Problem 9: The General Reciprocity Law∗

In 1974, the AMS held a symposium to focus on areas of importance in contemporary mathematics relevant to the Hilbert problems described by David Hilbert in his 1900 ICM address in Paris. The following article by Tate focuses on Hilbert’s ninth problem asking for the most general reciprocity law.

After discussing the history, Tate studies three post-war developments of class field theory (wherein his contribution is of great importance). These are: (i) The introduction of group cohomology into class field theory, (ii) Algebraic K-theory and symbols, and (iii) What could a “non-abelian class field theory” be? Tate also discusses the so-called Langlands conjectures for odd two-dimensional complex representations of the Galois group of a number field.

B Sury
Indian Statistical Institute
Bengaluru
Email: surybang@gmail.com

∗DOI: https://doi.org/10.1007/s12045-020-0943-9
The article is reproduced with permission from
The American Mathematical Society,
Tate, John. “Problem 9: The general reciprocity law,”
in Mathematical Developments Arising from Hilbert Problems.
PROBLEM 9: THE GENERAL RECIPROCITY LAW

J. Tate

The ninth problem of Hilbert concerns the “most general reciprocity law in an arbitrary algebraic number field”. It seems to me that he underestimated its potential when he referred to it as a “more special” problem in number theory and devoted only a short paragraph to it.

Let \( R \) be the ring of integers in an algebraic number field. If \( P \) is an odd prime ideal in \( R \), i.e., one whose residue class field is of odd characteristic, then the Legendre symbol \( \left( \frac{a}{P} \right) \) is defined, for \( a \in R \), to be one less than the number of incongruent solutions, \( x \), of the congruence \( x^2 \equiv a \pmod{P} \). In other words

\[
\left( \frac{a}{P} \right) = \begin{cases} 
1 & \text{if } x^2 \equiv a \pmod{P} \text{ is solvable and } a \not\equiv 0 \pmod{P}, \\
0 & \text{if } a \equiv 0 \pmod{P}, \\
-1 & \text{if } x^2 \equiv a \pmod{P} \text{ is not solvable},
\end{cases}
\]

If we view these congruences \( \pmod{P} \) as equations in the residue field \( \mathbb{F}_p \) and recall that the multiplicative group \((\mathbb{F}_p)^*\) of that field is cyclic of even order, then it is clear that

\[
\left( \frac{a}{P} \right) = f(a \pmod{P}),
\]

where \( f \) is the function on \( \mathbb{F}_p \) which vanishes at 0 and on \((\mathbb{F}_p)^*\) is the unique character of order 2. In particular, the symbol \( \left( \frac{a}{P} \right) \) is multiplicative in \( a \):

\[
\left( \frac{ab}{P} \right) = \left( \frac{a}{P} \right) \left( \frac{b}{P} \right).
\]

Thus, as function of \( a \), for fixed \( P \), the symbol \( \left( \frac{a}{P} \right) \) is easy to describe. The dual problem — how does \( \left( \frac{a}{P} \right) \) vary with \( P \) for fixed \( a \) — is the problem of quadratic reciprocity.
Consider the case \( R = \mathbb{Z} \), the ring of rational integers. Then \( P = p\mathbb{Z} \), where \( p \) is a positive odd prime number, and we write \( \left( \frac{a}{p} \right) \) instead of \( \left( \frac{a}{P} \right) \). For fixed \( p \) the symbol \( \left( \frac{a}{p} \right) \) depends only on the remainder when the integer \( a \) is divided by \( p \) and its value is easily tabulated for \( 0 \leq a < p \). But suppose we fix \( a \) and let \( p \) vary. What sort of function of \( p \) is \( \left( \frac{a}{p} \right) \)? If we write

\[
a = \pm q_1^{m_1} \cdots q_r^{m_r}
\]

where the \( q_i \) are odd positive primes, then

\[
\left( \frac{a}{p} \right) = \left( \frac{-1}{p} \right)^{m_0} \left( \frac{q_1}{p} \right)^{m_1} \cdots \left( \frac{q_r}{p} \right)^{m_r}.
\]

