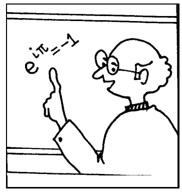
Classroom



In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Creating Math Problems*

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I have been asked several times, by teachers, students and friends, how I *create* math problems, specifically the ones that appear in olympiads. For me, olympiad problems are a by-product of my explorations of mathematics. In this write-up I will explain this with a particular example. We will start with a simple problem and create many more problems of different flavours and levels.

A Simple Problem

I once saw the following problem in a book.

Problem 1. *The numbers* $1, 2, 3, \ldots, n$ *are written on a board. We erase two of these numbers (of our choice) and write their sum back on the board. We keep repeating this process until there is only one number left on the board. What number is it?*

This is not a difficult problem. One can try to see what happens with a simpler example, say with only $1, 2, 3$ and 4 on the board

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to begin with. In this case, a few quick additions will convince us that no matter how we pick two numbers each time, the result in the end, is 10. One can also try for a few more sets of simpler examples to get the pattern. On the other hand, if we think about what exactly we are doing in the process then we realise that we are just adding all the numbers up, in different orders based on our choices. What if replace addition with some other operation? For example, with multiplication. Even in this case, the final number is always the sum of all the numbers that we had to begin with. The answer to the problem, therefore, is the sum $1 + 2 + \cdots + n = n(n + 1)/2$[1] .

[1] One can attribute this solution to the fact that addition of natural numbers is commutative and associative.

Our solution to the problem is complete, but here is where exploration starts. Can we replace the numbers with a different set of numbers? Sure, the solution will still be the same. Can we replace addition with some other operation? Even in this case, the solution is very similar, and the final answer would be the product of all the numbers we had at the beginning. What if we replace addition by subtraction?[2]

[2] Multiplication is also commutative and associative. Subtraction is neither commutative nor associative.

At first sight, this seems to go nowhere because we can potentially get different answers based on our choices. For example, if we start with the numbers $1, 2, 3, 4$, then the following two scenarios result in two different answers.

The numbers $1, 2, 3, \ldots, n$ are written on a board. We erase two of these numbers a and b (of our choice) and write $a - b$ back on the board. We keep repeating this process until there is only one number left on the board. Is this final number odd or even?

$$
\begin{array}{ccccccc}
1 & & \underline{\underline{2}} & & \underline{3} & & 4 \\
& 1 & & \underline{-1} & & \underline{\underline{4}} & \\
& & \underline{1} & & \underline{\underline{5}} & & \\
& & & 4 & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
\underline{\underline{1}} & & 2 & & \underline{3} & & 4 \\
& \underline{\underline{-2}} & & \underline{2} & & 4 & \\
& & \underline{\underline{-4}} & & \underline{4} & & \\
& & & -8 & & &
\end{array}
$$

In each row, we subtract $\underline{a}$ from $\underline{\underline{b}}$.

So the final number in the above two scenarios are 4 and -8. What other numbers can we get? If we play with a few more scenarios, we will start seeing a "pattern". In fact, we can only get an even number at the end (and further, the final number is between −9 and 9). Can we conclude that even if we had started with different set of numbers we would get an even number at the end? It is easy enough to try out a few more examples with small sets of numbers. If we try with $1, 2, 3, 4, 5$, we will see that we get only an odd number at the end. So we have a new problem.

Problem 2. *The numbers* $1, 2, 3, \ldots, n$ *are written on a board. We erase two of these numbers* a *and* b *(of our choice) and write* $a-b$ *back on the board. We keep repeating this process until there is only one number left on the board. Is this final number odd or even?*

If we carefully look at what we are doing in the process, we will see that the final number is of the form

$$s = \pm 1 \pm 2 \pm 3 \pm \cdots \pm n\,,$$

with at least one plus and one minus. We can rewrite this as

$$s = 1 + 2 + \cdots + n - 2(a_1 + a_2 + \cdots + a_k)\,,$$

where $a_1, a_2, \ldots, a_k$ are the numbers which appear with a minus sign. This shows that s and $n(n+1)/2$ have the same parity. The latter, and hence the former, is even if and only if 4 divides either n or $n+1$. Another way to look at this is to notice that for any integers a and b, the sum $a+b$ and the difference $a-b$ differ by an even number. In other words, $a - b \equiv a + b \pmod 2$. Therefore, the parity of the final number does not change if we replace subtraction by addition.

