>Yes</td>
<td>No, $1 \oplus 1 = 20$</td>
<td>Yes</td>
</tr>
<tr>
<td>$x \oplus y = 5x + 6y$</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$x \oplus y = (11/2)(x + y)$</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

**Table 1.** Some examples of binary laws of composition satisfying $1 \oplus 1 = 11$ or $2 \oplus 2 = 22$. 

The prospects of tremendous growth in economy if only the State and the Centre act in unison (see [2]). Let us forget politics and ask for some ‘genuine’ group operation “+” in which we have both $1 + 1 = 11$ and $2 + 2 = 22$. Indeed there are several such algebras. In this paper we characterize all fields $k$ having a polynomially defined group law + satisfying both $1 + 1 = 11$ and $2 + 2 = 22$. We also give an example of a rational group law having the same property.

This is not an article on politics. Instead, it is about plain simple old-fashioned algebra. Let us start from the two equations mentioned above: $1 + 1 = 11$ and $2 + 2 = 22$. This raises the question whether it is possible to define a new arithmetic “+$” satisfying the two equations $1 + 1 = 11$ and $2 + 2 = 22$? Yes, the young Danny, Mr. Yachuri and the Prime Minister Modi are all correct; it is certainly possible. As mentioned in the movie, it all depends upon what one means by the word called “plus”. It is a special case of what is known as a binary law of composition in algebra – a well-defined process of combining two given numbers to produce a unique third number. Now let me give some examples of such “additions” which demonstrate the validity of the above statements $1 + 1 = 1$ and $2 + 2 = 22$. To avoid confusion with the ordinary addition, let us use the special notation of circled-plus $\oplus$ for our new addition and retain the symbol “+” for the usual addition.
The most natural example of such an addition is the concatenation $\oplus$: the sum $a \oplus b$ is obtained by writing ‘a’ followed by ‘b’, e.g. $23 \oplus 37 = 2337$. Here, of course, $1 \oplus 1 = 11$ and $2 \oplus 2 = 22$, no problem. However, unlike our usual addition, this is not commutative: $37 \oplus 23 = 3723 \neq 23 \oplus 37$. Even worse, this is not a polynomial while our usual $x + y$ is a linear polynomial in two variables and universally defined over the field of rationals.

Example 2. Let us now give an example of a polynomially defined $\oplus$ such that $1 \oplus 1 = 11$ and $2 \oplus 2 = 22$. Define $x \oplus y$ by the rule $x \oplus y = 5x + 6y$ (5th item in the Table 1). Here, $2 \oplus 2 = 10 + 12 = 22$ and $1 \oplus 1 = 5 + 6 = 11$.

However, it is our innate feeling that this is not a genuine addition either. This operation is neither commutative nor associative. So we continue to ask for some ‘natural’ addition satisfying $1 \oplus 1 = 11$ and $2 \oplus 2 = 22$. But what do we mean by ‘natural’ here? We need to specify some of the more important (or ‘desirable’) properties valid for our classical ‘+’. The most important property of the addition is that it is a polynomially defined abelian group over, say the set of all real numbers. In particular, it has the following nice properties:

$$\{ x + y = y + x, (x + y) + z = x + (y + z), x + 0 = x \}.$$

The first two laws are known as commutativity and associativity, respectively. Notice the example given by the rule $x \oplus y = 5x + 6y$ is neither commutative nor associative. Let us check this example for associativity:

$$\begin{align*}
(1 \oplus 1) \oplus 1 & = 11 \oplus 1 = 55 + 6 = 61 \text{ but } 1 \oplus \\
1 \oplus (1 \oplus 1) & = 1 + 11 = 5 + 66 = 71.
\end{align*}$$

So this addition is neither commutative nor associative. Now ask the question: is there a field $k$ having a polynomially defined associative and commutative addition $\oplus$ satisfying both $1 \oplus 1 = 11$ and

In mathematics, a “good” theorem is not something that just happens to be true, but it is rather a nice mathematical statement that you want to be true. So adjust your definitions accordingly. – Shreeram S Abhyankar
2 ⊕ 2 = 22? In this note we show that the answer is an emphatic ‘yes’. There are infinitely many such fields. Here we characterize all fields \( k \) having a polynomially defined group law \( \oplus \) and satisfying both \( 1 \oplus 1 = 11 \) and \( 2 \oplus 2 = 22 \).

