The Life and Work of John Tate*

Jean-Pierre Serre

John Torrence Tate passed away in Cambridge, Massachussets on October 16, 2019, at the age of 94. An American citizen, he was an associate member of the French Academy since 1992[1].

After graduating from Harvard, he obtained his PhD at Princeton, in 1950, under the supervision of Emil Artin. He then worked as a professor at Harvard (1954–1990) and Austin (1990–2009), before returning to Harvard.

He made frequent long visits to France. In his autobiography ([TA]) he notes as special the year 1957–1958, which he spent in Paris ("… a great year for me"); it allowed him to participate in the Grothendieck seminar at IHES, as well as the courses and seminars at the Collège de France. He also took part in several Bourbaki meetings.

His work is centered on *number theory*, and its relationships with algebraic geometry. We owe him a large number of ideas and notions which have revealed themselves to be fundamental: Tate modules, Hodge-Tate decompositions, Mumford–Tate groups, Shafarevich-Tate group, Tate conjectures, Sato–Tate conjecture, etc. An excellent analysis of his work can be found in the report by J. Milne ([M2]). I will confine myself to summarizing those aspects that I know best.

- His thesis (defended in May 1950, cf. [T50], but only published in 1967) is a work suggested by Emil Artin: rewrite Hecke's results on zeta functions of number fields in the language of harmonic analysis on locally compact abelian groups. Hence the results are not new, but they give a modern treatment of old results. This thesis has had a lot of influence, because the point of view and the notation it introduces are perfectly adapted to the

Jean-Pierre Serre, (born September 15, 1926, Bages, France), is a French mathematician who was awarded the Fields Medal in 1954. In 2003 he was awarded the first Abel Prize by the Norwegian Academy of Science and Letters. Serre's mathematical contributions leading up to the Fields Medal were largely in the field of algebraic topology. He was a principal contributor to applications of algebraic geometry to number theory.

[1] This article is a translation of "La vie et l'oeuvre de John Tate" to be published in *Comptes Rendus Acad. Sci.*, 2020.

Keywords

Tate isogeny theorem, Hodge–Tate decompositions, Mumford–Tate groups; Shafarevich–Tate group; Tate conjectures, Sato–Tate conjecture.

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study of harmonic analysis on the adelic points of a reductive algebraic group over a global field (the case considered by Tate being that of the multiplicative group $G_m = GL_1$.

- His first original result is that of [T52]: *the Galois cohomology of the idèle class group is isomorphic*, with a shift of two, *to that of* **Z**. His proof uses a modified version of the cohomology of finite groups that he had introduced shortly before, without publishing it. One of the consequences is a cohomological criterion for an element which is locally a norm to be a global norm. This paper, which is just 4 pages long, earned him in 1956 the Cole Prize for Number Theory.

- In 1959, he obtained a surprising result ([T59]): the theory of elliptic functions due to Abel and Jacobi, formulated as power series in $q = e^{2i\pi z}$, can be *transferred to the p-adic case with exactly the same formulas*, once we interpret q as a non-zero p-adic number of absolute value < 1.[2] And so we obtain, essentially, all the p-adic elliptic curves having bad reduction at p. Tate, as was often the case later, did not immediately publish this beautiful result. He just wrote it to me ([CS], 4/8/1959) and made copies of his letter; copies which multiplied rapidly, so that his results quickly became "well-known"[3]. The first publication dates from 1993: [CW], no. 69.

[2]In fact, as Deligne and Rapoport have shown in [DR, §VII], the natural framework of Tate's construction is not a p-adic field, but the ring $\mathbf{Z}[[q]]$.

[3]This is reminiscent of the way in which Fermat and Pascal made their results known: Tate would have been comfortable in the 17th century.

- Tate immediately saw that the above results must have a more general form: there must exist *a theory of rigid analytic p-adic varieties*, much closer to algebraic geometry than the usual local theory (which we now call "soft"). Two years later (October 1961), it was done: he wrote a detailed version of this theory, which he sent me again in the form of a series of letters, which I immediately reproduced via IHES; they would be published nine years later in the journal *Inventiones mathematicae* ([T62b]). For more details, see sections 1–2 of Colmez's report ([CS]).