Here is more or less the same problem, but rephrased in a different way.

Problem 3. *A grasshopper is standing at the origin of a real line. It makes a total of* 2018 *moves. In the k-th move, it jumps from its position* x *to either* $x-k$ *or* $x+k$*. After* 2018 *moves, Calvin says that the grasshopper is at* $x = 1000$ *and Hobbes says that it is at* $x = 1001$*. Who among Calvin and Hobbes is right?*

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Invariants

Abstractly, in the problems above, we are replacing two numbers a and b by $a+b$ or $a-b$. In the first case, we keep the sum of all the numbers on the board a constant. Or in other words, the sum of the numbers is an *invariant*. In the second case, the parity of the sum of the numbers is an invariant. In both the cases, the number of entries on the board reduces all the way until we have only one number on the board. This way, we know that the process is always going to stop. How about creating a possibly infinite process? So instead of replacing a and b by one single number, we can replace them with two numbers. How shall we choose these two new numbers? One of the nice properties we had was that there was some invariant. So let us choose two new numbers such that their sum is $a + b$. This way we shall keep the sum of all the numbers on the board same all the time. The two new numbers should be derived from a and b. Here is one possible choice: replace a and b by $2a$ and $b - a$. We can also keep the numbers non-negative by always making sure that $b \geq a$. Let us now explore this case.

In explorations like this, we often look at special cases to get some insights into the structure or pattern of what we are dealing with. We did so in Problem 1 by looking at the special cases $n = 4$ and $n = 5$. In here, since the number of entries on the board does not reduce, we can simply look at just two numbers. To further simplify, let one of these numbers be 1. Let us look at what happens for a bunch of small values of the other number.

1	**1**
2	0
2	0

1	**2**
2	1

1	**3**
2	2
4	0
4	0

1	**4**
2	3
4	1

In each of the cases, the last *configuration* had already appeared sometime before, so the pattern will repeat. If we think about it more, it becomes clear that there has to be a repetitive pattern because there are only finitely many possible choices of the pair, namely, $(1, n), (2, n-1), (3, n-2), \ldots, (\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$. Note that we are using the fact that the sum of the two numbers is a constant.

One more observation we can make is that in case we had $(1, 1)$ or $(1, 3)$ in the beginning then we ended with 0 on the board (and it remains on the board thereafter). In the other two cases we got back to the original configuration. We can now rephrase our observation into a problem.

Problem 4. *Let n be a natural number. We start with two boxes, one containing just one marble and the other containing n marbles. In each* move*, we take the box with smaller number of marbles, double that number by transferring the same amount of marbles from the other box. If two boxes have the same number of marbles then we choose one of the boxes randomly to double the number of marbles in it. For what values of n will one of the boxes become empty after finitely many moves?*

One should always be aware of concluding too quickly from few special cases. For example, from so far what we have seen, we can possibly try to guess the answer to the above problem to be odd values of n. Let us consider one more example with $1, 5$ as the initial numbers. In this case, we next get $2, 4$ which repeats thereafter. So the answer to the above problem is not odd values of n. Note also that in case we start with 1 and 5, neither one of the numbers becomes zero nor do we go back to the original numbers. We shall come back to this later.

The exploration can take several routes at this stage. We shall go in one direction for the purpose of this article. We shall write a small piece of code (in Python) to see what the answer to the problem is!

```
for n in range(1, 1000):
    smaller_no = 1
    small_no_seen_so_far = [smaller_no]
    pattern_found = False
    while not pattern_found:
        a = 2 * smaller_no
        b = n + 1 - 2 * smaller_no
        smaller_no = min(a, b)
        if smaller_no == 0:
            print(n, end=', ')
            pattern_found = True
        if smaller_no in smaller_no_seen_so_far:
            pattern_found = True
        smaller_no_seen_so_far.append(smaller_no)
print('')
```

If we run this, we get the output $1, 3, 7, 15, 31, \ldots, 511$. So we can guess that one of the boxes will become empty if and only if n is of the form $2^k - 1$ for some natural number k. It is not difficult to show the 'if' part. If $n = 2^k - 1$, then after the first move, we will have 2 and $2^k - 2$ marbles in each of the boxes and after the next we will have 2^2, $2^k - 2^2$ marbles (provided $k \geq 2$). The pattern repeats for $k - 1$ moves, and we will have 2^{k-1} and $2^k - 2^{k-1}$ marbles at this stage. Note that the two numbers are the same so after the next move one of the boxes becomes empty.