**Theorem.** Let \( k \) be any subfield of the real or complex numbers. Then \( k \) has a polynomially defined group law \( \oplus \) satisfying both \( 1 \oplus 1 = 11 \) and \( 2 \oplus 2 = 22 \) if and only if \( k \) contains the quadratic extension field \( \mathbb{Q}[\sqrt{89}] \).

**Proof.** Let \( x \oplus y = g(x,y)k[x,y] \) be a non-constant associative polynomial over the field \( k \). Let the highest degree of \( x \) in the polynomial \( g(x,y) \) be, say \( n \). Comparing the \( x \)-terms in the equation \( g(x,g(y,z)) = g((x,y),z) \) we see that the degree of \( x \) in RHS is \( n^2 \) while the degree of \( x \) in LHS is just \( n \). Since \( k \) is an infinite field, the monomials are linearly independent and hence \( n^2 = n \) i.e. \( n = 0 \) or \( n = 1 \). But, if \( n = 0 \) then \( x \oplus y \) does not depend upon \( x \), impossible since a group operation depends upon both variables. Hence \( n = 1 \). Similarly the highest degree of \( y \) in \( g(x,y) \) is also one. Hence the most general polynomially defined group law \( g(x,y) \) over \( k \) is bilinear polynomial in \( x \) and \( y \):

\[
x \oplus y = axy + bx + cy + d,
\]

for some constants \( a, b, c, d \) in \( k \). It is easy to check that this binary law of composition will be associative if and only if \( b = c \) and \( ad = b^2 - b \) (see Box 1).

Thus we have

\[
x \oplus y = axy + bx + by + d.
\]

If \( a = 0 \), then \( b = 1 \) and hence \( x \oplus y = x + y + d \). But this will be a contradiction in our case since we demand \( 1 \oplus 1 = 11 \) and hence \( d = 9 \) but then \( 2 \oplus 2 = 13 \) and not 22. So \( a \neq 0 \) and the associative law will force \( d = (b^2 - b)/a \). Thus we have the most general polynomially defined group law satisfying \( 1 \oplus 1 = 11 \) and \( 2 \oplus 2 = 22 \), namely:

\[
x \oplus y = axy + b(x + y) + (b^2 - b)/a
\]
Here, we want the binary polynomial $x \oplus y = axy + bx + cy + d$ to be both associative and commutative. It is clear that commutativity implies symmetry in $x$ and $y$, i.e. the coefficients of $x$ and $y$ must be the same and hence $c = b$. Let us now calculate the expression $(x \oplus y) \oplus z$:

$$(x \oplus y) \oplus z = (axy + bx + cy + d)z + b(axy + bx + cy + d) + bz + d.$$

Again, for commutative and associative laws, we need the complete symmetry among the three variables $x, y,$ and $z$. In particular, the variables $x, y,$ and $z$ must have the same coefficients. In other words, we have $ad + b = b^2$ i.e. $ad = b^2 - b$. Here $a \neq 0$. For, if $a = 0$, then $b = 1$ and so $x \oplus y = x + y + d$ and this cannot satisfy both $1 \oplus 1 = 11$ and $2 \oplus 2 = 22$ as proved below. Thus the final form of the group law is $x \oplus y = axy + bx + by + d$ where $d = (b^2 - b)/a$ and this was exactly our claim.