Thus the problem for an arbitrary integer \( a \) reduces immediately to its special cases \( a = -1 \), \( a = 2 \), and \( a = q \), a positive odd prime \( \neq p \). In these cases \( \left( \frac{2}{p} \right) \) as function of \( p \) is given by the classical quadratic reciprocity law conjectured by Euler and proved by Gauss, as follows:

\[
\left( \frac{-1}{p} \right) = (-1)^{\frac{p-1}{2}}, \quad \left( \frac{2}{p} \right) = (-1)^{\frac{p-1}{2}}, \quad \text{and}
\]

\[
\left( \frac{q}{p} \right) = \left( \frac{q}{p} \right)^* = \left( \frac{p}{q} \right)^*, \quad \text{where} \quad p^* = (-1)^{\frac{p-1}{2}}.
\]

An easy consequence is that \( \left( \frac{a}{p} \right) \) depends only on the residue class of \( a \) (modulo \( 4a \)), a fact which is not at all obvious from the definition of \( \left( \frac{a}{p} \right) \).

Hilbert reinterpreted the quadratic reciprocity law and generalized it to an arbitrary algebraic number field \( K \). To do this he introduced a new symbol

\[
\left( \frac{a,b}{p} \right) = \begin{cases} 
1, & \text{if } ax^2 + by^2 = 1 \text{ has a solution } x,y \in K_p, \\
-1, & \text{if not.}
\end{cases}
\]

Here \( a \) and \( b \) are non-zero elements of \( K \), and \( K_p \) denotes the completion of \( K \) at \( P \). Contrary to the Legendre symbol, this Hilbert symbol is defined not only for odd primes, but for all primes \( P \) of \( K \), odd or even, and also for the “infinite primes”, for which \( K_p \) is the real or complex field. Hilbert proved:
(i) For fixed \( P \), the symbol \( \left( \frac{a \cdot b}{P} \right) \) is symmetric and bimultiplicative in \( a \) and \( b \).

(ii) If \( P \) is an odd prime and \( a \) an integer in \( K \) not divisible by \( P \), then

\[
\left( \frac{a \cdot b}{P} \right) = \left( \frac{a}{P} \right) \left( \frac{b}{P} \right)
\]

where \( \nu_P(b) \) is the exponent to which \( P \) appears in the prime factorization of \( b \).

(iii) For fixed \( a \) and \( b \)

\[
\prod_P \left( \frac{a \cdot b}{P} \right) = 1.
\]

This last product is really finite because by (ii) the \( P \)-factor in it is 1 if \( P \) is odd and prime to \( a \) and to \( b \).

It is easy to see that Hilbert's product formula (iii) generalizes the classical quadratic reciprocity law. If \( K = \mathbb{Q} \), the rational field, and we take for \( a \) and \( b \) two distinct odd positive prime numbers \( p \) and \( q \), the product formula (iii) becomes

\[
\left( \frac{p \cdot q}{\infty} \right) \left( \frac{p \cdot q}{2} \right) \left( \frac{p \cdot q}{p} \right) \left( \frac{p \cdot q}{q} \right) = 1,
\]

for by (ii) we have \( \left( \frac{p \cdot q}{p} \right) = 1 \) if \( r \) is an odd prime different from \( p \) and \( q \). The first, "\( \infty \)" factor is 1 because \( p \) and \( q \) are positive. Using (ii) and symmetry to evaluate the last two factors we find therefore

\[
\left( \frac{q}{p} \right) \left( \frac{p}{q} \right) = \left( \frac{p \cdot q}{2} \right) .
\]

By (i) the symbol \( \left( \frac{p \cdot q}{2} \right) \) depends only on \( p \) and \( q \) (mod 3), because integers \( a \equiv 1 \) (mod 3) are 2-adic squares. Taking special values of \( p \) and \( q \) we can therefore simply verify that

\[
\left( \frac{p \cdot q}{2} \right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} .
\]

The laws for \( \left( \frac{-1}{p} \right) \) and \( \left( \frac{2}{p} \right) \) follow similarly now, taking \( a = -1 \) or \( 2 \) and \( b = p \).

