# The Central Force Problem in *n* Dimensions*

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**The motion of a particle moving under the influence of a central force is a fundamental paradigm in dynamics. The problem of planetary motion, specifically the derivation of Kepler's laws motivated Newton's monumental work, *Principia Mathematica*, effectively signalling the start of modern physics. Today, the central force problem stands as a basic lesson in dynamics. In this article, we discuss the classical central force problem in a general number of spatial dimensions *n*, as an instructive illustration of important aspects such as integrability, super-integrability and dynamical symmetry. The investigation is also in line with the realisation that it is useful to treat the number of dimensions as a variable parameter in physical problems. The dependence of various quantities on the spatial dimensionality leads to a proper perspective of the problems concerned. We consider, first, the orbital angular momentum (AM) in *n* dimensions, and discuss in some detail the role it plays in the integrability of the central force problem. We then consider an important super-integrable case, the Kepler problem, in *n* dimensions. The existence of an additional vector constant of the motion (COM) over and above the AM makes this problem maximally super-integrable. We discuss the significance of these COMs as generators of the dynamical symmetry group of the Hamiltonian. This group, the rotation group in $n+1$ dimensions, is larger than the kinematical symmetry group for a general central force, namely, the rotation group in *n* dimensions.**

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## A Quick Recap of Classical Hamiltonian Dynamics

A Hamiltonian system with $n$ degrees of freedom has $n$ generalised coordinates $q_i$ (where $i = 1, 2, \ldots, n$), $n$ conjugate generalised momenta $p_i$, and a scalar function $H(q, p)$ (the Hamiltonian) of these variables. (We shall use $q$ and $p$ to denote the full set of coordinates and momenta, respectively.) $H$ may also have explicit time-dependence, but we restrict our attention to the autonomous case in which there is no such dependence. The time evolution of the dynamical variables is given by Hamilton's equations of motion,

$$\dot{q}_i = \partial H/\partial p_i, \quad \dot{p}_i = -\partial H/\partial q_i \quad (1 \leqslant i \leqslant n), \tag{1}$$

where an overhead dot denotes the time derivative, as usual. The Poisson bracket (PB) of any two functions $f$ and $g$ of the dynamical variables is defined as,

$$\{f, g\} = \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\Big). \tag{2}$$

(In this article, we do *not* use the summation convention for repeated indices. Sums over indices will be indicated explicitly.) $f$ and $g$ are said to be *in involution* if their PB vanishes identically, i.e., if $\{f, g\} \equiv 0$. The mutual independence of the $2n$ dynamical variables implies that they satisfy the canonical PB relations

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \tag{3}$$

where $\delta_{ij}$ is the Kronecker delta. A most useful identity is the 'chain rule' (also called the derivation law or the Leibnitz law) for PBs : if $f, g$ and $h$ are functions of the dynamical variables, then

$$\{f, gh\} = \{f, g\}\, h + g\, \{f, h\}. \tag{4}$$

An important relation satisfied by PBs is the Jacobi identity,

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0. \qquad (5)$$

Further, if $f(q)$ and $g(p)$ are functions of the coordinates and momenta respectively, then

$$\{f(q),\ p_j\} = \frac{\partial f}{\partial q_j} \ \text{ and } \ \{q_i,\ g(p)\} = \frac{\partial g}{\partial p_i}\,. \qquad (6)$$

It follows from (1) that any (differentiable) function $F(q, p)$ of the dynamical variables satisfies the equation of motion $dF/dt = \{F, H\}$. (Again, we restrict ourselves to the case in which $F$ has no explicit $t$-dependence.) A constant of the motion (COM) is any function $F$ of the dynamical variables that satisfies the condition $dF/dt = 0$ on all solution sets of (1). Thus, $F(q, p)$ is a COM if and only if $\{F, H\} = 0$, i.e., $F$ and $H$ are in involution. It is obvious that $H(q, p)$ is itself a COM, since $\{H,\ H\} \equiv 0$.

Hamilton's equations of motion, (1), comprise a set of $2n$ first-order, coupled, differential equations for the dynamical variables. In general, the equations are nonlinear in these variables. They possess a unique solution set $(q_i(t), p_i(t))$ for a given set of initial values $(q_i(0), p_i(0))$. At any subsequent instant of time, the solution set is represented by a point in the space of the $2n$ dynamical variables (called the phase space). As $t$ increases, this point traces out a trajectory, a one-dimensional curve, in the phase space. Since $H$ is a COM, every phase trajectory must lie on some level surface $H(q, p) = E$, where the value of the constant $E$ is determined by the given initial values of the dynamical variables. Obviously, every phase trajectory must also lie on the level surfaces of all the COMs, also called the 'first integrals', of the system. This implies that each phase trajectory actually lies on the intersection of these level surfaces. Each level surface is a $(2n - 1)$-dimensional hypersurface in the phase space. The intersection of the level surfaces of two different, functionally or algebraically independent constants of the motion (AICOMs) is $(2n - 2)$-dimensional, that of any three AICOMs is $(2n - 3)$-dimensional, and so on. Since a phase trajectory is a

1-dimensional object, there can only exist a maximum of $(2n-1)$ explicitly time-independent AICOMs for a Hamiltonian system with $n$ degrees of freedom.

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It follows that a primary task in the solution of Hamilton's equations is to find as many AICOMs as possible. The structure of these equations leads to the following strong result that specifies a necessary and sufficient condition for the full set to be 'integrable'—i.e., given the set of initial values $(q(0), p(0))$, for the solutions $(q(t), p(t))$ to be written in explicit form as functions of $t$ for all $t > 0$ :

- *Liouville-Arnold Condition for Integrability*: A Hamiltonian system with $n$ degrees of freedom is integrable if and only if there exist exactly $n$ AICOMs,

$$F_1(q, p),\ F_2(q, p),\ \ldots,\ F_n(q, p),$$

(where $F_1 = H$) *in involution* with each other. It is then guaranteed that there exists a canonical transformation from the set of variables $(q, p)$ to a new set $(\theta, I)$ of 'angle' variables $\theta_i$ and 'action' variables $I_i$ $(1 \leqslant i \leqslant n)$, such that all the $2n$ equations of motion can be solved explicitly.

Since the transformation is canonical, we have

$$\{\theta_i,\, \theta_j\} = 0, \quad \{I_i,\, I_j\} = 0, \quad \{\theta_i,\, I_j\} = \delta_{ij}. \tag{7}$$

The transformed Hamiltonian $K$ turns out to be a function of the action variables alone, and does not depend on the corresponding angle variables.