for some elements $a, b$ in $k$ with $a \neq 0$. Notice that there are two unknown parameters $a$ and $b$ to choose and we have two constraints i.e. $1 \oplus 1 = 11$ and $2 \oplus 2 = 22$. So we need to solve the two equations

$$a^2 + 2ab + (b^2 - b) = 11a4a^2 + 4ab + (b^2 - b) = 22a.$$  

Eliminating $b$ from the two equations\(^1\), we get $a^2 - 60a + 99 = 0$. The two solutions of this quadratic equation are $a = 30 \pm 3 \sqrt{89}$. Thus the field $k$ is consistent with the Modi–Yachuri equations $1 \oplus 1 = 11$ and $2 \oplus 2 = 22$ if and only if $\sqrt{89}$ exists in $k$. This completes the proof of the theorem.

Let us use this idea to give a finite example of such a field. Since 89 is a prime number, we take the finite field $k$ as $\mathbb{Z}[89]$ itself. Here $a = 30$ since $89 = 0$. The value of $b$ can be calculated and it turns out to be 5 (see Box 2; remember, here we work mod 89). Hence the group law $\oplus$ takes the bilinear form $x \oplus y = 30xy + 5x + 5y + 60$ (mod 89).

By our theory, this must be both commutative and associative.

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\(^1\)Indeed, resultant $(a^2 + 2ab + b^2 - b - 11a, 4a^2 + 4ab + b^2 - b - 22a, b) = a^4 - 60a^3 + 99a^2$. Since $a \neq 0$, we have the quadratic equation $a^2 - 60a + 99 = 0$. 

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Box 1. Conditions for Associativity

Here, we want the binary polynomial $x \oplus y = axy + bx + cy + d$ to be both associative and commutative. It is clear that commutativity implies symmetry in $x$ and $y$, i.e. the coefficients of $x$ and $y$ must be the same and hence $c = b$. Let us now calculate the expression $(x \oplus y) \oplus z$:

$$(x \oplus y) \oplus z = (axy + bx + by + d)z + b(axy + bx + by + d) + bz + d.$$
Box 2. Group Law in the Field Z[89]

Take \( a = 30 \) in the equations

\[
\begin{align*}
    a^2 + 2ab + (b^2 - b) &= 11a^4a^2 + 4ab + (b^2 - b) = 22a. \\
\end{align*}
\]

and solve for \( b \) (mod 89). To avoid excessive calculations, let us subtract the first equation from the second equation to get \( 3a^2 + 2ab = 11a. \) When \( a = 30 \) this equation becomes \( 2700 + 60b = 330 \) or \( b = (−2700 + 330)/60 = −237/6 = 30/6 = 5. \) Finally the value of the constant term \( d = (b^2 - b)/a = 20/30 = 2/3 = 60 \) since \( 3 \times 60 = 180 = 180 + 2 = 2 \) (mod 89). In other words, our desired group is \((Z[89], \oplus)\) where the group law is \( x \oplus y = 30xy + 5x + 5y + 60 \) (mod 89).

Commutativity is obvious because of the symmetry between the two variables \( x \) and \( y. \) Just for fun, let us verify the validity of the associative law.

\[
(x \oplus y) \oplus z = (30xy + 5x + 5y + 60) \oplus z
\]

\[
= 30(30xy + 5x + 5y + 60)z \\
+ 5(30xy + 5x + 5y + 60 + 5z + 60)
\]

\[
= 900xyz + 150xz + 150yz + 1800z + 150xy
+ 25x + 25y + 5z + 120
\]

\[
= 900xyz + 150xz + 150yz + 150xy + 25x
+ 25y + 1805z + 120
\]

\[
= 900xyz + 150xz + 150yz + 150xy + 25x
+ 25y + 25z + 120,
\]

which is symmetric in \( x, y \) and \( z \) and hence the addition \( \oplus \) is associative. Also, this operation has an identity element as well: \( x \oplus 77 = x \) for all \( x. \)

Indeed, \( x \oplus 77 = 2310x + 5x + 385 + 60 = 2315x + 445 = x \) (mod 89).

