What are Operads?*

Anita Naolekar

This article aims to provide an informal introduction to 'operads'. We work our way through motivations, examples, results and finally, give a sketchy description of the celebrated Koszul duality theory of operads.

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1. Historical Background

An operad is a mechanism that captures the results of computations of multilinear operations for various algebraic structures. Often, an additional feature allows one to keep track of the order in which computations are performed, and not just of the final result of the operations.

As history reveals, the first traces of the definition of operads is found in the article of M. Lazard [1] on formal groups. Later, the idea was primarily developed in Chicago in the 1960's. The theory of operads emerged as a tool in algebraic topology initially in the works of Frank Adams, Michael Boardmann and Rainer Vogt [2], Saunders McLane [3], Jim Stasheff [4], J. Peter May [5], and other topologists for understanding loop spaces. The name though was coined by Peter May, as he says, "The name 'operad' is a word that I coined myself, spending a week thinking about nothing else". This word is actually a contraction of two words—'operation' and 'monad'. About 30 years later, operads appeared in many other areas of mathematics, like differential geometry, non-commutative geometry, operator theory, category theory, quantum field theory, string topology, combinatorics, and even computer science.

Keywords
Operads, associative, commutative, Lie algebra, unitary condition, Koszul duality .

Loosely speaking, an operad is a sequence of sets $\{\mathcal{P}(n), n \geq 0\}$ (often with richer structures like vector spaces, topological spaces, chain complexes, simplicial sets, etc.) with composition

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laws, action of the symmetric group S_n on $\mathcal{P}(n)$ for all n, and a 'unit' in $\mathcal{P}(1)$, which intertwine by certain rules.

The most natural example is the sequence of sets $\{\mathrm{Map}(X^n, X), n \geq 0\}$, for any set X. Here $\mathrm{Map}(X^n, X)$ denotes the set of functions from the n-fold cartesian product of X to itself. There is more to this apparently ordinary sequence of sets than meets the eye. There are composition laws, symmetric group actions, and unitary conditions which all glue together in the form of an operad. Though in this case, each $\mathrm{Map}(X^n, X)$ is merely a set, operads can be defined over any symmetric monoidal category, i.e. a category where a 'product' makes sense, and this product is 'commutative' in some sense. Thus, we can define operads in the category of vector spaces, in the category of topological spaces, in the category of chain complexes, and so on. In the course of this article, we shall see examples of operads in both, the category of vector spaces (called algebraic operads) and also in the category of topological spaces (called topological operads).

We can define operads in the category of vector spaces, in the category of topological spaces, in the category of chain complexes, and so on.

Then comes the ubiquitous influence of representations. Representations of an operad $\mathcal{P}$ are named as $\mathcal{P}$-algebras or algebras over $\mathcal{P}$. Algebras which are defined by n-ary operations and relations, for some $n \geq 2$ arise as algebras over operads. We shall give the construction of operad $\mathcal{A}ss$ and operad $\mathcal{C}om$, 'algebras' over which are the associative and commutative algebras respectively. Besides the classical types of algebras that we are familiar with, e.g. associative algebras, commutative algebras and Lie algebras, in the last few decades, a lot of work is being done in trying to understand other types of algebras such as Poisson algebras, Gerstenhaber algebras, Jordan algebras, Batalin-Vilkovisky algebras, to name a few. There are 'algebras up to homotopy' or infinity algebras as well, like A_∞, L_∞ and so on. Operads are devices to encode a particular type of algebra. Given a type of algebra, the operad $\mathcal{P} = \{\mathcal{P}(n); n \geq 0\}$ which encodes this type of algebra has as $\mathcal{P}(n)$ the space of all n-ary operations.

One of the very first examples of operads was the Stasheff's *associahedra* operad, which was discovered by Jim Stasheff in 1961. Let X be a topological space and let $S^1 = \{z \in \mathbb{C} : ||z|| = 1\}$. Let

ΩX be its based loop space, i.e. ΩX is the space of all continuous functions $f : (S^1, 1) \longrightarrow (X, x_0)$ sending $1 \in S^1$ to the fixed basepoint $x_0 \in X$. There is a multiplication given by concatenation in ΩX, which is not associative, i.e. for loops f, g and h, $(fg)h$ and $f(gh)$ are different. However, there is a homotopy between them. Now, given four loops, there are five ways of parenthesizing them. Also, not only are there homotopies between any pair, but there are homotopies between homotopies, and so on. These higher homotopies are encoded in an operad called Stasheff's polytope or associahedra operad. This operad has been described in detail in Section 3. Stasheff also proved that a 'nice' connected space Y has the homotopy type of a based loop space ΩX for some space X, if and only if, Y is an 'algebra' over the associahedra operad.

With all this said and done, one may still be wondering, *why should we care about operads?*.