The remarkable reduction in the number of AICOMs required for complete integrability, from $(2n-1)$ of them to just $n$ of them (but in involution with each other), comes about as follows. The transformed Hamiltonian $K$ turns out to be a function of the action variables alone, and does not depend on the corresponding angle variables. Hamilton's equations of motion in the new variables simplify to

$$\left.\begin{array}{lllll} \dot{\theta}_i & = & \partial K(I)/\partial I_i & \equiv & \omega_i(I) \\ \dot{I}_i & = & -\partial K(I)/\partial \theta_i & \equiv & 0 \end{array}\right\} \ (1 \leqslant i \leqslant n). \tag{8}$$

It follows at once that each $I_i$ is a COM, and hence so is each $\omega_i(I)$. As a consequence, $\theta_i(t) = \omega_i\, t + \theta_i(0)$. In the $(2n+1)$-dimensional extended phase space spanned by the variables $(\theta, I, t)$, we thus have $n$ time-independent AICOMs $I_i$, together with $n$ explicitly time-dependent AICOMs $[\theta_i(t) - \omega_i\, t]$, where $1 \leqslant i \leqslant n$. Hence the system is completely integrated. In principle, all that remains is to invert the canonical transformation to obtain $(q(t), p(t))$.

It is relevant to add the following remarks:

(i) If the system has *less* than $n$ AICOMs in involution, it is not integrable in the sense described above for at least some initial conditions.

(ii) The system cannot have *more* than $n$ independent AICOMs such that all of them are in involution with each other.

(iii) The system may (and when integrable, does) have more than $n$ AICOMs, but not all of them will be in involution with each other.

(iv) The Liouville-Arnold condition does not specify *how* the COMs are to be found in any given instance.

If the system has *less* than $n$ AICOMs in involution, it is not integrable for at least some initial conditions.

We will call a Hamiltonian LA-integrable if the Liouville-Arnold condition stated above is satisfied.

Some further remarks will be of help in understanding the significance of the $n$ AICOMs $I_1, \ldots, I_n$ being in involution with each other. There is a mathematical structure called *symplectic geometry* that is very closely associated with classical Hamiltonian dynamics. The phase space of the variables $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ is endowed with a certain natural metric $\omega_{\alpha\beta}$, where the indices run from 1 to $2n$. This metric is given by

$$\omega_{\alpha\beta} = \begin{cases} \delta_{n+\alpha,\beta}, & 1 \leqslant \alpha \leqslant n \\ -\delta_{\alpha,\, n+\beta}, & 1 \leqslant \beta \leqslant n, \end{cases} \tag{9}$$

and all other components are zero. The Poisson bracket of any two functions of the phase space variables has a simple interpretation in terms of this metric: it is just the 'symplectic scalar product' of the gradients of the functions concerned, i.e.,

$$\{f, g\} = (\partial_\alpha f)\, \omega_{\alpha\beta}\, (\partial_\beta g), \tag{10}$$

where the repeated indices are to be summed over. If $f$ and $g$ are in involution, their gradients are orthogonal to each other. In other words, their level surfaces are orthogonal to each other. Applied to the action variables $I_i$ $(1 \leqslant i \leqslant n)$, the involution property $\{I_i, I_j\} = 0$ implies that the gradients of any $I_i$ and $I_j$ (where $i \neq j$) are orthogonal to each other in the symplectic sense. As a consequence, the corresponding set of differentials $(dI_1, \ldots, dI_n)$ forms a 'global' basis that spans the phase space, now restricted to the space of the $n$ angles $\theta_1, \ldots, \theta_n$. Restricting attention to bounded motion, the latter space is, in general, an $n$-dimensional torus, or $n$-torus for short.

We will not digress here into further details along these lines, as it would take us too far afield.

## The Central Force Problem in 2 and 3 Dimensions

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The conventional central force problem involves a classical, non-relativistic particle of mass $m$ moving under the influence of a central force $\mathbf{F}(\mathbf{r})$ derived from a scalar potential $V(r)$, according to

$$\mathbf{F}(\mathbf{r}) = -\nabla V(r) = -(dV/dr)\, \mathbf{e}_r. \tag{11}$$

Here $\mathbf{r}$ is the position vector of the particle, $r = |\mathbf{r}|$, and $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the radial direction. The Hamiltonian of the particle is given by

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(r), \tag{12}$$

where $\mathbf{p} = m\, d\mathbf{r}/dt$ is the linear momentum of the particle. The Cartesian components $x_i$ and $p_i$ $(1 \leq i \leq n)$ of $\mathbf{r}$ and $\mathbf{p}$ may be taken to be the generalised coordinates and momenta, respectively. The cases customarily treated are those corresponding to 2 and 3 spatial dimensions ($2D$ and $3D$). It is easy to establish that the central force problem is LA-integrable in $2D$ and in $3D$, as follows.

Consider first the $2D$ case, in which $r = (x_1^2 + x_2^2)^{1/2}$. The potential $V(r)$, and hence the Hamiltonian, is unchanged (or 'invariant') under any rotation of the coordinate axes about the origin. As is well known, this invariance leads to the conservation of the orbital angular momentum (AM). The AM has only a single component (i.e., it is a scalar) in $2D$, namely,

$$L = x_1\, p_2 - x_2\, p_1. \tag{13}$$

Using (6), it is straightforward to verify that $\{H, L\} = 0$, i.e., that $L$ is a COM. (One needs to use the fact that $\{V(r), p_j\} = V'(r)\, x_j/r$.) We thus have 2 independent COMs in involution, namely, $H$ and $L$. This suffices to guarantee that the Hamiltonian is LA-integrable. For any set of initial conditions, the phase trajectory lies on a 2-dimensional surface in the 4-dimensional phase space. This surface is the intersection of the 3-dimensional hypersurfaces $H$ = constant = $E$ (say) and $L$ = constant = $\ell$ (say). The problem is 'reduced to quadratures'. To specify the motion further, we need to solve the ordinary differential equation,

$$\frac{m}{2}\Big(\frac{dr}{dt}\Big)^2 + \frac{\ell^2}{2mr^2} + V(r) - E = 0, \tag{14}$$

for the radial coordinate $r$, with the appropriate initial conditions.

The problem becomes a little more intricate in $3D$. The Hamiltonian is as in (12), with $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. The angular momentum is now a 3-vector (more precisely, an axial vector) $\mathbf{L}$, with components

$$L_1 = x_2\, p_3 - x_3\, p_2, \quad L_2 = x_3\, p_1 - x_1\, p_3, \quad L_3 = x_1\, p_2 - x_2\, p_1. \tag{15}$$

The important point is that the components of $\mathbf{L}$ are *not* in involution with each other. Instead, they satisfy the PB relations

$$\{L_1, L_2\} = L_3, \quad \{L_2, L_3\} = L_1, \quad \{L_3, L_1\} = L_2. \tag{16}$$

This is, of course, the familiar angular momentum algebra—the Lie algebra of the rotation group $SO(3)$ in 3 dimensions: The generators of infinitesimal rotations of the coordinate axes are *represented* here by the components of the orbital AM; and the Lie algebra is *realised* by the PB relations between the generators.