Finally, let us calculate the crucial values of \( 1 \oplus 1 \) and \( 2 \oplus 2. \)

Here \( 1 \oplus 1 = 30 + 5 + 5 + 60 = 100 = 89 + 11 = 11 \) (mod 89).
Similarly, \( 2 \oplus 2 = 120 + 10 + 10 + 60 = 200 = 178 + 22 = 2 \times 89 + 22 = 22 \text{ mod } (89) \).

Thus both equations \( 1 \oplus 1 = 11 \) and \( 2 \oplus 2 = 22 \) are valid in the arithmetic of \( \mathbb{Z}[89] \).

However, \( 3 \oplus 3 \) will not be 33 in this arithmetic: \( 3 \oplus 3 = 270 + 15 + 15 + 60 = 4 \text{ (mod 89)} \).

This addition law \( x \oplus y \) is commutative, associative and has an identity element as well. Is this really a new math as questioned in the newspaper headlines [2]? No, this is just the old multiplication hidden in a new form. More formally, this group \((k, \oplus)\) is isomorphic to the usual multiplication over the field. Indeed, let \( f(x) = 30x + 5 \). Then

\[
\begin{align*}
f(x \oplus y) &= 30(30xy + 5x + 5y + 60) + 5 \\
&= 900xy + 150x + 150y + 1800 + 5 \\
&= 900xy + 150x + 150y + 1805 \\
&= 900xy + 150x + 150y + 25 \text{ (mod 89)} \\
&= (30x + 5)(30y + 5) \\
&= f(x) \cdot f(y).
\end{align*}
\]

So the semigroup \((\mathbb{Z}[89], \oplus)\) is isomorphic to the multiplicative semigroup \((\mathbb{Z}[89])\). For example:

\[
\begin{align*}
f(77) &= 30 \times 77 + 5 = 2315 = 2314 + 1 = 26 \times 89 + 1 = 1 \text{ (mod 89)}. \\
\text{Also,} \\
f(74) &= 30 \times 74 + 5 = 2225 = 25 \times 89 = 0 \text{ mod 89}).
\end{align*}
\]

Thus the element 77 acts like ‘1’ and the element 74 acts like ‘0’ in this semigroup.

In fact, in this algebra, \( x \oplus y = 74 \) if and only if \( x = 74 \) or \( y = 74 \) (mod 89).
An Example of a Rational Function $x \oplus y$ satisfying $1 \oplus 1 = 11$ and $2 \oplus 2 = 22$.

So far, we have been concentrating on polynomial functions. But we can go one step further and ask for commutative and associative functions $g(x, y) \in k(x, y)$, the field of binary rational functions over $k$. These are thoroughly studied in the literature under the name of formal semigroups and/or one-dimensional algebraic groups. For example, it is known that every rational group law over an algebraically closed field $k$ of characteristic 0 is of the form $L^{-1}G(L(x), L(y))$ where $G(x, y)$ is either $x + y$ or $x + y + xy$ and $L$ is a linear fractional transformation over $k$ such that $L(0) = 0$. (see e.g. [4], [5]). This being a pedagogical classroom note, we will be content by giving just one typical example of a rational function $x \oplus y$ satisfying the Danny–Modi–Yachuri equations:

Define $x \oplus y := (297 \, xy + 22x + 22y)/(22 + 9xy)$.

Here, $1 \oplus 1 = (297 + 44)/(22 + 9) = 341/31 = 11 \odot$ and $2 \oplus 2 = (1188 + 88)/(22 + 36) = 1276/58 = 22 \odot \ominus$

In fact, this rational formula is much more well-behaved than our polynomial example. Here we do have $x \oplus 0 = x$ as well.

Acknowledgements

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Suggested Reading

[1] 2+2=22; Alternative Math video: http://kstati.net/2-2-22/