During the 19th century some special \( \ell \)-th-power reciprocity laws were discovered for \( \ell \geq 2 \). Suppose \( K \) contains a primitive \( \ell \)-th root of unity. For a prime ideal \( P \) not dividing \( \ell \) the group \( U_\ell \) of \( \ell \)-th roots
of unity injects into the group \((R/P)^*\), which is cyclic of order \(NP - 1\). Hence we can define an \(i^{th}\) power-residue symbol at such primes \(P\) by

\[
\left( \frac{a}{P} \right)_i \in \mathbb{U}_i \cup \{0\}, \quad \text{and} \quad \left( \frac{a}{P} \right)_i \equiv a^\frac{NP-1}{i} \pmod{P}.
\]

The symbol \(\left( \frac{a}{P} \right)_i\) is then multiplicative in \(a\), and is 1 if and only if the congruence \(x^i \equiv a \pmod{P}\) is solvable and \(a \not\equiv 0 \pmod{P}\). The classical cubic and quartic reciprocity laws for \(K\) the field of third or fourth roots of unity were stated in terms of this symbol for \(i = 3\) or 4 and are of the same general nature as the quadratic law for the rational field.

Hilbert's 9th problem was to prove the reciprocity law for \(i^{th}\) power residues for an arbitrary number field \(K\) and for arbitrary \(i > 2\). It's not clear to me whether he meant implicitly that \(K\) should contain the \(i^{th}\) roots of unity or whether he really anticipated the freeing of the problem from that assumption. But at any rate it's clear from his writings at the time that he expected the solution to come from a generalization to arbitrary abelian extensions \(L/K\) of his own theory of quadratic extensions. (An abelian extension is a normal field extension with abelian Galois group.)

Building on work of Hilbert and Weber, Takagi created in the period 1900-1920 a general theory of abelian extensions of number fields. To state Takagi's fundamental results I will use the modern notion of the idele class group, instead of the generalized ideal class groups of Weber with which Takagi worked. With every number field \(K\) there is associated its idele class group \(C_K\), a locally compact abelian group. This group \(C_K\) is defined as the quotient of the restricted product of the multiplicative groups of the completions of \(K\) by the multiplicative group of \(K\). Each finite prime \(P\) of \(K\) determines a subset \([P]\) of \(C_K\) which we will call the class of \(P\). It is the set of images in \(C_K\) of the local parameters in \(K_P\). If \(L\) is a finite extension of \(K\), then there is a homomorphism \(N_{L/K}: C_L \rightarrow C_K\).

Takagi's main results can be summed up as follows:

1. The correspondence \(L \leftrightarrow N_{L/K} C_L\) is a bijection from the set of finite abelian extensions \(L\) of \(K\) (in a given algebraic closure \(\overline{K}\)) to the set of open subgroups of finite index in \(C_K\). For two abelian extensions \(L\) and \(L'\),

\[
L \subseteq L' \iff N_{L/K} C_L \supseteq N_{L'/K} C_{L'}.
\]
2. For each $L$ the abelian groups

$$\text{Gal}(L/K) \quad \text{and} \quad C^{N_L/K}_L$$

are abstractly isomorphic, and for each finite prime $P$ of $K$, the way in which $P$ decomposes in $L$ is entirely determined by the image $[P]_{L/K}$ in $C^{N_L/K}_L$ of the class $[P]$ of $P$.

With this work of Takagi the theory of abelian extensions - "class field theory" - seemed in some sense complete, yet there was still no general reciprocity law. It remained for Artin to crown the edifice with such a theorem. He conjectured in 1923 and proved in 1927 that for each abelian extension $L/K$ there is a natural isomorphism

$$C^{N_L/K}_L \xrightarrow{\sim} \text{Gal}(L/K)$$

which is characterized by the fact that for each finite prime $P$ of $K$ it carries $[P]_{L/K}$ onto the set $\text{Frob}_{L/K}(P)$ of "Frobenius substitutions" attached to $P$. (In the abelian case here under consideration, $\text{Frob}_{L/K}(P)$ is the set of automorphisms $\sigma \in \text{Gal}(L/K)$ such that $\sigma x \equiv x^P \pmod{P}$ for each $x$ in the ring of integers of $L$; it consists of a single element if $P$ is unramified in $L$.) When the $l^{th}$ roots of unity lie in $K$ and $L = K(\zeta^L_l)$, it follows immediately from the definitions that $\text{Frob}_{L/K}(P)$ is essentially the $l$-power residue symbol $(\frac{l}{P})_l$, and Artin's theorem implies then, that for fixed $a$ the value of $(\frac{l}{P})_l$ depends only on the class $[P]_{L/K}$ of $P$.