There are several advantages of using the language of operads. Here are a few:

- Many existing results for classical algebras, when seen from the operadic point of view, can be proved for other types of algebras.
- Abstraction in the language of operads enables us to simplify the statements as well as the proofs of the theorems.
- Operadic point of view has provided many new results, which were unknown even for classical algebras.

Though it is nice to know that operads have many advantages, in this article, we shall not impinge into establishing any of these. We shall rather concentrate on understanding them via several examples.

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2. Operads: Motivation and Definition

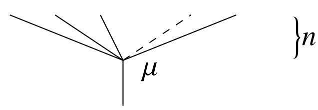
In this section, we talk about the most canonical example of operads, $\{\text{Map}(X^n, X), n \geq 0\}$, for a set X, and by exploring the composition products, and the actions of the symmetric groups, work our way towards the formal definition of an operad.

Given any set X and $n \geq 1$, let us look at the set Map(X^n, X) of all functions from the n-fold cartesian product of X to itself. These are also the set of all n-ary operations on X. We define Map(X^0; X) to be the null set. This sequence of sets {Map(X^n, X), $n \geq 0$} is the toy model for operads.

A tree is a connected finite graph with no cycles. A rooted tree is a tree with a designated vertex called the root. A planar tree is a rooted tree in which an ordering is specified for the edges of each vertex.

A tree is a connected finite graph with no cycles. A rooted tree is a tree with a designated vertex called the root. A planar tree is a rooted tree in which an ordering is specified for the edges of each vertex. Leaves are the edges which are connected to only one vertex, except the root. A convenient way to understand the operations in {Map(X^n, X), $n \geq 0$} is via planar trees. There is, however, a much deeper connection between operads and planar trees, besides representing operations, which we shall not delve into.

Let us denote n-ary operations (or n-inputs and 1-output) in Map(X^n, X) graphically by corollas, which are planar trees with only one vertex, i.e. the root and n leaves, as shown in the diagram below:



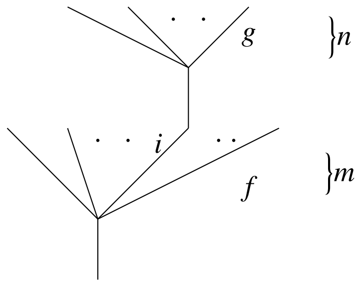
We can make sense of composing these n-ary operations in the following way. If f is a m-ary operation and g is a n-ary operation, then $f \circ_i g$ is a $(m+n-1)$-ary operation, for all $1 \leq i \leq m$, defined in the following way:

$$\circ_i : \text{Map}(X^n; X) \times \text{Map}(X^m; X) \longrightarrow \text{Map}(X^{n+m-1}; X)$$

, is given, for $1 \leq i \leq n$, by

$$\begin{aligned} & (f \circ_i g)(x_1, \ldots, x_{n+m-1}) \\ = \quad & f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+m-1}), x_{i+m}, \ldots, x_{n+m-1}). \end{aligned}$$

This is illustrated in the figure below.



$f \circ_i g$

Here the output of g is inserted as the i-th input of f.

These $\circ_i$ operations satisfy three 'natural' axioms. Let us try to understand these axioms again via planar trees, before we state the formal (somewhat intimidating at first sight) definition of an operad.

Associativity of the Composition

Let f, g, and h be m-ary, n-ary and p-ary operations respectively. Let $1 \leq i \leq m$, so that $f \circ_i g$ is a $(m + n - 1)$-ary operation. If we wish to compose once more, i.e. wish to make sense of $(f \circ_i g) \circ_j h$ for $1 \leq j \leq m + n - 1$, then three cases would arise, each corresponding to $1 \leq j \leq i - 1$, $i \leq j \leq i + n - 1$, and $i + n \leq j \leq n + m - 1$.

If $1 \leq j \leq i - 1$, then $(f \circ_i g) \circ_j h$ is same as inserting h at the j-th place first, and then inserting g at the $(i + p - 1)$-th place, i.e.

$$(f \circ_i g) \circ_j h = (f \circ_j h) \circ_{i+p-1} g.$$

Similarly, if $i \leq j \leq i+n-1$, then $(f \circ_i g) \circ_j h$ is same as inserting h on g at the $(j-i+1)$-th place first, followed by inserting $g \circ_{j-i+1} h$ on f at the i-th place, i.e.

$$(f \circ_i g) \circ_j h = f \circ_i (g \circ_{j-i+1} h).$$

Also, if $i + n \leq j \leq n + m - 1$, we have,

$$(f \circ_i g) \circ_j h = (f \circ_{j-m+1} h) \circ_i g.$$

The Symmetric Group Action

We can permute the inputs of any n-ary operation using the elements of the symmetric group S_n. For example, if f is a 4-ary operation, and $\sigma = (4132) \in S_4$, then $f\sigma$ is a 4-ary operation, defined by

$$f\sigma(a_1, a_2, a_3, a_4) = f(a_3, a_4, a_2, a_1).$$

. More generally, if f is a n-ary operation, and $\sigma \in S_n$, then

$$f\sigma(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

With this action on the set of all n-ary operations, we demand the following condition:

$$(f\sigma) \circ_i (g\tau) = (f \circ_i g)(\sigma \circ_i \tau),$$

where $\sigma \circ_i \tau \in S_{m+n-1}$ is the block permutation with the i-th entry of σ replaced by the block τ.