We start with two boxes, one containing just one marble and the other containing 10^{10} marbles. In each move, we take the box with smaller number of marbles, double that number by transferring the same amount of marbles from the other box. Show that none of the boxes ever become empty.

We also observed on route to this solution that if one of the boxes becomes empty only if in the stage before the number of marbles in both the boxes are equal, say equal to m. But the total number of marbles is a constant, so we get $n + 1 = 2m$. We now have a special case of the problem which has a simple solution.

Problem 5. *We start with two boxes, one containing just one marble and the other containing 10^{10} marbles. In each move, we take the box with smaller number of marbles, double that number by transferring the same amount of marbles from the other box. Show that none of the boxes ever become empty.*

Before going back to Problem 4, we shall look at one more special case. We have seen that for one of the boxes to become empty, the total number of marbles should be even. If $n+1$ is even and one of the boxes becomes empty, then both the boxes will have $(n+1)/2$ marbles each before one of them becomes empty. How can we reach the stage with the boxes having $(n+1)/2$ marbles each? We must have doubled the number of marbles in one of the boxes to reach that stage. So $(n+1)/2$ is even! (Here we are assuming $n \geq 2$.) So here is another variation of the problem with a somewhat simple solution.

Problem 6. *We start with two boxes, one containing just one marble and the other containing $10^{10}+1$ marbles. In each move, we take the box with smaller number of marbles, and double that number by transferring the same amount of marbles from the other box. Show that none of the boxes ever become empty.*

Now we go back to solving Problem 4. We have to show that if n is not of the form $2^k - 1$ then we will never have an empty box. From what we have argued so far, we can assume that n is odd. So after the first move, we will have 2 and $n-1$ marbles in the boxes. Note that both these numbers are even and hence all the numbers we will get hereafter will be even. So we can divide each of them by 2 and then consider the case with the smaller numbers, namely, with 1 and $(n-1)/2$. It is not difficult to conclude the result from here by simple induction.

We start with two boxes, one containing just one marble and the other containing $10^{10}+1$ marbles. In each move, we take the box with smaller number of marbles, double that number by transferring the same amount of marbles from the other box. Show that none of the boxes ever become empty.

Recall that Problem 4 was a special case of a special case! We considered only two numbers, and we also fixed one of the two numbers. Here is a more general version of the problem (which has a very similar solution).

Problem 7. *Let $n \geq 2$ be a natural number. There are n numbers $a_1, a_2, \ldots, a_n$ written on a blackboard such that the greatest common divisor (gcd) of these numbers is* 1. *In each* move*, Calvin chooses two of these numbers, say x and y with $x \leq y$, and replaces them with $2x$ and $y-x$. Show that Calvin can make $n-1$ of these numbers zero if and only if $a_1 + a_2 + \cdots + a_n$ is a power of* 2.

We have added the condition on the gcd of the numbers for convenience.

Repeating Patterns

We now know that if n is a natural number not of the form $2^k - 1$ and we start with the numbers 1 and n, then after a finite number of moves we will have a repeating pattern. However, we saw that only in some cases, the numbers return back to 1 and n. So the following problem is a natural one to pose.

Problem 8. *Written on the blackboard are two natural numbers a and b. In each move, we replace the existing numbers x and y, say with $x \leq y$, by $2x$ and $y - x$. For what values of a and b do we get back to the same numbers after finitely many moves?*

We can make a simple change in our code to check this for $a = 1$ and $b = n$ with $1 \leq n \leq 1000$. We will then see that we get back to the original numbers whenever n is even. We will leave it to the readers to explore further and find the solution to this problem[3], and instead of giving a solution to this, we shall explore a different property.

[3]We get back to the original numbers if $a = da_0$ and $b = db_0$ with $gcd(a_0, b_0) = 1$ and one of a_0, b_0 is even.

Suppose that we start with 1 and $2n$ for some natural number n. We know that after finitely many moves, we will be back to the same numbers again. A natural question to ask is: what is the smallest number of moves after which we get back to the same numbers? It is again easy to write a code to see the pattern.

```
for n in range(2, 11):
    a = 2
    count = 1
    while a > 1:
        count += 1
        a = min(2 * a, 2 * n + 1 - 2 * a)
    print((2 * n, count))
```

From the result we have the following answers for small values of n.