Making this dependence explicit yields all known reciprocity laws. For this reason Artin said when he first conjectured his theorem that it had to be viewed as the general reciprocity law, for arbitrary fields, with or without roots of unity, even though it sounds a bit strange at first.

How did Artin guess his reciprocity law? He was not looking for it, not trying to solve a Hilbert problem. Neither was he, as would seem so natural to us today, seeking a canonical isomorphism, to make Takagi's theory more functorial. He was led to the law in trying to show that a new kind of $L$-series which he introduced really was a generalization of the usual $L$-series. The usual $L$-series,

$$L(s, X) = \prod_p (1 - X([P]) N_P^{-s})^{-1},$$

were associated with idele class characters $X: C_K \to \mathbb{C}$. Artin's $L$-series

License of copyright restrictions may apply to redistribution; see https://www.ams.org/publications/ebooks/terms
\[ L(\sigma, R) = \prod_p \det(1 - R(\text{Frob}_{L/K}(p))N_p^{-\sigma})^{-1} \]

were associated to representations \( R : \text{Gal}(L/K) \longrightarrow \text{GL}(n, \mathbb{C}) \) of Galois groups over \( K \), abelian or not. (In these Euler products, the factors corresponding to ramified primes need special definitions.) Clearly, in order to identify his \( L \)-series with the usual \( L \)-series in case of an abelian extension \( L/K \), Artin needed an identification of the abelian Galois group \( \text{Gal}(L/K) \) with a quotient of \( C_K \) which would make \( \text{Frob}(p) \) correspond to \([p]\); then each representation \( R \) of degree 1 would correspond to a character \( \chi \) such that \( \chi([p]) = R(\text{Frob}(p)) \). Not only was the idea of Artin's reciprocity law inspired by analysis, but also its proof.

Artin thanks Tschebotarow for one of the basic ideas of his proof, the use of cyclotomic fields in the way Tschebotarow had used them in proving his famous density theorem.

With Artin's general reciprocity law, the basic structure of class field theory was complete. There soon followed, by Hasse, the generalization of Hilbert's norm-residue symbol \( (\frac{a,b}{\mathfrak{p}}) \) to arbitrary \( \mathfrak{p} \), when the \( \mathfrak{p} \)-th roots of 1 are in \( K \). Also the structure of the Brauer Group and central simple algebras over number fields was determined by Hasse with R. Brauer and E. Noether in the early 1930's. From then until the end of World War II, the main progress in these matters was Chevalley's introduction of ideles and idele classes which enabled smoother formulations and clarified the local-global aspect of class field theory.

In the rest of this talk I want to discuss three post-war developments, each of which represented an extension or generalization of an important aspect of the reciprocity law.

A generalization of the group isomorphism aspect of reciprocity came about as a result of the introduction of the methods of group cohomology into class field theory, by Hochschild, Nakayama and Weil. Artin and I polished these methods in his seminars and, using a trick of Nakayama's, I showed that the cup-product with the fundamental class gives isomorphisms

\[ H^{r-2}(G, \mathbb{Z}) \longrightarrow H^r(G, C_L), \quad \text{for all } r \in \mathbb{Z}, \]

where \( G \) is the Galois group of an arbitrary finite Galois extension \( L/K \). Here the cohomology groups are those for a finite \( G \), in which homology is interpreted as negative dimensional cohomology: \( H^{-1}(\xi, M) = H^{-1-\xi}(\xi, M) \) for
Problem 9: The General Reciprocity Law

$r \geq 1$. Moreover, $H^0(G,M) = H^0/\mathcal{N}_G M$ is the “reduced” $H^0$, consisting of fixed elements modulo norms. For $G$ abelian, we have $H^0(G,\mathbb{Z}) = H_1(G,\mathbb{Z}) = G$, and for $r = 0$, the isomorphism displayed above is just the reciprocity law

$$G \xrightarrow{\sim} \mathcal{N}_L^G / \mathcal{K}_L^G.$$  

Further developments by Nakayama and me led to a general duality theorem for the cohomology of infinite Galois groups like those of the maximal algebraic extension unramified outside a certain set of primes, and these Galois-cohomological results have been reinterpreted and generalized in terms of etale cohomology by M. Artin and B. Mazur.