Since the components of **L** are not in involution with each other, the AM vector **L** can contribute only one COM to the set of AICOMs in involution. This can be taken to be any component $(\mathbf{L}\cdot\mathbf{n})$ of **L**, where **n** is an arbitrary unit vector. The usual convention is to choose $L_3$.

The third AICOM required for LA-integrability in $3D$ is provided by the square of the total angular momentum, given by

$$L^2 = L_1^2 + L_2^2 + L_3^2. \tag{17}$$

$L^2$ is a *Casimir invariant*: i.e., it is a COM (as is easily verified) that is in involution with all three generators of rotations, $L_1, L_2$ and $L_3$. Moreover, it is clearly not solely a function of any single component of the AM such as $L_3$. The set of three AICOMs in involution, namely $(H, L_3, L^2)$, then guarantees that the central force problem in $3D$ is LA-integrable.

The motion in the physical 3-dimensional space is always restricted to some plane determined by the initial conditions, because the constancy of $\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\mathbf{r}\times d\mathbf{r}/dt$ implies that the plane formed by the instantaneous position vector **r** and its increment $d$**r** remains the same as time elapses.

But rotational invariance, which makes *every* component of **L** a COM, does more. $H, L_1, L_2, L_3$ comprise four AICOMs in the 6-dimensional phase space. ($L^2$ is not counted in this set because it is a function of the COMs $L_1, L_2$ and $L_3$ that have already been included.) Every phase trajectory lies on some 2-dimensional surface that is the intersection of the level surfaces of the four AICOMs $H, L_1, L_2$ and $L_3$. The problem is again reduced to quadratures: the differential equation for the radial coordinate $r$ is again given by (14), where $\ell^2$ now stands for the value of $L^2$ as given by (17). The motion in the physical 3-dimensional space (the configuration space of the one-particle system at hand) is always restricted to some plane determined by the initial conditions, because the constancy of $\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\mathbf{r}\times d\mathbf{r}/dt$ implies that the plane formed by the instantaneous position vector $\mathbf{r}$ and its increment $d\mathbf{r}$ remains the same as time elapses.

The interesting question that arises naturally is: What happens to the central force problem if the spatial dimensionality $n$ increases beyond 3? What is the role played by rotational invariance via the conservation of orbital angular momentum? We now address these questions.

## Integrability of the $n$-dimensional Case

Consider a particle of mass $m$ moving under the influence of a central potential $V(r)$ in $n$-dimensional space, where $n$ is any positive integer $> 3$. The Hamiltonian is the same as in (12), but $\mathbf{r}$ and $\mathbf{p}$ now have Cartesian components $(x_1, \ldots, x_n)$ and $(p_1, \ldots, p_n)$, respectively. These dynamical variables satisfy the canonical PB relations given by (3), with $q_i \equiv x_i$.

(i) Is $H$ LA-integrable, i.e., are there exactly $n$ AICOMs in involution?

(ii) Further, is $H$ *super-integrable*, which requires the existence of $s$ AICOMs with no explicit $t$-dependence, where $n < s \leq 2n - 1$? If so, what is the value of $s$?

As in the 3D case, we note that $H$ is unchanged under rotations about the origin of coordinates in $n$-dimensional space. The corresponding generators of rotations, the components of the orbital angular momentum, must be COMs. Unlike 3D, however, the orbital AM is not a vector, but an antisymmetric tensor of rank 2. Its Cartesian components are given by:

$$L_{ij} = x_i\, p_j - x_j\, p_i \quad (1 \leq i, j \leq n). \tag{18}$$

Since $L_{ii} \equiv 0$ and $L_{ij} = -L_{ji}$, we only need to deal with the $\binom{n}{2} = \frac{1}{2}n(n-1)$ components $L_{ij}$ where $1 \leq i < j \leq n$. [$\binom{n}{r}$ is the modern notation for ${}^nC_r$, and we will use this notation.] The 3D case corresponds to the identification

$$(L_1, L_2, L_3) \equiv (L_{23}, -L_{13},\, L_{12}) \quad \text{(3D case)}. \tag{19}$$

The AM reduces to a vector $\mathbf{L}$ in that case alone: note that $\binom{n}{2} = n$ for $n = 3$. Returning to the $n$D case, we may use the canonical PBs (3) and the defining relation (18), to verify that $\{L_{ij}, H\} = 0$ for each component of the AM tensor. The set $\{L_{ij}\}$ thus provides $\binom{n}{2}$ COMs. But this seems to pose a problem, because $\binom{n}{2} = 2n = 10$ already for $n = 5$; and when $n > 5$, the number of components $\binom{n}{2}$ is greater than the dimensionality $2n$ of the phase space itself! In fact, $\binom{n}{2}$ increases *quadratically* with increasing $n$, while $2n$ increases linearly. Since $2n - 1$ is the upper bound on the possible number of $t$-independent AICOMs, it follows that all the $L_{ij}$'s *cannot* be algebraically (or functionally) independent of each other. There *must* exist identities connecting different members of the set, reducing the number of algebraically independent members to a certain value $r$ that increases no faster than linearly with $n$. We will return to this point in the next section.

We need to find, first, the Poisson bracket algebra satisfied by the components of the AM. Once again, using the defining relation (18), the chain rule (4) and the canonical PBs (3), we get

$$\{L_{ij},\, L_{kl}\} = \delta_{ik}\, L_{jl} + \delta_{jl}\, L_{ik} - \delta_{il}\, L_{jk} - \delta_{jk}\, L_{il}. \qquad (20)$$

This is the angular momentum algebra in $n$ dimensions, i.e., it is the Lie algebra $\mathfrak{so}(n)$ of the infinitesimal generators of the Lie group $SO(n)$, here represented by the components of the orbital AM as given by (18). It follows immediately from (20) that

$$\{L_{ij},\, L_{kl}\} = 0 \ \text{ when } i, j, k, l \ \text{ are all unequal.} \qquad (21)$$