$2n$	2	4	6	8	10	12	14	16	18	20
no. of moves	1	2	3	3	5	6	4	4	9	6

The pattern is not clear. In fact, looking at the first few cases is not really going to help much. But we looked at them nevertheless for two reasons. First, the special cases are what we explore always in the beginning. And second, we wanted to demonstrate that the special cases may not always help!

Let us look once more at what a move consists of. If we had the numbers x and y with $x \leq y$ then we replace them with $2x$ and $y - x$. We also know that $x + y = 2n + 1$ is a constant. In the first section, we had talked about the difference between addition and subtraction. That is what we shall use here. Note that $y - x = 2y - (y + x) = 2y - (2n + 1)$. So, our move is the same as doubling the two numbers and subtracting $2n+1$ from the larger one (which will be more than $2n+1$). So if we look modulo $2n+1$, then we are just doubling the two numbers. Therefore, if we start with the numbers 1 and $2n$, then after k moves the numbers on the board are 2^k and -2^k modulo $2n+1$. Hence the smallest number of moves to get back to the original numbers equals k where $2^k \equiv -1 \pmod{2n + 1}$ or $2^k \equiv 1 \pmod{2n + 1}$. With this knowledge, we can pose the following problem.

Problem 9. *Written on the blackboard are two natural numbers* 1 *and* 124*. In each move, we replace the existing numbers x and y, say with $x \leq y$, by $2x$ and $y - x$. What is the smallest value of $k \geq 1$ such that after k moves, we have the numbers* 1 *and* 124 *on the blackboard?*

The answer to this question is 50. It follows from the fact[4] that, for any natural number l, the smallest $d \geq 1$ for which $2^d \equiv 1 \pmod{5^l}$ is $d = 4 \cdot 5^{l-1}$.

[4] 2 is a generator of $\mathbb{Z}/5^k\mathbb{Z}$.

Two Variations

Written on the blackboard are natural numbers 1 and n. In each move, if the existing numbers x and y are not equal, say $x < y$, then we replace them with $2x + 1$ and $y - x - 1$. We stop when the two numbers are equal. Show that we stop after k moves if and only if n equals $2^{k+2} - 3$.

Let us get back to Problem 4. We had a condition that if both the boxes had an equal number of marbles, then we choose one of the boxes randomly for doubling the number of marbles in it. If we consider the equal numbers as the stopping point, then until we stop we will have x and y marbles in the boxes with $x < y$. So we can replace them by $2x + 1$ and $y - x - 1$ without resulting in negative numbers. Note that in this case, having zero marbles is not a stopping point. So the question now is for what values of n will the process stop if we start with the numbers 1 and n?

Our approach to answer this is similar to the ones above. We can first write a piece of code to make our guess.

```
for n in range(3, 1000):
    a = 3
    smaller_numbers = []
    count = 1
    while a not in smaller_numbers and a != n:
        count += 1
        smaller_numbers.append(a)
        a = min(2 * a + 1, 2 * n - 2 * a - 1)
    if a == n:
        print((2 * n, count))
```

From this we guess that if $n = 2^{k+2} - 3$ then after k moves, we will have equal numbers. So we have our new problem (which we will leave it to the readers to solve).

Problem 10. *Written on the blackboard are natural numbers* 1 *and n. In each move, if the existing numbers x and y are not equal, say x < y, then we replace them with* $2x + 1$ *and* $y - x - 1$. *We stop when the two numbers are equal. Show that we stop after k moves if and only if n equals* $2^{k+2} - 3$[5] .

[5]Hint: If a_i and b_i are the numbers after the i-th move, try to write them as a linear combination of a_0, b_0 and 1 and analyse the coefficients.

Finally, we shall look at a case where the sum of the two numbers is not an invariant. Let us start with two numbers a and b on a blackboard. Instead of keeping $a + b$ a constant, we can let it grow. To keep things simple, in a move, let us replace one of the numbers of our choice by their sum. Starting from two random natural numbers, can we possibly reach some interesting numbers? For example, here is a problem that explores one such direction.

Problem 11. *Two natural numbers are written on a blackboard. A move consists of replacing one of the numbers by the sum of the two. Prove that, starting with any two coprime numbers on the board, one can make a number of appropriate moves to end up with two squares on the board*[6] *.*

[6] One can use Dirichlet's theorem on primes in arithmetic progression, combined with Chinese remainder theorem and Hasse's principle to solve this problem.