When the ground field $K$ contains the $4^{th}$ roots of unity, the explicit reciprocity law for $4^{th}$ power residues involves the norm residue symbols $\left( \frac{a_1}{p} \right)$ for primes $p$ dividing $\mathfrak{a}$, just as the original quadratic reciprocity law involves the symbol $\left( \frac{p}{2} \right) = (-1)^{\frac{p-1}{2}}$ for odd $p$ and $q$, and also $\left( \frac{p^2}{2} \right) = (-1)^{\frac{p-1}{2}}$. In the 1960’s a completely new light was shed on the norm residue symbol as a result of the development of algebraic $K$-theory. The norm residue symbol satisfies several formal identities, the most important of which are, in abbreviated notation, the following:

$$(a_1 a_2 b) = (a_1, b)(a_2, b), \quad (b, a) = (a, b)^{-1}, \quad \text{and} \quad (a, 1-a) = 1.$$  

Work of Steinberg on central extensions of algebraic groups led him to consider all functions of two non-zero variables in an arbitrary field with values in an abelian group satisfying the above identities, and such a function is called a Steinberg cocycle. Milnor defined for every associative ring $\mathfrak{a}$ with $1$ an abelian group $K_G\mathfrak{a}$, and interpreted Steinberg’s results as showing that, when $\mathfrak{a}$ is a commutative field the group $K_G\mathfrak{a}$ is generated by elements $(a, b)$, for $a, b \in A^*$, which do satisfy the above relations. Matsumoto proved that these relations were enough to define $K_G\mathfrak{a}$ for a field $\mathfrak{a}$; in other words $K_G\mathfrak{a}$ is the target group for the universal Steinberg cocycle. Suppose $F_p$ is the completion of a number field $F$ at a prime $p$, and let $U_p$ denote the group of roots of unity in $F_p$. Excluding the case $F_p = \mathbb{Q}$, the group $U_p$ is finite, say of order $m_p$. C. Moore showed that the most general continuous Steinberg cocycle on $F_p$ is the norm residue symbol $\left( \frac{a_b}{p} \right)_{F_p}^{F_{p^m}}$, and that, viewing these as cocycles on $F$ for varying $p$, the only relation among them was that coming from the $m$-th power reciprocity law, where $m$ is the order of

\[ \left( \frac{a_b}{p} \right)_{F_p}^{F_{p^m}} \]
the group of roots of unity $U_p$ in $F$. This result can be expressed by means of an exact sequence,

$$0 \rightarrow \text{Ker } h \rightarrow K_F^h \rightarrow \bigoplus_{p \text{ non-complex}} U_p \rightarrow U_p \rightarrow 0,$$

where $h$ is the map induced by the symbols $(\frac{a,b}{p})_{mp}$. The new thing in this is that the order $m_p$ of the local symbols considered varies with $p$; classically, one never considered symbols of different order simultaneously.

After Moore's determination of $\text{Coker } h$, it was natural to consider the structure of $\text{Ker } h$. Bass told me about the problem in case $F = \mathbb{Q}$, the rational field, and asked me if I could see any relation to classical number theory. I thought not, but a few days later found a proof that $\text{Ker } h = 0$ for $F = \mathbb{Q}$, and soon realized that the method, an induction over the primes, using the euclidean algorithm, was just the formal part of Gauss' first proof of quadratic reciprocity; Gauss had determined the structure of $K_{\mathbb{Q}}/\mathbb{Z}[\mathbb{Z}]$ without realizing it! Using a result of Bass, Garland proved $\text{Ker } h$ finite by showing $K_p$ finite, by analytic methods, where $A$ is the ring of integers in $F$. Later, Quillen gave a definition of higher $K$-groups $K_p$ which enabled him to prove a general finiteness theorem and to prove the exactness of a localization sequence which showed in particular that $\text{Ker } h$ was the kernel of the map of $K_p$ onto its easily computable "tame part" (and not only the image of that kernel, as Bass had shown earlier).

These connections between the reciprocity law and cohomology theory and algebraic $K$-theory which we have just discussed are interesting and important, but I think the biggest problem after Artin's reciprocity law was to extend it in some way to non-abelian extensions, or, from an analytic point of view, to identify his non-abelian L-series. Artin conjectured that these functions are holomorphic in the whole $s$-plane except for a pole of known order at $s = 1$. He proved that they satisfied a functional equation and that a power of each was meromorphic.