When there is a common index between the first pair and the second pair, the PB is non-vanishing. For instance, if $i = k$ and $j, l$ are distinct and unequal to $i$, we have

$$\{L_{ij},\, L_{il}\} = L_{jl}. \qquad (22)$$

We can now write down the set of components of the AM that are in involution with each other. This set is not unique, but the *number* of elements it has is unique. Suppose the dimensionality is even, so that $n = 2\nu$ where $\nu$ is a positive integer. A possible set in involution that immediately suggests itself is the following one with 'non-overlapping' indices,

$$L_{12},\ L_{34},\ L_{56},\ \ldots,\ L_{2\nu-1,\,2\nu}. \tag{23}$$

A little thought shows that no other component $L_{ij}$ can be in involution with the set of components listed in (23). Hence this set represents a maximal set of components in involution. Note that there are exactly $\nu = \frac{1}{2}n$ members in this set. These components form what is known as the *Cartan sub-algebra* of the Lie algebra $\mathfrak{so}(2\nu)$. The *rank* of this Lie algebra is $\nu$. Now suppose $n$ is odd, so that $n = 2\nu + 1$. The set (23) is again a maximal set of components in involution. An alternative set in this case would be

$$L_{23},\ L_{45},\ L_{67},\ \ldots,\ L_{2\nu,\,2\nu+1}. \tag{24}$$

Once again, there are exactly $\nu$ members in either set, but now $\nu = \frac{1}{2}(n-1)$. Either of the sets (23) or (24) comprises the Cartan sub-algebra of the Lie algebra $\mathfrak{so}(2\nu+1)$. It is more convenient to choose the set in (24) when $n$ is odd, and we shall do so. The rank of this Lie algebra is also $\nu$. Thus, we have obtained $\nu = \frac{1}{2}n$ (respectively, $\frac{1}{2}(n-1)$) AICOMs in involution with each other, when $n$ is even (respectively, odd).

Next, we seek mutually independent *functions* of the $L_{ij}$'s that are in involution with each other, as well as with the already-identified AICOMs in (23) (respectively, (24)). Clearly, these functions must also involve $L_{ij}$'s that are *not* in these lists—or else, they would not be functionally independent of the $L_{ij}$'s already counted in as AICOMs in involution. In analogy with the $3D$ case, we may turn to the Casimir invariants of the Lie algebras $\mathfrak{so}(2\nu)$ or $\mathfrak{so}(2\nu+1)$ for this purpose. These invariants are COMs that are (i) functions of the $L_{ij}$'s, and (ii) in involution with all $\binom{n}{2}$ of them. Now, it follows from a result in the theory of Lie algebras that the number of independent Casimir invariants is equal to the rank of the algebra. In the present instance, the rank is given by $\nu = [\frac{1}{2}n]$, the largest integer $\leqslant \frac{1}{2}n$. It turns out that the corresponding Casimir invariants are symmetric polynomial

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functions of the $L_{ij}$'s of the following orders. For $n = 2\nu$, the orders are $2, 4, \ldots, 2\nu - 2$, and also $\nu$; while for $n = 2\nu + 1$, the orders are $2, 4, \ldots, 2\nu$. (These are the so-called *Betti numbers* of the groups $SO(2\nu)$ and $SO(2\nu+1)$, respectively.) In principle, the Casimir invariants, the $L_{ij}$'s in (23) or (24), and the Hamiltonian suffices to form a set of exactly $n$ AICOMs in involution. This is the number needed for LA-integrability of the central force problem in $n$ dimensions. It is more convenient and instructive, however, to replace the Casimir invariants with a set of AICOMs that are *quadratic* functions of the $L_{ij}$'s. The latter fulfil all the requirements specified, and their mutual functional independence is manifest (see [2] in 'Suggested Reading').

Consider the partial sum of squares of the $L_{ij}$'s

$$C_N = \tfrac{1}{2} \sum_{i,j=1}^{N} L_{ij}^2, \tag{25}$$

where $N$ is any positive integer $\geqslant 2$. Using the algebra (20) and the chain rule (4), it is straightforward to check that

$$\{C_N,\, L_{ij}\} \begin{cases} = \;\; 0 & \text{for } N \leqslant i < j \;\text{ or }\; i < j \leqslant N, \\ \neq \;\; 0 & \text{for } i \leqslant N < j. \end{cases} \tag{26}$$

The cases of even $n$ and odd $n$ differ slightly from each other. When $n$ is an even number ($= 2\nu$), consider the set of $\nu$ COMs given by,

$$C_2,\, C_4,\, C_6,\, \ldots,\, C_{2\nu}. \tag{27}$$

It is evident that these quantities are algebraically independent of each other (since new components of the AM are involved whenever $N$ increases). Using (26), it is straightforward to check that they are in involution with each other and with each of the AM components in (23). But $C_2$ is just $L_{12}^2$, and must be omitted because it is solely a function of the first member $L_{12}$ of the set in (23). This leaves $\nu - 1$ AICOMs ($C_4$ to $C_{2\nu}$) that are in involution with each other. Together with the set of AM components listed in (23), we thus have $\nu + \nu - 1 = n - 1$ AICOMs in involution with each other. When the Hamiltonian $H$ in (12) is also included in the set, we get precisely $n$ AICOMs in involution (the largest number possible), namely,

$$\underbrace{H,}_{1}\ \underbrace{L_{12},\, L_{34},\, \ldots,\, L_{2\nu-1,2\nu},}_{\nu}\ \underbrace{C_4,\, C_6,\, \ldots,\, C_{2\nu}}_{\nu-1}. \qquad (28)$$

The criterion for LA-integrability is satisfied. The central force problem in any even number of dimensions is, therefore, integrable.

When $n$ is an odd number $(= 2\nu + 1)$, the maximal set of AICOMs in involution may be chosen as follows (see [2]): the $\nu$ AM components listed in (24), the $\nu$ quadratic forms in (27), and the Hamiltonian. This yields $\nu + \nu + 1 = n$ AICOMs in involution with each other (the largest number possible), namely,

$$\underbrace{H,}_{1}\ \underbrace{L_{23},\, L_{45},\, \ldots,\, L_{2\nu,2\nu+1},}_{\nu}\ \underbrace{C_2,\, C_4,\, \ldots,\, C_{2\nu}}_{\nu}. \qquad (29)$$

Hence the central force problem in any odd number of dimensions is also LA-integrable.

## Super-integrability in the $n$-dimensional Case

Next, we turn to the question of super-integrability of the central force problem in $n$ dimensions. Exactly how many explicitly time-independent AICOMs $(= s)$ are there in this instance? As already mentioned, all the $\binom{n}{2}$ components of the AM cannot be algebraically independent, and there must exist identities between them. We must, therefore, find, first, the number $r$ of independent components among the set of $L_{ij}$'s. As we have an explicit expression for the orbital AM components in terms of the coordinates and momenta, it is not difficult to find the simple identity satisfied by these components.

