On Tate and His Mathematics – A Glossary*

The subject of number theory has developed into a sophisticated branch of mathematics, interacting deeply with other key areas. John Tate is a prime architect of this development, through his accomplishments spanning seven decades. The influence of Tate's ideas on number theory – and, in fact, in mathematics as a whole – is so pervasive that it will be difficult to even define the notions involved in a way understandable to the uninitiated.

Tate has been bestowed with many honours. Already, in 1956, he was awarded the Cole Prize of the American Mathematical Society for outstanding contributions to number theory. In 1995, he received the Steele Prize for Lifetime Achievement from the AMS and the Wolf Prize (shared with M. Sato) in 2002 for his "creation of fundamental concepts in algebraic number theory". He was twice an invited speaker at the ICMs (1962-Stockholm and 1970-Nice). Tate was elected to the U.S. National Academy of Sciences in 1969. He was named a foreign member of the French Académie des Sciences in 1992 and an honorary member of the London Mathematical Society in 1999. The Abel Prize was awarded to him in 2010.

Essential mathematical ideas and constructions initiated by Tate and later named after him abound; some mathematical terms bearing his name are:
Tate modules, Tate elliptic curves, Tate uniformization, Tate's q, Lubin-Tate theory, Honda-Tate theory, Sato-Tate conjecture, Barsotti-Tate groups, Hodge-Tate decomposition, Tate cohomology, Poitou-Tate duality, Serre-Tate parameters, Tate cycles, Tate conjectures, Mumford-Tate group, Mumford-Tate conjectures, Tate twists, Tate motives, Tate spaces, Tate ring, Tate algebra, Tate residue, Cassels-Tate pairing, Neron-Tate heights, Shafarevich-Tate groups, Koszul-Tate resolution, etc.

We first give a glimpse into Tate's personality through some of his interviews, after which we go on to describe in simple terms some areas to which he has contributed enormously.

During an interview after being awarded the Abel Prize in 2010, Tate was asked about Serge Lang who gave many of the names to notions mentioned above. In his typically unassuming manner, he said:
"*He (Lang) started Princeton graduate school in philosophy, a year after I started in physics, but he, too, soon switched to math. He was a bit younger than I and had served a year and a half in the U.S. Army in Europe after the war, where he had a clerical position in which he learned to type at incredible speed, an ability which served him well in his later book writing. He had*

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many interests and talents. I think his undergraduate degree from Caltech was in physics. He knew a lot of history and he played the piano brilliantly. It has been noted that there are many mathematical notions linked to my name. I think that is largely due to Lang's drive to make information accessible. He wrote voluminously. I did not write easily and did not get around to publishing; I was always interested in thinking about the next problem. To promote access, Serge published some of my stuff and, in reference, called things "Tate this" and "Tate that" in a way I would not have done had I been the author. Throughout his life, Serge addressed great energy to disseminating information; to sharing where he felt it was important. We remained friends over the years."

In that same interview, Tate was asked whether he was more of a theory builder or a problem solver, and he said:
"*I suppose I'm a theory builder or maybe a conjecture maker. I am not a conjecture prover very much, but I don't know. It is true that I'm not good at solving problems. For example, I would never be good in the Math Olympiad. There speed counts and I am certainly not a speedy worker. That's one pleasant thing in mathematics: It doesn't matter how long it takes if the end result is a good theorem. Speed is an advantage, but it is not essential.*"

In his article, Dinesh Thakur has referred in passing to a large number of Tate's contributions. Here, we give a brief description of some of them. While discussing these notions, we particularly choose some easy-to-understand concrete applications even though for a purist they may not be the deepest ones.

Tate's Thesis

Tate's PhD thesis written in 1950 under the direction of Emil Artin at Princeton University introduced the idea of employing harmonic analysis or Fourier analysis, to problems in classical number theory. This is ubiquitous now – it was also an inspiration for the Langlands Program. Tate's thesis built on the PhD thesis in 1946 of Margaret Matchett, another student of Artin. Tate worked with the locally compact abelian group of adeles (which is a 'restricted' product of all the completions of the field of rational numbers or of more general number fields). In this manner, the product structure of adeles makes it possible to obtain the 'zeta function' or more general 'Hecke L-functions' which are integrals over the adeles factorized as Euler products. In this manner, the crucial functional equations of these L-functions which were proved by Hecke, are quickly obtained from the Poisson summation formula. As alluded to above, these ideas were generalized by others to develop the so-called Langlands Program. Tate's work provides a new point of view of Hilbert's (abelian) class field theory. Langlands Program is a (as yet largely conjectural) non-abelian generalization of class field theory. Tate's approach is significant because it improves the classical point of view and achieves conceptually satisfactory explanations for the classical results. For instance, the adelic framework explains the role of the gamma function in the functional equation of the Riemann zeta function. At the same time, it suggests a way of unifying harmonic analytic and arithmetic considerations in the general context of automorphic forms, by working in the adelic context.