- He returned to number fields in his 1962 presentation to the International Congress at Stockholm ([T62a]): *Duality theorems* for Galois cohomology of local and global fields (so-called Poitou–Tate theory, because Georges Poitou had obtained essentially the same results, roughly at the same time). The statement of the local theorem is simple:

Let k be a local field, and let $\Gamma_k = \mathrm{Gal}(k_s/k)$, where k_s is a separable closure of k. Let M be a finite Γ_k module of order prime to the characteristic of k. Let $M' = \mathrm{Hom}(M, k_s^\times)$ be the *Tate dual* of M. For all r, the cup-product defines a bilinear pairing

$$H^r(\Gamma_k, M) \times H^{2-r}(\Gamma_k, M') \to H^2(\Gamma_k, k_s^\times) \cong \mathbf{Q}/\mathbf{Z}$$

and the theorem says that this bilinear map gives a duality between the two cohomology groups $H^r(\Gamma_k, M)$ and $H^{2-r}(\Gamma_k, M')$. We have $H^r(\Gamma_k, M) = 0$ if $r \neq 0, 1, 2$.

There is an analogous result[4] when M (resp. M') is replaced by the group of k_s-points of an abelian variety (resp. of its dual). The main difference is that the exponent $2 - r$ is replaced by $1 - r$; in addition, cohomology groups in dimension 0 are compact groups, while they are discrete groups in dimension 1; the cohomology groups in dimension $\geqslant 2$ are zero.

[4]Tate proved this result when char(k) = 0; the case when char(k) $\neq$ 0 is due to J. Milne cf. [M1].

- A nice complement to local class field theory: the paper [T65a], is written in collaboration with his former pupil J. Lubin. The authors show how the isomorphism of the local class field can be constructed explicitly from a certain formal group; it is a theory analogous to that of global class field theory for $\mathbf{Q}$ and for imaginary quadratic fields. Application: a very natural proof of *the existence theorem* in the local case.

- In 1964, at the Woods Hole Summer School, Tate set out *a series of conjectures* which would have a great influence, cf. [T65b]. One of the most striking conjectures is the one about algebraic cycles of codimension d in a smooth projective variety X over a field k which is of finite type over the prime field. Let ℓ be a prime number not dividing the characteristic of k, and let $H^{2d}(X)_\ell$ be the etale cohomology group of X/k_s with coefficients in $\mathbf{Q}_\ell(d)$. Any codimension d cycle defines an element $c(z)$ of $H^{2d}(X)_\ell$ which is invariant under the Galois group $\Gamma_k = \mathrm{Gal}(k_s/k)$. Tate conjectures that, conversely, *any element of* $H^{2d}(X)_\ell$ *invariant under* Γ_k *is a* $\mathbf{Q}_\ell$*-linear combination of classes of type* $c(z)$. This statement is particularly interesting when k is a finite field: the invariance under Γ_k then means invariance under the action of the Frobenius automorphism of X.

As Tate notes, there is a certain analogy between this statement and the classical Hodge conjecture for complex algebraic varieties[5].

[5] Curious difference: the case $d = 1$ of the Hodge conjecture has been proved a long time ago, whereas it is not so for the Tate conjecture in this case.

- It was also at Woods Hole that Tate announced what is sometimes called the *Serre–Tate theorem*: lifting an abelian variety infinitesimally is equivalent to lifting its p-divisible group. There is a brief summary in [T65b], but the details of the proof can be found in [CS], 1/10/1964. It is clear from [CS] that the theorem, in its general form, is entirely due to Tate. My contribution was only ([CS], 12/10/1963) to have first proved it in the special case where the abelian variety is ordinary, using the Greenberg functor; as Tate did not like this method, this prompted him to find another one, much more general in scope.