Consider the product

$$L_{12}\, L_{34} = (x_1\, p_2 - x_2\, p_1)(x_3\, p_4 - x_4\, p_3). \qquad (30)$$

A little algebra shows that we may write this in the form

$$L_{12}\, L_{34} = L_{13}\, L_{24} - L_{14}\, L_{23}. \qquad (31)$$

Since $1, 2, 3, 4$ are distinct indices, we may replace them with the distinct indices $i, j, k, l$ to obtain the identity

$$L_{ij} L_{kl} = L_{ik} L_{jl} - L_{il} L_{jk}. \tag{32}$$

Equation (32) is satisfied whenever $i, j, k, l$ are distinct indices, for all values of the indices from 1 to $n$. When any two of the indices are equal, the identity reduces to a triviality. Note that all six pairwise combinations of the four indices $i, j, k, l$ occur in (32). Hence the AM component corresponding to any pair can be written in terms of the other five components.

A remark is in order here on the nature of the identity (32). We know that the components of the AM satisfy the Lie algebra $\mathfrak{so}(n)$, and that they are the infinitesimal generators of the Lie group $SO(n)$. One might imagine that (32) is a general relation between these generators that is valid for all representations of the latter. This is not so; (32) is representation-dependent. We shall not digress into further technical details here. In the case of interest to us, namely, the representation of the orbital AM in terms of canonical coordinates and momenta in phase space, the validity of the relation has been established explicitly.

We must now determine the actual number of *independent* identities of the form (32), and subtract it from $\binom{n}{2}$ to arrive at the number $r$ of independent AM components. Once again, all relations of the form (32) cannot be independent! There are $\binom{n}{4}$ ways of choosing 4 distinct indices $i, j, k, l$ from $n$ indices, and each choice leads to a relation like (32). But $\binom{n}{4}$ exceeds the number $\binom{n}{2}$ of components $L_{ij}$ for all $n > 6$.

A brute force method of finding the number $r$ of functionally independent $L_{ij}$'s is as follows. Write down the $\binom{n}{2} \times 2n$ *Jacobian matrix* of the $\binom{n}{2}$ components $L_{12}, L_{13}, \ldots, L_{n-1,n}$ with respect to the $2n$ phase space variables $x_1, \ldots, x_n, p_1, \ldots, p_n$. By elementary row operations, reduce to zero all the elements of as many rows as possible. The number of such rows is the nullity of the matrix, while the number of the rest of the rows is the rank of the matrix. This rank is precisely $r$, the number of algebraically independent components of the AM tensor. We have carried out this exercise using the *RowReduce* package in *Mathematica* for the cases from $n = 3$ up to $n = 7$. The results fit the formula $r = 2n - 3$.

Alternatively, we first deduce the number $\mu(n)$ of independent identities of the form (32) among the $L_{ij}$'s as follows. $\mu(n)$ can at best be a quadratic in $n$, to match (and to compensate for) the quadratic dependence on $n$ of the total number of components, which is $\binom{n}{2}$. We know that $\mu(2) = 0$ (there is only one component, $L_{12}$, in 2D) and $\mu(3) = 0$ (there is no relation between $L_{12}, L_{13}$ and $L_{23}$ in 3D). Therefore, $\mu(n)$ must be of the form $c(n-2)(n-3)$, where $c$ is a constant. We know, further, that $\mu(4) = 1$, because there is a single relation (31) between the $L_{ij}$'s in this case. This yields $c = \frac{1}{2}$, so that

$$\mu(n) = \tfrac{1}{2}(n-2)(n-3) = \binom{n-2}{2}. \qquad (33)$$

Hence the number of functionally independent components of the AM tensor in $n$ dimensions is

$$r = \binom{n}{2} - \binom{n-2}{2} = 2n - 3. \qquad (34)$$

But there is a simple and direct argument that yields the value of $r$ directly. Pick any one of the indices from 1 to $n$, say 1 (the same argument works for any other choice). We then have $n - 1$ independent components of the form $L_{1j}$, where $j$ runs from 2 up to $n$. No member of this set can be written entirely in terms of the other members of the set. Now pick any index other than 1, say 2 (the same argument works for any other choice). We then have $n-2$ independent components of the form $L_{2k}$, where $k$ runs from 3 up to $n$. Once again, none of these components can be written entirely in terms of the members of the sets $\{L_{1j}, 2 \le j \le n\}$ and $\{L_{2k}, 3 \le k \le n\}$. Therefore, these two sets comprise a total of $n - 1 + n - 2 = 2n - 3$ functionally independent components. Every other component $L_{jk}$, where $2 \le j \le n$ and $3 \le k \le n$, can be written in terms of these $2n - 3$ independent components by virtue of the identity (32): we have, explicitly,

$$L_{jk} = (L_{1j}L_{2k} - L_{1k}L_{2j})/L_{12}. \qquad (35)$$

| (a) | (b) | (c) | (d) | (e) | (f) | (g) |
|---|---|---|---|---|---|---|
| Space dim. | No. of comps. $L_{ij}$ | No. of relns. bet. $L_{ij}$ | # of in-dep. $L_{ij}$ ($r$) | COMs in involn. | Indep. COMs ($s$) | Phase space dim. |
| 2 | 1 | – | 1 | 2 | 2 | 4 |
| 3 | 3 | 0 | 3 | 3 | 4 | 6 |
| 4 | 6 | 1 | 5 | 4 | 6 | 8 |
| 5 | 10 | 3 | 7 | 5 | 8 | 10 |
| 6 | 15 | 6 | 9 | 6 | 10 | 12 |
| 7 | 21 | 10 | 11 | 7 | 12 | 10 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $n$ | $\binom{n}{2}$ | $\binom{n-2}{2}$ | $2n-3$ | $n$ | $2n-2$ | $2n$ |

**Table 1.** Number of COMs involving the orbital angular momentum in various dimensions.

This proves rigorously that the number of independent AM components is precisely $r = 2n - 3$, corroborating (34).

We are ready, finally, to deduce the extent of super-integrability that the central force problem in $n$ dimensions possesses, owing to the conservation of orbital angular momentum. Recall that the problem has already been shown to be LA-integrable for both even and odd values of $n$. We now find that $H$ and $\{L_{ij}\}$ together provide

$$s = 1 + r = 2n - 2 \tag{36}$$

$t$-independent AICOMs. We conclude that: *Rotational invariance alone suffices to make the general central force problem in n dimensions just one AICOM short of maximal super-integrability.*

Rotational invariance alone suffices to make the general central force problem in $n$ dimensions just one AICOM short of maximal super-integrability.

*Table* 1 summarises the various numbers involved for different values of $n$.

As in 3D, the path of the particle subjected to a central force in $n$D remains confined to some (2-dimensional) plane in $n$-dimensional space, for any set of initial conditions. A simple way to establish this is as follows. Let $\mathbf{r}(0)$ and $\mathbf{p}(0)$ be the initial position and momentum vectors of the particle. The dynamics is unchanged by rotations of the coordinate axes about the origin. Rotate the coordinate axes about the origin till $\mathbf{r}(0)$ lies along the 1-direction.

