Classroom

In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Ramanujan-type Numbers

Diophantine geometry is a fascinating interplay between algebra, geometry and number theory. The author was introduced to it in a blackboard-less one-on-one talk given in the aerodrome at IIT Kanpur during a thunderstorm! The speaker was Professor V. Srinivas of the TIFR. At that point in time, the latter was a graduate student at the University of Chicago and happened to be in Kanpur.

A ‘classical’ story has Ramanujan telling Hardy that the number 1729 is the smallest number that is the sum of two cubes in two different ways. (We have $1729 = 10^3 + 9^3$ and $1729 = 12^3 + 1^3$.) One of my friends mis-narrated this story; he said that 1729 is the only such number, instead of saying it is the smallest such number. Now one can quickly point out that $1729 \cdot n^3$ is also (in an obvious way) the sum of two cubes in two different ways. So, this error in narration is quickly corrected.

A more sophisticated question is to ask whether it is possible to write other solutions $(W : X : Y : Z)$ of the equations $X^3 + Z^3 = W^3 + Y^3$, where the numbers have no common factors.

Keywords
Geometry, algebra, number theory, cubes eliminating variable, quadratic equation.

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Apparently Magic Solution

Here is a solution! The number 9261729 can be written as \(210^3 + 9^3\) and also \(172^3 + 161^3\).

More generally, we can write the solution (for any integers \(p\) and \(q\)):

\[
\begin{align*}
W &= p^2 - 29 \cdot pq + 189 \cdot q^2 \\
X &= 2 \cdot p^2 - 44 \cdot pq + 252 \cdot q^2 \\
Y &= 2 \cdot p^2 - 40 \cdot pq + 210 \cdot q^2 \\
Z &= p^2 - 13 \cdot pq + 21 \cdot q^2
\end{align*}
\]  

(1)

The reader is urged to check that this actually gives a solution of \(W^3 + Y^3 = X^3 + Z^3\) by substitution. After the substitution of \(p\) and \(q\), one may need to clear the common factors to get a solution of the required type.

It should be reasonably clear that such a formula cannot be obtained by ‘inspection’. How did it come about?

Algebra Behind the Solution

There is a “purely algebra” method of deriving the above solution (1), which we now explain. It too will make use of some ‘magical’ substitutions which (the author thinks!) can only be explained by geometric insights which are outlined in the third section.

Reducing the Degree

The first idea used in the calculation is the identity

\[
(a - b) \cdot (a^2 + ab + b^2) = a^3 - b^3.
\]

The equation \(W^3 + Y^3 = X^3 + Z^3\) can be written as \(W^3 - Z^3 = X^3 - Y^3\). Using the identity above this becomes,

\[
(W - Z) \cdot (W^2 + W \cdot Z + Z^2) = (X - Y) \cdot (X^2 + X \cdot Y + Y^2)
\]
So, we can think of this as being a pair of equations
\[ W - Z = u \cdot (X - Y) \]
\[ u \cdot (W^2 + W \cdot Z + Z^2) = X^2 + X \cdot Y + Y^2 \]

We note that for the given solution, we have \( W - Z = 9 - 1 = 8 \), while \( X - Y = 12 - 10 = 2 \). So we can set \( u = 4 \) and try to find other solutions of the pair of equations:
\[ W - Z = 4 \cdot (X - Y), \quad (2) \]
\[ 4 \cdot (W^2 + W \cdot Z + Z^2) = X^2 + X \cdot Y + Y^2. \quad (3) \]

**Eliminating a Variable**

Our next step is to eliminate one variable using first equation (2). The identities that make this easier to do are:
\[
(a + b)^2 + (a - b)^2 = 2 \cdot (a^2 + b^2),
\]
\[
(a + b)^2 - (a - b)^2 = 4 \cdot ab.
\]

This allows us to re-write a linear combination of \( a^2 + b^2 \) and \( ab \) in terms of \( (a + b)^2 \) and \( (a - b)^2 \). This is also useful since we can write \( a \) and \( b \) in terms of \( a + b \) and \( a - b \).

In particular, we note that
\[
4 \cdot (W^2 + W \cdot Z + Z^2) = 3 \cdot (W + Z)^2 + (W - Z)^2.
\]

This is useful since we have \( W - Z = 4 \cdot (X - Y) \) from equation (2). Hence, if we put \( W + Z = T \), then we can replace \( W - Z \) in equation (3) to obtain
\[
3 \cdot T^2 + 16 \cdot (X - Y)^2 = X^2 + X \cdot Y + Y^2
\]
which simplifies to
\[
3 \cdot (T^2 + 5 \cdot X^2 + 5 \cdot Y^2 - 11 \cdot X \cdot Y) = 0
\]

Note that if we have values of \( T \) (which is \( W + Z \)) and \( X \) and \( Y \) which satisfy this equation, then we can obtain
\[
W = \frac{T + 4(X - Y)}{2}
\]
\[
Z = \frac{T - 4(X - Y)}{2}
\]
by reversing the above equations.

So, we only need to find solutions of the single equation,

$$T^2 + 5 \cdot X^2 + 5 \cdot Y^2 - 11 \cdot X \cdot Y = 0. \quad (4)$$

This is a quadratic equation in three variables ($T : X : Y$) with the solution $(5 : 6 : 5)$ corresponding to Ramanujan’s solution $(W : X : Y : Z) = (9 : 12 : 10 : 1)$ for the original problem.