- In 1966, Tate proved an astonishing theorem about abelian varieties over a finite field: *they are determined, up to isogeny, by the eigenvalues of their Frobenius endomorphisms.* He was so happy that he wrote to me in French[6] and, for once, he published it without further delay ([T66a]). The essential step consists in proving that if A is an abelian variety of positive dimension over a finite field, the ring End(A) is not reduced to $\mathbf{Z}$. This is proved by showing that, if $\mathrm{End}(A) = \mathbf{Z}$, the abelian variety A has so many isogenies that there exists an infinity of mutually non-isomorphic polarized abelian varieties, all of the same dimension, and with the same degree of polarization – which is impossible. This method was used again, seventeen years later, by Faltings, the base field this time being a number field, cf. [F1].

[6] "The answer is so beautiful that I want to write it in French", [CS], 2/17/1966.

- It was at the same time (1965/1966) that Tate gave, in a seminar at the College de France, a series of ten lectures on *p-divisible groups* and their applications to abelian varieties ([S], p. 321–324). We find there for the first time the surprising properties of the field $\mathbf{C}_p$, the completion of an algebraic closure of a local field k of unequal characteristic, as well as *the Hodge–Tate decomposition* of $T_p(G) \otimes \mathbf{C}_p$, when G is a p-divisible group and $T_p(G)$ is its Tate module. When G is the p-divisible group associated with an abelian variety A over k having good reduction, this amounts to saying that $H^1_{\mathrm{et}}(A, \mathbf{Q}_p) \otimes \mathbf{C}_p$ decomposes into the direct sum of $H^1_{\mathrm{Zar}}(A, \Omega^0_A) \otimes \mathbf{C}_p$ and $H^0_{\mathrm{Zar}}(A, \Omega^1_A) \otimes \mathbf{C}_p(-1)$.

  Tate also conjectured that an analogous decomposition exists for the cohomology (in any dimension) of a smooth projective variety over a local field ([T66b]); this was proved later by Fontaine-Messing ([FM]), with a restriction on the residual characteristic, and then, in the general case, by Faltings ([F2]).

Other Works by Tate

- A very simple definition of the *canonical height* (also called the Néron -Tate height) of the rational points of an abelian variety; he showed that it is a quadratic form, as conjectured by Néron : [CS], 24/10/1962.
- *Sato-Tate conjecture* for elliptic curves on **Q**: [CS], 05/08/1963. It is the prototype of a series of conjectures on Galois representations associated with motives.
- *Good reduction criterion* for abelian varieties, and applications to CM type varieties: [T68].
- The structure (with F. Oort) of *finite flat group schemes of order* p: [T70].
- *p-adic analogues of the Birch and Swinnerton-Dyer conjecture* (with B. Mazur): [T87].
- The determination (with H. Bass) of the *Milnor K-groups* K_n *of number fields*, for $n \geqslant 3$, which leads to the determination of $K_n/2K_n$ in those cases: [T73].
- A contribution to the study of some *local non-commutative rings* resembling the regular commutative local rings (with M. Artin and M. Van der Bergh); there are elliptic curves involved: [T90].

In addition to two American Mathematical Society awards (Cole Prize 1956 and Steele Prize 1995), Tate received two major international awards: the Wolf Prize (Israel 2002), shared with Mikio Sato, and the Abel Prize (Norway 2010).

Tate's Publications

[CW] ***Collected Works of John Tate*** **(B. Mazur and J.-P. Serre edit.), two volumes, *Amer. Math. Soc*., 2016.**

[T50] J. Tate, *Fourier analysis in number fields and Hecke's zeta functions*, PhD. thesis, Princeton, 1950 ; in *Algebraic Number Theory*, Acad. Press (1967), pp.305–347 ; [CW], no 1.

[AT] E. Artin and J. Tate, *Class Field Theory*, Princeton, 1951–1952; W A Benjamin, New York, 1990 ; revised edition, *Amer. Math. Soc*., Chelsea Publ., 2009.

[T52] J. Tate, *The higher dimensional groups of class field theory*, *Ann. of Math*., 56, 1952, pp. 294-297 ; [CW], no 7.

[T59] –, *Rational points on elliptic curves over complete fields*, unpublished manuscript, Harvard, 1959 ; reproduced in 1993 in [CW], no 69.

[T62a] –, *Duality theorems in Galois cohomology of number fields*, in *Proc. Internat. Congr. Mathematicians* (Stockholm 1962), Inst. Mittag-Leffler, Djursholm (1963), pp.288–295 ; [CW], no 18.