Then $x_1(0) \neq 0$, while $x_2(0) = \cdots = x_n(0) = 0$. Next, resolve $\mathbf{p}(0)$ into a component $p_1(0)$ along the 1-direction, and a component $\mathbf{p}_\perp(0)$ normal to this direction. Now rotate the coordinate axes about the 1-axis till the 2-axis coincides with the direction of $\mathbf{p}_\perp(0)$. Thus, the initial position and momentum have components

$$\left.\begin{array}{rcl} \mathbf{r}(0) &=& (x_1(0), 0, 0, \ldots, 0), \\ \mathbf{p}(0) &=& (p_1(0), p_2(0), 0, \ldots, 0). \end{array}\right\} \quad (37)$$

Hamilton's equations for the position coordinates are

$$\dot{x}_i = \partial H/\partial p_i = p_i/m, \quad 1 \leq i \leq n. \quad (38)$$

Setting $t = 0$, we find that $\dot{x}_i(0) = 0$ for $3 \leq i \leq n$. As both $x_i$ $(3 \leq i \leq n)$ and its time derivative vanish at $t = 0$, these coordinates never 'take off', and remain equal to zero for all $t$. The path of the particle is confined to the $(x_1, x_2)$ plane. That is, the path of the particle in configuration space is a planar curve.

This conclusion is corroborated by looking at the components of the momentum. Hamilton's equations give

$$\dot{p}_i = -\partial H/\partial x_i = -x_i V^{'}(r)/r, \quad 1 \leq i \leq n. \quad (39)$$

Again, setting $t = 0$ we find that $\dot{p}_i(0) = 0$ for $2 \leq i \leq n$. Since we already know that $p_i(0) = 0$ for $3 \leq i \leq n$, we may conclude that $p_i(t)$ remains equal to zero for all time for these components. (Note that we cannot draw the same conclusion for $p_2(t)$, since $p_2(0) \neq 0$ in general.) The motion, therefore, occurs entirely in the $(x_1, x_2, p_1, p_2)$ subspace of the $(2n)$-dimensional phase space.

What kind of path does the particle follow in the $(x_1, x_2)$ plane of configuration space? The answer depends on the potential $V(r)$. Once again, the problem is reduced to quadratures: the radial coordinate satisfies the same equation as it does in 2D and 3D, namely, (14). The quantity $\ell^2$ is again the value of the square of the total AM, which is now given by

What kind of path does the particle follow in the $(x_1, x_2)$ plane of configuration space? The answer depends on the potential $V(r)$.

$$L^2 \equiv C_n = \tfrac{1}{2} \sum_{i,j=1}^{n} L_{ij}^2. \quad (40)$$

Whether (14) can be solved *analytically* or not depends on the potential $V(r)$. For certain special potentials, this is possible, and we then have maximal super-integrability ($s = 2n - 1$). The reason why this happens lies in the existence of some extra symmetry ('dynamical symmetry') in the problem, over and above rotational symmetry. The most prominent examples of this possibility are the cases of the $1/r$ potential, or the Kepler problem, and the $r^2$ potential, or the $n$-dimensional isotropic harmonic oscillator. We will consider, below, the former of these two cases, as the symmetry involved here is less manifest in this instance than it is in the latter. (It turns out to be a nonlinear realisation of the symmetry.)

## The Kepler Problem in 3 Dimensions

Let us recapitulate briefly the standard Kepler problem in 3D, in order to prepare the ground for the problem in $n$D (which is the focus of our attention here). The Hamiltonian of the Kepler problem is

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} - \frac{\kappa}{r}, \tag{41}$$

where $\kappa$ is a positive constant. (We restrict our attention to an attractive potential, as this is the case relevant to bounded motion in space.) It is a standard exercise in classical mechanics to show that the negative energy ($E < 0$) solutions of the equations of motion correspond to periodic motion in elliptical orbits satisfying Kepler's three laws of planetary motion.

We are interested here in the super-integrability of the Hamiltonian (41). It is well known that this is brought about by the existence of another vector COM besides the angular momentum $\mathbf{L}$, namely, the Laplace-Runge-Lenz vector given by

$$\mathbf{A} = (\mathbf{p} \times \mathbf{L}) - \frac{m\kappa\mathbf{r}}{r}. \tag{42}$$

It is straightforward to verify that the PB $\{\mathbf{A}, H\}$ vanishes identically. (In doing so, it is helpful to replace $-\kappa/r$ with $H - \mathbf{p}^2/(2m)$.) On the face of it, it appears that we now have too many COMs to be accommodated in 6-dimensional phase space, because $H, \mathbf{L}$ and $\mathbf{A}$ comprise a set of seven COMs. But two distinct relations connect $\mathbf{A}$ to $H$ and $\mathbf{L}$. First, we have

$$\mathbf{A} \cdot \mathbf{L} = 0, \tag{43}$$

showing that $\mathbf{A}$ lies always in the plane of the orbit of the particle in configuration space. Second, we find that the magnitude of $\mathbf{A}$ is determined by the values of $H$ and $L^2$ via the relation

$$A^2 = 2mHL^2 + m^2\kappa^2. \tag{44}$$

The vector $\mathbf{A}$, therefore, contributes just one more AICOM, rather than three, to the set $H, L_1, L_2, L_3$. This brings the total number of explicitly $t$-independent AICOMs to five, making the Kepler problem in 3D *maximally* super-integrable.

The geometrical significance of the Laplace-Runge-Lenz vector is as follows. $\mathbf{A}$ is directed along the semi-major axis of the elliptical orbit of the particle, from the origin of coordinates (the centre of attraction) towards the perigee[1]. The magnitude of $\mathbf{A}$ specifies the eccentricity of the ellipse, which is given by $\varepsilon = A/(m\kappa)$. But the most important aspect is the following: the existence of the vector COM $\mathbf{A}$ does not merely add an AICOM to $H$ and $\mathbf{L}$ to make the Hamiltonian maximally super-integrable.

[1]The point of closest approach to the origin.