In his characteristically modest way, Tate had this to say about his thesis, in an interview after receiving the Abel Prize in 2010:

"*Well, first of all, it was not a new result, except perhaps for some local aspects. The big global theorem had been proved around 1920 by the great German mathematician Erich Hecke. Artin suggested to me that one might prove Hecke's theorem using abstract harmonic analysis on what is now called the adele ring. In a way it was just a wonderful exercise to carry out this idea. And it was also in the air. So often there is a time in mathematics for something to be done. My thesis is an example. Iwasawa would have done it had I not.*"

In answer to a question of whether mathematics is an art or a science, Tate says:
"*It is both. There certainly is an artistic aspect to mathematics. It's just beautiful. Unfortunately it's only beautiful to the initiated, to the people who do it. It can't really be understood or appreciated much on a popular level the way music can. You don't have to be a composer to enjoy music, but in mathematics you do. That's a really big drawback of the profession. A non-mathematician has to make a big effort to appreciate our work; it is almost impossible. Yes, it's both. Mathematics is an art, but there are stricter rules than in other arts. Theorems must be proved as well as formulated; words must have precise meanings. The happy thing is that mathematics does have applications which enable us to earn a good living doing what we would do even if we weren't paid for it. We are paid mainly to teach the useful stuff.*"

Let us try to get a very brief overview of what class field theory is about.

Class Field Theory – A Peek Through Some Examples.

Class field theory had its origins in Gauss's quadratic reciprocity and was developed over a century. In simple terms, number-theoretic problems over integers (for instance, the solutions of Diophantine equations) lead usually to developing the arithmetic of rings of integers in extension fields K of the rational numbers in the absence of the analogue of uniqueness of factorization. This non-uniqueness is measured by a finite group called the ideal class group of K and class field theory shows that the maximal, unramified, abelian extension (the so-called Hilbert class field of K) is a finite extension which has Galois group isomorphic to the class group of K.

Rather than get into the technicalities of this theory in this note, we give a few concrete ways in which it gets applied.

In order to study the solutions of polynomial equations $f = 0$ over integers, mathematicians have built algebraic objects K_f extending the rational numbers which have field structures on them (that is, where one can add, multiply, divide, etc.) and where the polynomial f must necessarily have all its solutions. They study these fields using their symmetries – known as the Galois group of the polynomial which is a group of permutations of the roots in this extension field. The behaviour of usual prime numbers in this extension field gives concrete number-theoretic information. For instance, if the polynomial is $f(X) = X^n - 1$, then the corresponding field K_f is generated over the rational numbers by the n-th roots of unity. This field has its own 'quota' O_f of integers generated by the usual integers along with the n-th roots of unity. The arithmetic of this more general set of "integers" is an essential tool to study number theory. The analogue of a prime number is a 'prime ideal' which is not a single element but comes as a package of elements. The prime numbers viewed in this bigger ring O_f may be factorable! For instance, the prime number 2 is factored as $(1 + i)(1 - i)$ in the ring of Gaussian integers which corresponds to the polynomial $X^2 + 1$.

These more general "integers" have a form of unique factorization (in terms of prime ideals rather than prime numbers). However, the ideals may not be determined by single elements (that is, may not be principal), unlike $\mathbb{Z}$. One has a finite group which measures this obstruction to "principality" – called the class group of O_f. Even when we have a positive obstruction, these ideas are often sufficient to yield concrete number theoretic results over the integers. For instance, the fact that the equation $x^2 + 5 = y^3$ does not have integer solutions x, y can be deduced by knowledge about the class group of the ring O_f where $f(X) = X^2 + 5$.

It has been recognized that when the extension fields have an abelian Galois group (that is, commutative group of permutations of the roots), it is determined by the set of usual prime numbers which split completely (that is, factor completely) in this ring. This is one concrete way to view abelian class field theory. For instance, in the case of $X^n - 1$, the set of primes p splitting completely in the corresponding ring are precisely those p which satisfy the congruence $p \equiv 1 \pmod{n}$.

A more precise and concrete way of describing abelian class field theory is the assertion:
The set $Spl(f)$ of prime numbers splitting completely in O_f can be described by congruences with respect to a modulus depending only on f if, and only if, K_f is an abelian extension of $\mathbb{Q}$.

The Artin reciprocity law is a precise version of ($\Leftarrow$), and ($\Rightarrow$) says that "congruence conditions" will not suffice to characterize a reciprocity law for non-abelian extensions.

Here are some concrete results demonstrating the application of class field theory.