The Unitary Condition

There exists a 1-ary operation, namely the identity map, which behaves as the unit of the $\circ_i$ operations, i.e. if f is a n-ary operation, and $\mathbf{1} = \mathrm{Id}_{\mathbf{X}}$, then

$$f \circ_i \mathbf{1} = \mathbf{f} = \mathbf{1} \circ_{\mathbf{1}} \mathbf{f}.$$

An abstraction of the above example is what leads us to the formal definition of the *symmetric operad*.

Definition 2.1. A symmetric operad is a sequence of sets

$$\mathcal{P} = \{\mathcal{P}(0), \mathcal{P}(1), \mathcal{P}(2), \ldots,\}$$

together with composition functions

$$\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \longrightarrow \mathcal{P}(m + n - 1), \quad 1 \leq i \leq n,$$

satisfying the following identities:

(i) (Associativity) $\forall f \in \mathcal{P}(n)$, $g \in \mathcal{P}(m)$ and $h \in \mathcal{P}(r)$

$$(f \circ_i g) \circ_j h = \begin{cases} (f \circ_j h) \circ_{i+p-1} g & 1 \leq j \leq i-1 \\ f \circ_i (g \circ_{j-i+1} h) & i \leq j \leq m+i-1 \\ (f \circ_{j-m+1} h) \circ_i g & m+i \leq j \leq m+n-1 \end{cases}$$

(ii) (Equivariance) Each $\mathcal{P}(n)$ has a right action of S_n,

$$\begin{aligned} \mathcal{P}(n) \times S_n &\longrightarrow \mathcal{P}(n) \\ (f, \sigma) &\mapsto f\sigma, \end{aligned}$$

such that $\forall f \in \mathcal{P}(n)$, $g \in \mathcal{P}(m)$, and $\sigma \in S_n$ and $\tau \in S_m$

$$(f\sigma) \circ_i (g\tau) = (f \circ_i g)(\sigma \circ_i \tau),$$

where $\sigma \circ_i \tau$ is the block permutation, where the i-th entry of σ is replaced by the entire block τ.

(iii) There exists an unique element $\mathbf{1} \in \mathcal{P}(1)$, such that for all n, for all $f \in \mathcal{P}(n)$, and $1 \leq i \leq n$,

$$f \circ_i \mathbf{1} = f = \mathbf{1} \circ_1 f.$$

Remark 2.2. As has been mentioned earlier, the $\mathcal{P}(n)s$ as in the definition above need not be just sets. They can have richer structures. For example, $\mathcal{P}(n)$s can be vector spaces (in which case the operad is called an algebraic operad), or topological spaces, or graded modules, or chain complexes, or simplicial sets or any kind of objects where a 'product' makes sense. We should accordingly ask for the compositions $\circ_i$ and the symmetric group actions to be maps in the appropriate category. For example, if $\mathcal{P}(n)$s are vector spaces, we demand $\circ_i$'s and the symmetric actions to be linear maps, if $\mathcal{P}(n)$s are topological spaces, we demand these are continuous functions and so on.

Remark 2.3. A non-symmetric operad consists of all the data of a symmetric operad, except for the symmetric group actions. We shall see some examples of non-symmetric operads in the sequel.

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3. Some Important Examples of Operads

The first example cited here is the algebraic version of the toy model that we discussed in Section 2. It is also important in defining the notion of 'algebras over operads'.

Example 3.1. Endomorphism Operad: Let V be a vector space over the ground field $\mathbb{K}$. The *endomorphism operad* End_V is defined as follows:

$$End_V(n) = Hom_{\mathbb{K}}(V^{\otimes n}, V), \ \ \forall \ n \geq 0,$$

where $V^{\otimes n}$ denotes the n-fold tensor product of V over $\mathbb{K}$, and by convention $V^{\otimes 0} = \mathbb{K}$. The composition Maps

$$\circ_i : End_V(n) \otimes End_V(m) \longrightarrow End_V(m+n-1)$$

are defined as follows:

$$\begin{aligned}&(f \circ_i g)(a_1, \ldots, a_{m+n-1})\\ =&f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+m-1}), a_{i+m}, \ldots, a_{m+n-1}),\end{aligned}$$

for all $a_1, \ldots, a_{m+n-1} \in V$, and the action of S_n on $Hom(V^{\otimes n}, V)$ is given by