- *The existence of the COM* $\mathbf{A}$ *also enlarges the group of transformations under which the set of solutions of Hamilton's equations for a given value of the energy $E$ is mapped to itself.*

Individual solutions belonging to this set transform into each other. This group is known as the *dynamical symmetry group* of the given Hamiltonian. (See *Box* 1.) The Lie algebra of this group is as follows. Define the vector COM

$$\mathbf{K} = \mathbf{A}/\sqrt{2m|E|}. \tag{45}$$

(Recall that we are concerned here with solutions corresponding to $E < 0$, i.e., motion in elliptical orbits in physical space.) Then, over and above the AM algebra (16), we also have (for any given $E < 0$)

$$\left.\begin{array}{lll}\{K_1, , L_2\} = K_3, & \{K_3, , L_1\} = K_2, & \{K_2, , L_3\} = K_1 \\ \{K_1, , K_2\} = L_3, & \{K_3, , K_1\} = L_2, & \{K_2, , K_3\} = L_1.\end{array}\right\} \quad (46)$$

Equations (16) and (46) for the PBs of the components of $\mathbf{L}$ and $\mathbf{K}$ represent the Lie algebra $\mathfrak{so}(4)$ of the six generators of the 4-dimensional rotation group $SO(4)$. This fact emerges more clearly if we define the COMs $\mathbf{M} = \frac{1}{2}(\mathbf{L} + \mathbf{K})$ and $\mathbf{N} = \frac{1}{2}(\mathbf{L} - \mathbf{K})$. Then, the three components of $\mathbf{M}$ satisfy the 3D angular momentum Lie algebra, as do the three components of $\mathbf{N}$; while they are in involution with each other, i.e., $\{M_i, N_j\} = 0$. It only remains to invoke the group isomorphism $SO(3) \otimes SO(3) = SO(4)$.

Thus, $SO(4)$ is the dynamical symmetry group of the 3D Kepler problem in the $E < 0$ sector (as far as continuous symmetries are concerned). For a given value of $E$, the transformations generated by $L_i$ (spatial rotations) take us from one elliptical orbit to another, both in the original plane and in all other planes; the transformations generated by $K_i$ change the shape (i.e., the eccentricity) of the orbit.

## The Kepler Problem in *n* Dimensions

Let us move on to the main concern here, namely, the situation in $n$D. The Hamiltonian is the same as in 3D, i.e., as in (41), but $\mathbf{r}$ and $\mathbf{p}$ are now $n$-component vectors. The orbital angular momentum is an antisymmetric tensor of rank 2 whose general component $L_{ij}$ is given by (18). As we have seen, the set of functional relations (32) between the $L_{ij}$'s reduces the $\binom{n}{2}$ components of the tensor to $2n - 3$ $t$-independent AICOMs. Together with $H$, this makes $2n - 2$ such AICOMs, one short of maximal superintegrability.

Remarkably enough, the Kepler Hamiltonian for general $n$ continues to have an additional vector COM. This is the $n$-component Laplace-Runge-Lenz vector $\mathbf{A}$ whose $i^{\text{th}}$ component is given by (see [3] and [7]),

$$A_i = \sum_{j=1}^{n} L_{ij}\, p_j - \frac{m\kappa x_i}{r} \quad (1 \le i \le n). \tag{47}$$

It is straightforward to check that $\{A_i, H\} = 0$, i.e., that each $A_i$ $(1 \le i \le n)$ is a COM. We can go on to show that the existence of the vector COM **A** ensures that the basic features of the Kepler problem continues to be valid in $n$ dimensions:

(i) All three of Kepler's laws regarding elliptical orbits remain valid in $n$ dimensions.

(ii) **A** lies in the plane of the orbit of the particle.

(iii) The magnitude of **A** satisfies (44), $L^2$ now being given by (40).

(iv) There are exactly $n-2$ independent conditions that are analogous to the condition $\mathbf{A} \cdot \mathbf{L} = 0$ that applies in 3D.

(v) Together with the relation expressing $A^2$ in terms of $H$ and $L^2$, this implies $n-1$ conditions on **A**. The Laplace-Runge-Lenz vector therefore provides just one additional AICOM (as it does in 3D), making the Kepler problem in $n$ dimensions maximally super-integrable.

As before, our interest here is in the extension of the symmetry group of the Kepler Hamiltonian that is associated with the COM **A**. The AM components, of course, satisfy the algebra of Poisson brackets given by (20). Repeating that equation for ready reference, we have

$$\{L_{ij}, L_{kl}\} = \delta_{ik}\, L_{jl} + \delta_{jl}\, L_{ik} - \delta_{il}\, L_{jk} - \delta_{jk}\, L_{il} \quad (1 \le i, j, k, l \le n). \tag{48}$$

Defining the vector $\mathbf{K} = \mathbf{A}/\sqrt{2m|E|}$ as in the 3D case, we find the PB relations

$$\{K_j, L_{kl}\} = \delta_{jl}\, K_k - \delta_{jk}\, K_l \ \text{ and } \ \{K_i, K_j\} = L_{ij}. \tag{49}$$

All indices run from 1 to $n$ in (48) and (49). It is clear that the $\binom{n}{2}$ components of the angular momentum tensor and the $n$ components of the Laplace-Runge-Lenz vector form a closed algebra of Poisson brackets. Moreover, $\binom{n}{2} + n = \binom{n+1}{2}$. Indeed, we may define

$$K_i \equiv L_{i,n+1} \quad (1 \le i \le n) \tag{50}$$

to find that (48) and (49) can be combined and re-written in the form given by Eq. (48), but where *all the indices now run from* 1 *to* $n + 1$. That is,

$$\{L_{ij},\, L_{kl}\} = \delta_{ik}\, L_{jl} + \delta_{jl}\, L_{ik} - \delta_{il}\, L_{jk} - \delta_{jk}\, L_{il} \;\; (1 \le i, j, k, l \le n+1). \tag{51}$$

But this is precisely the Lie algebra $\mathfrak{so}(n+1)$ satisfied by the $\frac{1}{2}n\,(n+1)$ generators of the rotation group $SO\,(n+1)$ in $n+1$ dimensions. We conclude that $SO\,(n+1)$ is the dynamical symmetry group of the Kepler Hamiltonian in $n$ dimensions, in the case of the closed orbits (or $E < 0$). In other words:

Transformations of the dynamical symmetry group maps the solutions of the equations of motion corresponding to a given value of the total energy $E$ to other solutions with the same value of $E$.

- *The set of continuous canonical transformations of the* $2n$ *phase space variables* $x_i$, $p_i$ $(1 \le i \le n)$ *that leave the Hamiltonian of the Kepler problem unchanged (for any given* $E < 0$*) is in one-to-one correspondence with the set of rotations in* $(n+1)$*-dimensional space.*

What do the transformations of the dynamical symmetry group do? They map the solutions of the equations of motion corresponding to a given value of the total energy $E$ to other solutions with the same value of $E$. In configuration space, this includes transformations that change the direction of the major axis of an elliptical orbit, and/or its plane, and/or its eccentricity, without changing $E$.

While invariance under rotations in the $n$-dimensional space is manifest in the central force problem, this extended symmetry in phase space is not. That is why dynamical symmetry is sometimes referred to as hidden symmetry.