R. Brauer proved in the early 1930's that they were meromorphic. Only in recent years has there been further progress.

In 1967, Langlands was studying the analytic theory of automorphic forms on general reductive algebraic groups and saw a formal relation between Artin's $L$-series and some Euler products arising in the theory of Eisenstein series. This led him to some general conjectures, of which the following is a superspecial case:
Let \( R: \text{Gal}(L/\mathbb{Q}) \to \text{GL}(2, \mathbb{C}) \) be an irreducible representation of degree 2 of the Galois group of a finite Galois extension \( L \) of \( \mathbb{Q} \). Let \( N \) be the conductor of \( R \), as defined by Artin, and let \( X: \mathbb{Z} \to \mathbb{C} \) be the Dirichlet character such that for each prime number \( p \) unramified in \( L \) we have \( X(p) = \det R(\text{Frob}_L(p)) \). (The existence of \( X \) follows from the famous theorem of Kronecker-Weber which states that every abelian extension of \( \mathbb{Q} \) is contained in a cyclotomic extension.) Suppose \( X(-1) = -1 \). Let the Artin \( L \)-series attached to \( R \) be

\[
L(s,R) = \sum_{n \geq 1} a_n n^{-s},
\]

and put

\[
f_R(z) = \sum_{n \geq 1} a_n e^{2\pi i nz}.
\]

Then \( f_R \) should be a holomorphic "new-form" of weight 1, character \( X \), and level \( N \). In particular, \( f_R \) is holomorphic in the upper half-plane \( \text{Im} \ z > 0 \), and should satisfy

\[
f_R \left( \frac{az+b}{cz+d} \right) = (cz+d)X(d)f_R(z), \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \text{ and } c \equiv 0 (N).
\]

Moreover, every holomorphic new-form \( f \) of weight 1 should be of the form \( f_R \) for a Galois representation \( R \) as above.

In general, Langlands sees a pattern in which there should be such a statement for representations of any degree \( n \) of Galois groups over any global field \( K \). For \( n = 1 \), and any \( K \), the relationship envisaged by Langlands is just Artin's reciprocity law. For \( n = 2 \) and \( K = \mathbb{Q} \), it is the conjecture above, if the representation \( R \) is "odd" in the sense that \( X(-1) = -1 \). (For "even" representations the picture involves non-holomorphic modular forms.) In fact, Langlands' vision goes farther, embracing arbitrary reductive algebraic groups, not only \( GL(n) \), and the \( \ell \)-adic representations of Galois groups arising from the cohomology of algebraic varieties as well as Artin representations. A. Weil had the idea in special cases, especially concerning the \( L \)-series attached to elliptic curves.

What is the situation regarding proofs of these conjectures? Almost nothing is known beyond \( GL(2) \), and I will limit the discussion here to the case of \( GL(2) \) and Artin representations. In 1969 Langlands proved that the function \( f_R \) above is a new form if and only if the \( L \)-series \( L(s,\text{Rom}_1) \) is entire (i.e., satisfies Artin's conjecture), for every representation.
of degree 1 of Galois groups over \( \mathbb{Q} \); in fact, he proved the analog of
that statement for arbitrary Artin-representations of degree 2 over any
global field. To do this he had "only" to show that the constant in the
functional equation of \( L(s, RR_R) \) behaved in a certain way as function of
\( R_R \), and then apply the generalization of Hecke's fundamental theorems
relating Dirichlet series with functional equations to automorphic forms
which had been developed by Jacquet-Langlands and, in a different form,
by Weil. In 1973, Deligne and Serre proved that every holomorphic new
form \( f \) of weight 1 was of the form \( f_R \) for an "odd" Galois representation
\( R \) over \( \mathbb{Q} \) of degree 2 as above. This result, which can be viewed as a
degree 2 analog of the Kronecker-Weber theorem, is proved by methods
which work only over \( \mathbb{Q} \), not over an arbitrary \( K \). Deligne had associated
\( l \)-adic representations to modular forms of weight \( \geq 2 \). Starting from a
form of weight 1, Serre considered for each prime \( l \) the Deligne
representation \( R_l \) corresponding to a form \( f_l \) of higher weight which
was congruent to \( f \) (mod \( l \)). Assuming the Petersson conjecture for \( f \),
Serre could then show that an infinity of these \( R_l \)'s were congruent (mod \( l \))
to one fixed Artin representation \( R \), whose \( L \)-series corresponded to \( f \).
Deligne then realized that Serre's construction of \( R \) would go through
with a weak form of the Petersson conjecture which had been proved
especially by Rankin. This gave the theorem; and as a by-product, a
tortuous proof of the full Petersson conjecture in weight 1, where it is
not a consequence of the conjectures of Weil which were proved by Deligne.