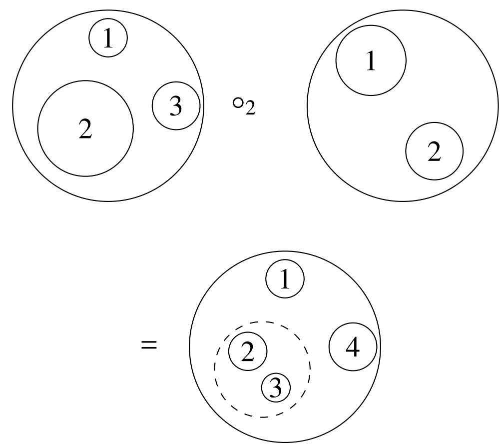
$$\begin{aligned}Hom(V^{\otimes n}, V) \times S_n &\longrightarrow Hom(V^{\otimes n}, V)\\ (f, \sigma) &\mapsto f\sigma,\end{aligned}$$

where $(f\sigma)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

The next example has been used to understand n-fold loop spaces. It appears in the works of Boardmann-Vogt [2], J. Peter May [5], and subsequently many others.

Example 3.2. Little n-discs Operad: The *little n-disks operad* $\mathcal{D}_n$ is a topological operad defined as follows.

Let $D^n = \{x \in \mathbb{R}^n, \|x\| \leq 1\}$ be the n-dimensional unit disc. An element in $\mathcal{D}_n(r)$ is completely determined by a family of r embeddings $\alpha_i : D^n \longrightarrow D^n, \ i = 1, \ldots, r$, satisfying the non-intersecting condition. In other words, $\mathcal{D}_n(r)$ is made up of r non-intersecting subdiscs in the interior of D^n. Each $\mathcal{D}_n(r)$ is a topological space, where an element of the space $\mathcal{D}_n(r)$ can be realized as a configuration of r discs in the unit disc. There is an enumeration of these r interior discs as a part of the structure. The operadic composition $\circ_i$ is given by insertion of a disc in the i-th interior disc, as is illustrated in the following figure.



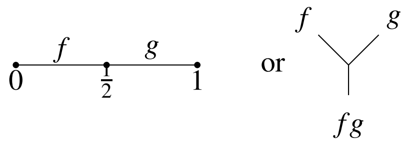
The action of the symmetric group S_r on $\mathcal{D}_n(r)$ is given by permuting the labelling of the r interior discs.

The next example has been defined by Stasheff [4] with some insights from Frank Adams. Stasheff uses it to characterize loop spaces.

Example 3.3. The Associahedra Operad: Let X be a topological space and let ΩX be its based loop space, that is, ΩX is the space of all continuous functions $f : (S^1, 1) \longrightarrow (X, *)$ sending $1 \in S^1$ to the fixed base-point $* \in X$. If $f, g \in \Omega X$, $fg \in \Omega X$ is defined as the loop which traverses half the time in f and rest half in g. Precisely, the product on ΩX, say fg is given by

$$(fg)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

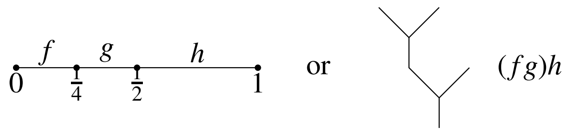
This is conveniently expressed graphically in terms of planar trees as below:



This gives us a binary operation on the based loop space ΩX of X.

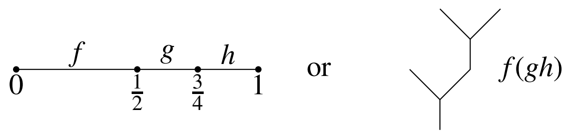
This product is clearly not associative:

$(fg)h$:

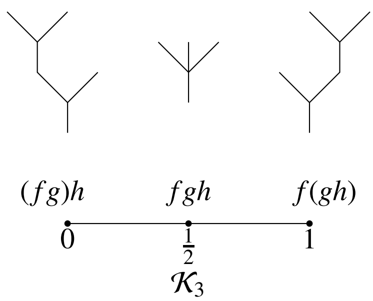


whereas,

$f(gh)$:



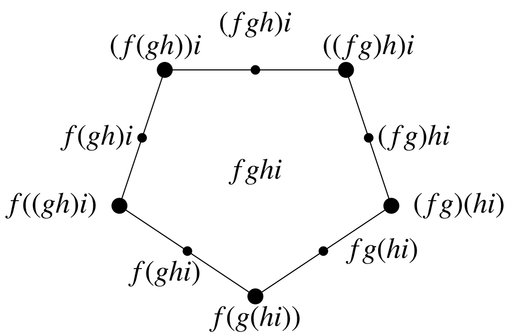
But the two loops $f(gh)$ and $(fg)h$ are homotopic. This can be represented by a line segment, where the vertices represent the two loops $(fg)h$ and $f(gh)$, and each point in between represent an intermediate loop. In particular, the midpoint represents the loop in which all f, g and h are traversed in equal (one-third) time, and can be represented by the corolla, as below. This is denoted by $\mathcal{K}_3$.