**Solving Quadratic Equations**

The third idea behind the solution is that if we have a quadratic equation (4), *and* a solution of it, then we can ‘complete squares’ to find a parametric solution. This step is somewhat complex and is better explained in the next section.

For now, let us note that putting $X = 2T - Y$ in the above quadratic expression gives:

$$T^2 + 5 \cdot (2 \cdot T - Y)^2 + 5 \cdot Y^2 - 11 \cdot (2 \cdot T - Y) \cdot Y$$

$$= 21 \cdot (T^2 + Y^2 - 2 \cdot T \cdot Y)$$

$$= 21 \cdot (Y - T)^2$$

This suggests that we have an identity of the form

$$T^2 + 5 \cdot X^2 + 5 \cdot Y^2 - 11 \cdot X \cdot Y$$

$$= (a \cdot T + b \cdot X + c \cdot Y) \cdot (X + Y - 2 \cdot T) + 21 \cdot (Y - T)^2$$

for suitable constants $a$, $b$ and $c$. Solving for these constants, we get the identity:

$$T^2 + 5 \cdot X^2 + 5 \cdot Y^2 - 11 \cdot X \cdot Y$$

$$= (10 \cdot T + 5 \cdot X - 16 \cdot Y) \cdot (X + Y - 2 \cdot T) + 21 \cdot (Y - T)^2$$

We can thus re-write the above quadratic equation as

$$(10 \cdot T + 5 \cdot X - 16 \cdot Y) \cdot (X + Y - 2 \cdot T) + 21 \cdot (Y - T)^2 = 0$$
This equation can be again broken into the pair of linear equations (for a suitable constant $r$):

$$10 \cdot T + 5 \cdot X - 16 \cdot Y = r \cdot (T - Y) \quad (5)$$

$$r \cdot (X + Y - 2 \cdot T) = 21 \cdot (Y - T). \quad (6)$$

From equation (6), we obtain

$$X = \frac{(21 - r) \cdot Y + (2 \cdot r - 21) \cdot T}{r}. \quad (7)$$

Substituting this into equation (5) and clearing denominators, we obtain

$$(r^2 - 21 \cdot r + 105) \cdot Y - (r^2 - 20 \cdot r + 105) \cdot T = 0.$$

Hence, it is clear that $(T : Y) = (r^2 - 21 \cdot r + 105 : r^2 - 20 \cdot r + 105)$ is a solution. Substituting these values back into the (7) for $X$ gives us

$$X = r^2 - 22 \cdot r + 126.$$

Now, we use the formulae (mentioned above)

$$W = T/2 + 2 \cdot (X - Y) \quad \text{and} \quad Z = T/2 - 2 \cdot (X - Y).$$

This gives the algebraic solution (1) after substituting $r = p/q$ and clearing denominators.

**Geometry Behind the Algebra**

The above algebra is still quite mysterious! Looking at it geometrically makes things a bit simpler.

The projective space $\mathbb{P}^3$ is the space of tuples $(W : X : Y : Z)$, considered equivalent up to multiplication by a non-zero rational number. (We also exclude the point $(0,0,0,0)$.)

In this projective space, the surface $S$ is defined by the equation $W^3 - Z^3 = X^3 - Y^3$.

This surface $S$ contains the projective line $L$ which consists of all points of the form $(a : b : b : a)$. This line $L$ is defined by the linear equations $W - Z = 0$ and $X - Y = 0$. 
The Ramanujan point \((W : X : Y : Z) = (9 : 12 : 10 : 1)\) and this line determine the project plane \(P\) defined by the linear equation \((W - Z) = 4(X - Y)\).

The Ramanujan point \((W : X : Y : Z) = (9 : 12 : 10 : 1)\) and this line determine the project plane \(P\) defined by the linear equation \((W - Z) = 4(X - Y)\). The intersection of this plane \(P\) and the surface \(S\) consists of the line \(L\) and the curve \(C\) defined by the pair of equations

\[
W - Z = 4 \cdot (X - Y) \text{ and } 4 \cdot (W^2 + W \cdot Z + Z^2) = X^2 + X \cdot Y + Y^2
\]

This curve is sometimes called the residual intersection of the plane with \(S\) after leaving out the line \(L\). Since \(C\) is a projective plane curve we see immediately that it is a “conic” since it is defined within the plane by a single quadratic equation. The Ramanujan point is a point on this conic \(C\).

It is a standard fact that given a point on a conic \(C\) we can produce a parametric solution of the conic as follows. Each line through the point meets the conic at one other point. This gives a bijection between points on \(C\) and lines through the point; the tangent line corresponds to the point itself. This is the basis of the last bit of algebra above.

There are many more results regarding points on surface (one defined by a homogeneous equation of degree 3) in projective space.

Conclusion

This example can be seen to demonstrate the power of algebraic geometry to provide insight into the solution of certain types of Diophantine problems. The intrepid reader may want to try to use some more algebraic geometry to check whether integer solutions are dense in the space of solutions over real numbers.

The Cubic Surface [2] has been an object of study many times, especially the 27 lines on it and the associated Double Six [4]. A more complete solution of the equation studied in this article is provided by Noam Elkies.
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Suggested Reading