For the sake of completeness, we mention that the corresponding dynamical symmetry group in the case of a parabolic orbit (or $E = 0$) is the Euclidean group of translations and rotations in $n$ dimensions. In the case of hyperbolic orbits (or $E > 0$), the dynamical symmetry group is the special Lorentz group $SO\,(1, n)$ in $(n+1)$-dimensional space-time. Each of these groups has the same number of generators as $SO\,(n+1)$, namely, $\frac{1}{2}n\,(n+1)$.

While invariance under rotations in the $n$-dimensional space is manifest in the central force problem, this extended symmetry in phase space is not. That is why dynamical symmetry is sometimes referred to as *hidden symmetry*.

A few remarks are in order on the quantum mechanical counterpart of the foregoing. As is well known, functions of the phase space variables must now be replaced by operators, Poisson brackets by commutators (divided by $i\hbar$), Hamilton's equations by the Schrödinger equation for the wave function (or Heisenberg's equations of motion for the operators), and so on. Functions of the phase space variables that are in involution are replaced by operators that commute with each other. Attention must be paid to the ordering of non-commuting operators. We have seen that the conservation of orbital AM reduces the general central force problem in classical mechanics to quadratures, i.e., to the solution of the differential equation for the radial coordinate $r(t)$. In quantum mechanics, the problem reduces to the solution of the Schrödinger equation for the radial wave function. Turning to the case of an attractive $1/r$ potential, the dynamical symmetry group for the Kepler problem in $n$ dimensions remains $SO(n+1)$, with the same generators as in the classical case. As a consequence of a symmetry larger than rotational invariance, the bound-state or discrete spectrum of the Hamiltonian enjoys the property of 'accidental' degeneracy, i.e., the energy levels do not depend on the orbital AM quantum number $\ell$. It is interesting to note that, in $n$ spatial dimensions, the eigenvalues of $L^2$ turn out to be $\hbar^2\,\ell\,(\ell+n-2)$, which reduces to the familiar expression $\hbar^2\,\ell\,(\ell+1)$ when $n=3$.

Finally, one may ask whether the Kepler problem has an even larger dynamical symmetry group than $SO(n+1)$. This seems to be unlikely because we have already identified $2n-1$ time-independent COMs, i.e., we have maximal super-integrability. A better argument is that all possible transformations that take an elliptical orbit to another elliptical orbit has been accounted for by the group $SO(4)$ (in the 3D case) or $SO(n+1)$ (in the $n$D case). This heuristic answer turns out to be correct. On the other hand, suppose we broaden the definition of dynamical symmetry to include transformations that connect phase trajectories corresponding to different energies, not just within the individual sectors $E < 0$, $E = 0$ and $E > 0$, but also across these sectors. Then, remarkably enough, such an enlarged set of transformations also form a Lie group, the special conformal group $SO(2, n+1)$. We defer discussion of this aspect to another occasion!

**Box 1. Dynamical Symmetry**

Noether's theorem connects symmetry, invariance and conservation laws in a wide variety of dynamical systems. It is most familiar in the context of the Lagrangian formalism—a continuous symmetry transformation that leaves the action invariant leads to a conserved quantity. The classical, Hamiltonian version of this result is as follows.

If a function $G(q, p)$ of the dynamical variables is the generator of an infinitesimal canonical transformation $q_i \to q_i + \delta q_i$ and $p_i \to p_i + \delta p_i$ then

$$\delta q_i = \epsilon \{q_i, G(q, p)\} \text{ and } \delta p_i = \epsilon \{p_i, G(q, p)\},$$

where $\epsilon$ is the infinitesimal parameter. If, further, $G(q, p)$ is a COM, then the transformation belongs to the dynamical symmetry group of the Hamiltonian concerned. In other words, the dynamical symmetry group of a Hamiltonian is the maximal group of canonical transformations such that all its generators are COMs.

In the central force problem in 3D, we have time-translation invariance, leading to the conservation of energy. The Hamiltonian $H$ is, of course, the generator of infinitesimal translations in time. Time translations comprise the one-parameter group $T(1)$. There is no conservation of linear momentum (the particle moves under the influence of a force!) There is, however, rotational invariance, leading to the conservation of the angular momentum (AM) $\mathbf{L}$ about the centre of force. The generators of infinitesimal rotations are $(L_1, L_2, L_3)$. These generators satisfy the Lie algebra $\mathfrak{so}(3)$ of the 3D rotation group $SO(3)$, as given by (16).

In classical dynamics, the bilinear operation associated with the Lie algebra is the Poisson bracket. The set of COMs $(H, L_2, L_2, L_3)$ correspond to the generators of *kinematical*, space-time symmetries of the general central force problem in 3D—namely, invariance under time translation and spatial rotations, respectively. Discrete symmetries such as parity invariance must also be included, but here we restrict our attention to continuous symmetries. Since all components of the AM are in involution with $H$, the kinematical symmetry group is a direct product of the form $T(1) \otimes SO(3)$. It is customary to drop the factor $T(1)$ (i.e., it is understood to be present) and to refer to $SO(3)$ as the kinematical symmetry group of the central force problem in 3D.

*Contd.*

**Box 1.** ***Contd.***

In the special case of the $1/r$ potential (the Kepler problem), there is a larger, 'hidden' symmetry group of transformations in phase space that leave the Hamiltonian unchanged. In the sector restricted to bounded motion ($E < 0$) in an attractive $1/r$ potential, the largest group with this property is the *dynamical* symmetry group $SO(4)$, whose Lie algebra $\mathfrak{so}(4)$ has six generators: the three components of the AM $\mathbf{L}$, and the three components of the vector $\mathbf{A}/\sqrt{-2mH}$, where $\mathbf{A}$ is the Laplace-Runge-Lenz vector. This algebra is represented by the Poisson bracket relations in (16) and (46).

In the case of the general central force problem in $n$ dimensions, we again have time-translation invariance, the generator of time translations being $H$ as always. Rotational invariance leads to the conservation of AM. The generators of rotations in $n$-dimensional space are the $\binom{n}{2}$ components of the AM tensor $L_{ij}$, which satisfy the Lie algebra $\mathfrak{so}(n)$ as given by (20). The kinematical symmetry group is now $SO(n)$. Once again, the $1/r$ potential (or the Kepler problem) enjoys an additional 'hidden' symmetry, owing to the existence of a $n$-component Laplace-Runge-Lenz vector. The components of this vector, together with the $L_{ij}$'s, generate the Lie algebra $\mathfrak{ao}(n+1)$, as given by (51). The dynamical symmetry group of the Kepler problem in $n$ dimensions is $SO(n+1)$, for the class of bound state solutions.

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