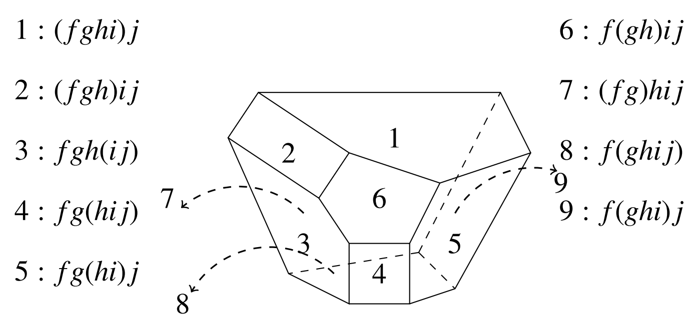
There is an interesting number theoretic connection. The n-th Catalan number C_n is the number of different ways a word consisting of $n+1$ letters can be completely parenthesized. Precisely, $C_n = \frac{(2n)!}{(n+1)!n!}$.

The n-th Catalan number C_n is the number of different ways a word consisting of $n + 1$ letters can be completely parenthesized.

When we take the product of four loops, f, g, h, i, say, there are Catalan number $C_3(= 5)$ number of ways in which we can parenthesize. Hence, we get a pentagon, vertices of which are the product of four loops, differently parenthesized. There are homotopies between some of them, as shown in the figure below, due to which we get the edges of the pentagon. Interestingly, there are homotopies between the homotopies, which gives us the face of the pentagon. This is what we name as $\mathcal{K}_4$. Every vertex of the polytope $\mathcal{K}_4$ can be represented by planar trees with four leaves, and the interior $fghi$ by a corolla with four leaves.



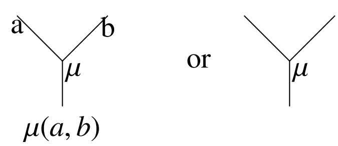
The vertices of the next polytope $\mathcal{K}_5$ consists of all the ways in which we can parenthesize five loops. Hence, $\mathcal{K}_5$ has the Catalan number $C_4(= 14)$ vertices. There are 21 edges, which correspond to homotopies between loops, and there are 9 faces, which correspond to homotopies between homotopies, and the interior which correspond to a homotopy between the homotopies between homotopies.



Each of the 9 faces can be indexed by a loop, as has been indicated in the above diagram. Taking $\mathcal{K}_1 = \Phi$, and $\mathcal{K}_2$ to be a point, and following the same line of thought we get a sequence of polytopes $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \ldots$ and so on. With composition induced by the inclusion of faces, this sequence of polytopes forms a non-symmetric operad.

4. The Classical Algebras

For the sake of completeness, let us quickly recall some examples of algebras which we commonly come across. An algebra is a vector space V over a field $\mathbb{K}$, equipped with a linear map $\mu : V \otimes V \longrightarrow V$, which we often refer to as multiplication. We often denote $\mu(a, b) := a \cdot b$. Let us think of this 2-ary map μ as a planar tree with two leaves.



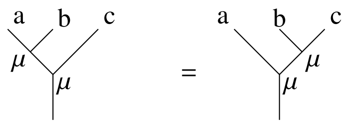
4.1 *Associative, Commutative, Lie*

Here are some examples of classical algebras:

1. If the multiplication $\mu : V \otimes V \longrightarrow V$ satisfies the associative condition, that is,

$$\mu(\mu(a, b), c) = \mu(a, \mu(b, c)),$$

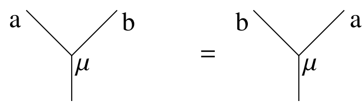
we call the algebra an **associative** algebra. Expressing via planar trees, we get



2. If μ satisfies the commutative condition, that is,

$$\mu(a, b) = \mu(b, a),$$

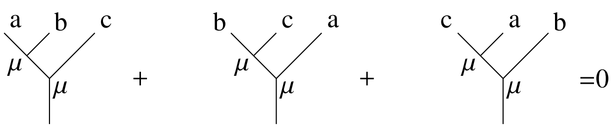
we call the algebra a **commutative** algebra. Graphically,



3. If μ satisfies the *Jacobi* condition

$$\mu(\mu(a, b), c) + \mu(\mu(b, c), a) + \mu(\mu(c, a), b) = 0,$$

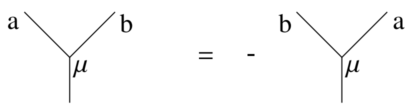
i.e.



and the *skew-symmetric* condition

$$\mu(a, b) = -\mu(b, a),$$

i.e.



we call the algebra a **Lie** algebra.