Putting together the results of Langlands and Deligne-Serre we see
that there is a one-to-one correspondence between "odd" Artin
representations \( R \) over \( \mathbb{Q} \) of degree 2 such that \( RR_R \) satisfies the Artin
Conjecture for all \( R_R \) of degree 1, and holomorphic new forms of weight 1.
If we try to find explicit examples of this correspondence, there is a
catch on each side, indicated by the underlined phrase. It is easy to
construct representations \( R \), but in general we don't know they satisfy
Artin's Conjecture. On the other hand, while it is relatively easy to
construct modular forms of weight \( \geq 1 \), and the Riemann-Roch theorem
tells us exactly how many of them there are at each level, it is not so
easy to exhibit forms of weight 1, and the Riemann-Roch formula fails to
decide how many of them there are at a given level. Of course if the
representation \( R \) is monomial, that is, is induced from a degree 1
representation \( R' \) over a quadratic extension \( K \) of \( \mathbb{Q} \), then the Artin
Conjecture is satisfied (because \( L(s, R) \) is equal to \( L(s, R') \)), an ordinary
abelian \( L \)-series for \( K \) and consequently, by Langlands' result, \( f_R \) is
a modular form. But this "dihedral" case was known to Hecke in 1926!
He constructed the $f_R$'s by theta-functions belonging to binary quadratic forms - forms associated with ideals in the quadratic field $K$. At the time he wrote quite explicitly that the way to get modular forms is from arithmetic, and in particular, from $L$-series with a functional equation in which the $\Gamma$-factor, which in general is of the form

$$\Gamma\left(\frac{s}{2}\right)^{a} \Gamma\left(\frac{s+1}{2}\right)^{b} \Gamma(s)^{c},$$

is of the very simple form $\Gamma(s)$. He then listed all the cases of abelian $L$-series with such a $\Gamma$-factor, thus getting just the dihedral "odd" $R$'s (besides the reducible "odd" $R$'s which give Eisenstein series). Hecke wondered whether in this way he got all modular forms of weight 1, though he doubted this was so. Now, at the same time, in the same University, Hamburg, Artin was defining his non-abelian $L$-series $L(s, \rho)$ and proving they satisfied a functional equation, in which the $\Gamma$-factor was sometimes $\Gamma(s)$, namely, when $R$ is "odd" of degree 2. Thus, if either had really understood what the other was doing, the above special conjecture might very well have been found by Artin and Hecke 40 years before Langlands hit on it. Perhaps mathematics wasn't ready. As Hilbert wrote in the introduction to his Mathematical Problems:

"Wenn uns die Beantwortung eines mathematischen Problems nicht gelingen will, so liegt häufig der Grund darin, dass wir noch nicht den allgemeineren Gesichtspunkt erkannt haben, von dem aus das vorgelegte Problem nur als einziges Glied einer Kette verwandter Probleme erscheint."

The way Langlands found the conjecture seems a good example of what Hilbert had in mind.

Another reason for the relationship's eluding Artin and Hecke may be the fact that explicit non-dihedral numerical examples are hard to find. Indeed at the time of the DeKalb conference, none was known! I concluded the oral presentation of this paper there by explaining that, in the hope of finding one, I had looked for non-dihedral $R$'s of low conductor, $N$, and had found an $R$ with $N = 135 = 7 \cdot 19$ which I hoped might be amenable to computation. After the talk, Atkin suggested that the labor involved might be considerably reduced by systematic use of the involutions $w_7$ and $w_{19}$. Armed with his theory of the $w$'s, four Harvard students, D. Flath, R. Kottwitz, J. Tunnell, J. Weisinger and I succeeded in the next month in proving (by relatively easy hand computation) the existence of the corresponding new form $f_R$ of weight 1 and level 135 predicted by
Langlands. By the theorem of Deligne-Serre, this produced the first example of an Artin L-series which is known to be holomorphic in spite of the fact that no power of it is a product of abelian L-series.