[T62b] –, *Rigid analytic spaces*, notes IHES (1962) ; Invent. math. 12 (1971), pp.257–289 ; [CW], no 36.

[T65a] –, *Formal complex multiplication in local fields* (with J. Lubin), Ann. of Math. 81 (1965), pp.380–387 ; [CW], no 20.

[T65b] –, *Algebraic cycles and poles of zeta functions*, in *Arithmetical Algebraic Geometry*, Harper and Row, New York (1965), pp.93–110 ; [CW], no 21 (completed in [T94]).

[T65c] –, *Elliptic curves and formal groups* (with J. Lubin and J-P. Serre), unpublished manuscript, reproduced in [CW], no 22.

[T66a] –, *Endomorphisms of abelian varieties over finite fields*, Invent. math. 2 (1966), pp.134–144 ; [CW], no 27.

[T66b] –, *p-divisible groups*, in *Proc. Conf. Local Fields* (Driebergen 1966), Springer-Verlag (1967), pp.158–163 ; [CW], no 30.

[T68] –, *Good reduction of abelian varieties* (with J-P. Serre), Ann. of Math. 88 (1968), pp.492–517; [CW], no 33.

[T70] –, *Group schemes of prime order* (with F. Oort), Ann. Sci. E.N.S. 3 (1970), pp.1–21, [CW], no 34.

[T73] –, *The Milnor ring of a global field* (with H. Bass), Lecture Notes in Mathematics 342 (1973), pp.349–446; [CW], no 37.

[T76] –, *Relations between K_2 and Galois cohomology*, Invent. math. 36 (1976), pp.257–274 ; [CW], no 45.

[T81] –, *On Stark's conjecture on the behavior of $L(s,\chi)$ at $s = 0$*, J. Fac. Sci. Univ. Tokyo Math. 28 (1981), pp.963–978 ; [CW], no 54.

[T84] –, *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$* (Lecture notes edited by D. Bernardi and N. Schappacher), Progress in Mathematics 47 (1984), Birkhäuser Boston.

[T87] –, *Refined conjectures of the "Birch and Swinnerton-Dyer type"* (with B. Mazur), *Duke Math. J*. 54 (1987), pp.711–750; [CW], no 59.

[T90] –, *Some algebras associated to automorphisms of elliptic curves* (with M. Artin and M. Van der Bergh), in *The Grothendieck Festschrift* I, Birkhäuser Boston, 1990, pp.33–85 ; [CW], no 61.

[T94] –, *Conjectures on algebraic cycles in ℓ-adic cohomology*, in *Motives* Part 1, pp.71–83, A.M.S. (1994) ; [CW], no 65.

[TA] –, Autobiography, in [HP] below, pp.249–257.

Other Publications

[C] P Colmez, *Tate's work and the Serre-Tate correspondence*, *Bull. Amer. Math. Soc*. 54 (2017), pp.559–573.

[CS] P. Colmez and J-P. Serre (edit.), *Correspondance Serre-Tate*, 2 vol., *Documents Mathematiques* 13-14, Soc. Math. France, 2015.

[DR] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, in *Modular Functions of One Variable* II, L.N. 349, Springer-Verlag 1973, pp.143–316.

[F1] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. math. 73 (1983), pp.349–366 ; Erratum, ibid. 75 (1984), p. 381.

[F2] – , *p-adic Hodge theory*, J.A.M.S. 1 (1988), pp.255–299.

[FM] J-M. Fontaine and W. Messing, *p-adic periods and p-adic etale cohomology*, A.M.S. Contemp. Math. 67 (1987), pp. 179–207.

[HP] H. Holden and R. Piene (edit.), *The Abel Prize 2008-2012*, Springer-Verlag 2014.

[M1] J.S. Milne, *Weil-Châtelet groups over local fields*, Ann. Sci. E.N.S. 3 (1970), pp. 273–284.

[M2] – , *The work of John Tate*, in [HP], pp. 259–334 (contains a detailed analysis and a bibliography of Tate's publications).

[S] J-P. Serre, *Résumé des cours de* 1965-1966, in Oeuvres II, pp. 315-324.

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