4.2 *n-ary Algebras*

All the examples considered so far were defined by binary operations and their governing laws. On the other hand, there are generalizations of binary algebras which are generated by n-ary operations, for $n \geq 3$. We can widen this definition of an algebra to include 'algebras' which are determined by n-ary operations and relations. In recent times a lot of work is being done on algebras where the defining operation is a n-ary operation, $n \geq 3$, for example, n-ary Lie algebras, brace algebras, etc.

5. Algebras Over Operads

Operads are important via their representations, which are called algebras over these operads. An algebra of a certain type, usually defined by generating operations and relations, defines an algebraic operad, say $\mathcal{P}$, algebras over which are the type of algebras we started with. Let us first make sense of a morphism between operads. We talk about algebraic operads here, but one can define algebras over operads in any symmetric monoidal category, in a similar way.

An algebra of a certain type, usually defined by generating operations and relations, defines an algebraic operad.

Definition 5.1. Let $\mathcal{P} = \{\mathcal{P}(n),\ n \geq 1\}$ and $Q = \{Q(n),\ n \geq 1\}$ be two (algebraic) symmetric operads. A morphism $\phi : \mathcal{P} \longrightarrow Q$ consists of a sequence of linear maps $\phi(n) : \mathcal{P}(n) \longrightarrow Q(n)$ such that it respects the compositions, actions and the unit. More precisely, $\phi_1(\mathbf{1}_{\mathcal{P}}) = \mathbf{1}_Q$, and for all $n, m \geq 1$, and for all $\sigma \in S_n$, the following diagrams commute:

$$\begin{array}{ccc} \mathcal{P}(n) \otimes \mathcal{P}(m) & \xrightarrow{\circ_i} & \mathcal{P}(n+m-1) \\ {\scriptstyle \phi_n \otimes \phi_m} \downarrow & & \downarrow {\scriptstyle \phi_{n+m-1}} \\ Q(n) \otimes Q(m) & \xrightarrow{\circ_i} & Q(n+m-1) \end{array}$$

$$\begin{array}{ccc} \mathcal{P}(n) & \xrightarrow{\sigma} & \mathcal{P}(n) \\ {\scriptstyle \phi_n} \downarrow & & \downarrow {\scriptstyle \phi_n} \\ Q(n) & \xrightarrow{\sigma} & Q(n) \end{array}$$

Definition 5.2. A vector space A is said to be an algebra over an operad $\mathcal{P}$ (or a $\mathcal{P}$-algebra) if there exists a morphism of operads $\rho : \mathcal{P} \longrightarrow \mathrm{End}_A$.

Thus, if A is a $\mathcal{P}$-algebra, there exists

$$\rho_n : \mathcal{P}(n) \to \mathrm{End}_A(n), \quad \forall n \geq 0,$$

which are linear, equivariant and unital. Note that $\rho_n : \mathcal{P}(n) \to \mathrm{Hom}(A^{\otimes n}, A)$ induces a map $\tilde{\rho}_n \in \mathrm{Hom}(\mathcal{P}(n) \otimes_{S_n} A^{\otimes n}, A)$, given by

$$\tilde{\rho}_n(\mu, a_1, \ldots, a_n) = \rho_n(\mu)(a_1, \ldots, a_n),$$

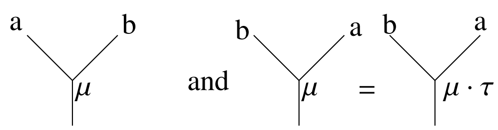
for all $\mu \in \mathcal{P}(n)$ and for all $a_1, \ldots, a_n \in A$.

If μ is an arbitrary element in $\mathcal{P}(n)$, then $\rho_n(\mu) : A^{\otimes n} \longrightarrow A$ gives a n-ary operation on A. Thus a morphism $\rho : \mathcal{P} \longrightarrow \mathrm{End}_A$ associates to each "abstract" n-ary operation on A, a real n-ary operation on A. This endows A with an algebraic structure. In fact, to any type of algebras (with operations having one output), one can associate a specific operad, whose algebras in turn, are the type of algebras we started with.