Suppose we are interested in determining which primes p can be expressed as $x^2 + 5y^2$ for integers x, y. This reduces to the question of whether p splits completely in the Hilbert class field of $\mathbb{Q}(\sqrt{-5})$. As this latter field is $\mathbb{Q}(\sqrt{-5}, i)$, an abelian extension, the conditions on p reduce to certain congruence conditions, viz.,

$$p \equiv 1 \; or \; 9 \; (mod \; 20).$$

That is, the primes expressible as $x^2 + 5y^2$ or precisely those satisfying the above congruence conditions.

On the other hand, consider the analogous problem as to which primes are expressible as $x^2 + 23y^2$. In this case, the class field of $\mathbb{Q}(\sqrt{-23})$ is a non-abelian extension field (with symmetry = Galois group S_3 over $\mathbb{Q}$).
Therefore, the condition for a prime to be of the form $x^2 + 23y^2$ is NOT expressible as a set of congruence conditions.

In this case, it turns out that the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$ is obtained by adjoining a root of the polynomial $x^3 - x^2 + 1$. Then, one may deduce that a prime p is expressible as $x^2 + 23y^2$ if, and only if, the two polynomials $x^2 + 23$ and $x^3 - x^2 + 1$ have roots modulo p.

Actually, the Langlands Program (the non-abelian conjectural generalization of abelian class field theory) has been worked out in this case and one can rephrase these conditions on p in terms of a certain coefficient of a particular function (called a modular form, that is the non-abelian analogue of the Legendre symbol).

More precisely, p is of the form $x^2 + 23y^2$ if and only if, the coefficient of q^p is 2 in the product

$$q \prod_{n \geq 1} (1 - q^n)(1 - q^{23n}).$$

We give another type of concrete application of class field theory. For given positive integers a, b, we can explicitly find the proportion of prime numbers which divide a number of the form $a^n + b^n$ and belong to a certain arithmetic progression. For instance, the proportion of prime numbers dividing some number of the form $6^n + 1$ and belonging to the arithmetic progression $15k + 7$ is $17/192$.

Birch and Swinnerton-Dyer Conjecture

This conjecture concerns elliptic curves.
A polynomial relation $f(x, y) = 0$ over a field defines a curve. One is interested in the set of points with x, y in a given field, which we take to be $\mathbb{Q}$ in what follows. A non-singular curve (its projectivized model) is determined by a non-negative integer called its genus. The curves of genus 0 are easier to study and lead us to solving linear and quadratic equations. On the other hand, if the genus is at least 2, Faltings proved (what had been conjectured by Mordell)

that the number of points with co-ordinates in $\mathbb{Q}$ is finite. This can be thought to be a big step in the direction of Fermat's last theorem also. Tate had contributed to this development too. However, the most interesting and elusive curves to study are those of genus 1 – the so-called elliptic curves. The set of $\mathbb{Q}$-points (if non-empty) in this case has an abelian group structure. Mordell had proved that the set $E(\mathbb{Q})$ is finitely generated; so, it is isomorphic to $\mathbb{Z}^r \oplus$ (finite, abelian). The 'rank' r of this abelian group is a mysterious object. Even knowing when it is non-zero has often surprising consequences. One instance is that for the curve $y^2 = x^3 - n^2x$ for a positive integer n. The rank in this case is positive, if, and only if, n is the area of a right-angles triangle with rational sides. The Birch and Swinnerton–Dyer conjecture is a million-dollar Clay Prize problem; it provides an analytic way of getting hold of the rank.

One may write the equation of an elliptic curve E over $\mathbb{Q}$ as $y^2 = x^3 + ax + b$ with a, b integers such that this cubic has distinct roots. Equivalently, the discriminant $\Delta = -4a^3 - 27b^2 \neq 0$. For any prime p not dividing the discriminant, consider the number N_p of solutions of the congruence $x^3 + ax + b \equiv 0 \pmod{p}$. One forms the 'partial' L-function

$$L(E, s) = \prod_p (1 - (p - N_p)p^{-s} + p^{1-2s})^{-1} ,$$

where the product is over odd primes not dividing Δ. This is considered a function of the complex variable s and the product converges for $Re(s) > 3/2$. A long-standing conjecture due to Hasse has been proved now showing that $L(E, s)$ has an analytic continuation to all complex numbers s. Then, we have:

Birch and Swinnerton-Dyer Conjecture: *The Taylor expansion of $L(E, s)$ around $s = 1$ is of the form $c(s - 1)^r$+ higher order terms with $c \neq 0$.*
This order of zero is the rank of $E(\mathbb{Q})$.
One has a refined version which involves the 'completed' L-function where terms corresponding to primes dividing 2Δ are also included. In this case, the main terms is of the form $c_0(s - 1)^r$ where c_0 is given in terms of the finite part of $E(\mathbb{Q})$ and also something called the Tate-Shafarevich group which somehow measures a local to global failure.

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