6. The Symmetric Operads *Com*, $\mathcal{A}ss$, *uCom*, $u\mathcal{A}ss$

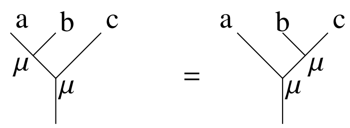
6.1 *The Operad* Com

If we start with a commutative associative non-unital algebra, then what is the operad $\mathcal{P}$, such that algebra over $\mathcal{P}$, or $\mathcal{P}$-algebras are precisely commutative associative non-unital algebras? The answer is the operad *Com*. Let (A, μ) be a commutative associative non-unital algebra. The two 2-ary abstract operations

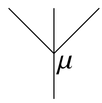


correspond to $\mu(a, b) = ab$, and $\mu \cdot \tau(a, b) = ba$, where τ is the transposition $(12) \in S_2$. Now, due to commutativity, the two conditions are the same. Thus *Com*(2) is the one-dimensional vector space generated by this operation, with trivial S_2-action. Moreover, there is only one 1-ary operation, namely, Id_A. Thus $Com(1) = \mathbb{K}$.

Operations with 3 or more inputs are obtained by composing 2-ary and 1-ary operations. Using associativity, for $n = 3$; we have



Hence 3-ary operations can be viewed as 3-corollas,



Again using commutativity of the algebras, we see that that *Com*(3) is generated by only one 3-ary operation, hence, $Com(3) = \mathbb{K}$. Arguing in the same way, we get that $Com(n) = \mathbb{K}$, for $n \geq 1$ and $Com(0) = 0$, as there are no 0-ary operations. The action of the symmetric group on each *Com*(n) is assumed to be trivial.

On the other hand, it can be shown that the algebras over the operad *Com*, as described above, are commutative associative non-unital algebras. Let A be a vector space which is an algebra over *Com*. By definition, this implies that there is a sequence of maps $\rho_n : Com(n) \longrightarrow \mathrm{End}_A(n)$ for all $n \geq 0$, satisfying certain associative, equivariant and unitary conditions. This implies that there exists

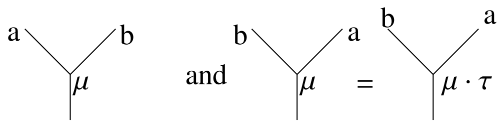
$$\rho_n : \mathbb{K} \longrightarrow \mathrm{Hom}(A^{\otimes n}, A),$$

for all $n \geq 0$. Let $\mu := \rho_2(1_k) : A^{\otimes 2} \to A$. To verify that μ is commutative, we observe that as the action on *Com* is trivial, $\mu\sigma = \mu$, where σ is the non-trivial element in S_2. To verify associativity, note that by definition of an operad morphism,

$\rho_3(1_k \circ_1 1_k) = \mu \circ_1 \mu$, and also $\rho_3(1_k \circ_2 1_k) = \mu \circ_2 \mu$. Thus, $\mu \circ_1 \mu = \mu \circ_2 \mu$, essentially proving associativity of the binary operation μ.

6.2 *The Symmetric Operad $\mathcal{A}ss$*

This is the operad associated with the associative non-unital algebras. As previously, $\mathcal{A}ss(0) = 0$ and $\mathcal{A}ss(1) = \mathbb{K}$. Since we do not have commutativity, the two binary operations



are not the same. Hence, $\mathcal{A}ss(2)$ is two-dimensional, more precisely, $\mathcal{A}ss(2) = \mathbb{K}[S_2]$. Regarding 3-ary operations, using associativity, we get 6 corollas. The action of S_3 changes the ordering of the leaves. Thus each corolla can be identified by an element of the symmetric group S_3. Thus, $\mathcal{A}ss(3) = \mathbb{K}[S_3]$, with the regular action of S_3. Arguing similarly, we obtain that: $\mathcal{A}ss(n) = \mathbb{K}[S_n]$, for $n \geq 1$ where the symmetric group action is the regular action, and $\mathcal{A}ss(0) = 0$.

On the other hand, we can show that algebras over the operad $\mathcal{A}ss$, as defined above, are precisely associative non-unital algebras. Let A be an algebra over $\mathcal{A}ss$. This precisely means that there is an operad morphism $\rho : \mathcal{A}ss \longrightarrow \mathrm{End}_A$, i.e. there exists sequence of maps $\rho_n : \mathbb{K}[S_n] \longrightarrow \mathrm{Hom}(A^{\otimes n}, A)$ satisfying associative, equivariant, and unitary conditions. The map $\mu = \rho_2(\sigma) \in \mathrm{Hom}(A^{\otimes 2}, A)$, where σ is the trivial element in S_2, provides an associative multiplication on A, which follows from the following fact. By definition of operad morphisms, $\rho_3(\sigma \circ_1 \sigma) = \rho_2(\sigma) \circ_1 \rho_2(\sigma)$. Also, $\rho_3(\sigma \circ_2 \sigma) = \rho_2(\sigma) \circ_2 \rho_2(\sigma)$. But, $\sigma \circ_1 \sigma = \sigma \circ_2 \sigma$, which is the trivial element in S_3. As, $\rho_2(\sigma) := \mu$, we have $\mu \circ_1 \mu = \mu \circ_2 \mu$. This is precisely the associative condition of μ.

6.3 *The Operad* *uCom*

The operad *uCom* is associated with unital commutative associative algebras.

The operad *uCom* is associated with unital commutative associative algebras. This is identical to the operad *Com* except that $uCom(0) = \mathbb{K}$. The reader may check that algebras over this operad are precisely unital commutative associative algebras.

6.4 *The Operad* $u\mathcal{A}ss$

The operad $u\mathcal{A}ss$ is the one whose algebras are associative unital algebras. This operad is identical to the operad $\mathcal{A}ss$, except in arity 0. We define $u\mathcal{A}ss(0) = \mathbb{K}$. It is not difficult to see that the algebras over this operad are associative unital algebras.

7. Two Phenomenal Results

The A_∞-spaces were introduced by Jim Stasheff in the early sixties. An A_∞-space is a topological space Y together with a family of maps

$$\theta_n : \mathcal{K}_n \times Y^n \longrightarrow Y, \ \ \forall n \geq 1$$

which glue together to make Y into an algebra (note that Y is here a space, as we are over a topological operad) over the associahedra operad $\mathcal{K} = \{\mathcal{K}_n\}_{n \geq 1}$.

Here are two path-breaking results, which were proved in during the 60's–70's, when algebraic topologists were studying iterated loop spaces. These results show the importance of the little n-discs operad and the associahedra operad in algebraic topology.

Theorem 7..1. *A connected space Y (of the homotopy type of a CW-complex) has the homotopy type of a based loop space ΩX for some space X if and only if Y admits the structure of an A_∞-space, i.e. Y is an algebra over the associahedra operad. [4]*

Theorem 7..2. Recognition Principle*: (Boardman-Vogt [2], J. P. May [5]) A group-like space X (i.e. $\pi_0(X)$ is a group) has the homotopy type of a n-fold loop space if and only if it is an algebra over the little n-discs operad.*

8. Waiving at Koszul Duality Theory of Operads

In this last section, we very briefly discuss a very important notion in the theory of operads, namely that of Koszul duality. Let V be a vector space over $\mathbb{K}$. Let $S(V)$ be the symmetric algebra on V and let $\Lambda(V)$ be the exterior algebra on V thought of as a graded vector space. Then the graded vector space $S(V) \otimes \Lambda(V^*)$ can be endowed with a suitable differential, to make it into an acyclic complex, i.e. with vanishing homologies. In his work [6], S. B. Priddy showed that for certain quadratic associative algebras A, there exists an algebra $A^!$, called the *Koszul dual* of A, such that $A \otimes A^!$ is an acyclic complex. Such algebras are called *Koszul algebras*.

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The Koszul duality between Lie algebras and commutative algebras was found in the works of D. Quillen [7]. Later, V. Ginzburg and M. Kapranov, in their celebrated work [8] have shown that such dualities can be constructed for various other types of algebras.

Ginzburg and Kapranov extended Koszul duality to operads: A quadratic operad $\mathcal{P}$ admits a Koszul dual operad $\mathcal{P}^!$, and when the 'bar complex' of $\mathcal{P}$ is quasi-isomorphic to $\mathcal{P}^!$, the operad is called a Koszul operad. The Koszul dual of a quadratic $\mathcal{P}$-algebra is a $\mathcal{P}^!$-algebra, which leads us to define a Koszul duality theory for these algebras. If $\mathcal{L}ie$ denotes the operad encoding Lie algebras, then we have some amazing facts:

$$\mathcal{A}ss^! = \mathcal{A}ss, \qquad \mathcal{L}ie^! = \mathcal{C}om, \qquad \mathcal{C}om^! = \mathcal{L}ie.$$

Let me close this episode with one application of Koszul duality theory, among many others. Given an isomorphism between two vector spaces, one can transfer the algebra structure, if any, on one to the other, so that the algebra structures now become isomorphic. In homotopical algebra, we are interested in quasi-isomorphisms, which are maps of chain complexes inducing isomorphisms in homology. Given two chain complexes and a quasi-isomorphism between them, given one algebra structure on one of them, one can always define an algebra structure on the other, but

Given an isomorphism between two vector spaces, one can transfer the algebra structure, if any, on one to the other, so that the algebra structures now become isomorphic.

these are not the same structures. In some sense, they are same only up to homotopy. This result is called HTT or homotopy transfer theorem. Though the computations are easy for the classical cases, using Koszul duality theory of operads one can give a unified treatment to this theorem.

Interested readers can look up to the following articles: [9], [10], and [11]. For a more advanced treatment, a good reference is the book by J.-L. Loday and B. Vallette [12].

Acknowledments

I would like to thank the referee for his/her helpful comments.

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